PROFINITE ITERATED MONODROMY GROUPS OF UNICRITICAL POLYNOMIALS

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Abstract. Let $f(x) = ax^d + b \in K[x]$ be a unicritical polynomial with degree $d \geq 2$ which is coprime to $\operatorname{char} K$. We provide an explicit presentation for the profinite iterated monodromy group of f, analyze the structure of this group, and use this analysis to determine the constant field extension in $K(f^{-\infty}(t))/K(t)$.

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1. Introduction

Let K be a field and let $f(x) \in K(x)$ be a rational function of degree $d \geq 2$ coprime to $\operatorname{char} K$. We write $f^n := f \circ f \circ \cdots \circ f$ to denote the n-fold composition of f. Given an element β in some extension of K and a choice $K(\beta)^{\mathrm{sep}}$ of separable closure, let $K(f^{-n}(\beta))$ denote the splitting field of $f^n(x) = \beta$ in $K(\beta)^{\mathrm{sep}}$ over $K(\beta)$.

If β is not post-critical, then $f^{-n}(\beta)$ contains d^n distinct elements in $K(\beta)^{\mathrm{sep}}$, and together the sets $f^{-n}(\beta)$ naturally carry the structure of a regular rooted d-ary tree T_d^∞ with root β where each $\alpha \in f^{-n}(\beta)$ is a child of $f(\alpha)$. The absolute Galois group $\operatorname{Gal}(K(\beta)^{\mathrm{sep}}/K(\beta))$ fixes K, hence the coefficients of f, and thus permutes the nodes at each level of this tree while respecting the tree structure. This gives us an *arboreal Galois representation*

$$\rho : \operatorname{Gal}(K(\beta)^{\mathrm{sep}}/K(\beta)) \to \operatorname{Aut}(T_d^\infty).$$

These Galois representations encode interesting arithmetic dynamical information, much of which is yet to be understood. Arboreal representations were first introduced and studied by Odoni [Odo85b; Odo85a; Odo88] who used the Chebotarev density theorem to link these representations to the density of prime divisors in orbits. See [Jon13] for a survey of arboreal representations up to 2013 and [Ben+19, Sec. 5] for updates through 2018.

Of particular interest is when the function f is *post-critically finite* or PCF: when every critical point has a finite forward orbit. In the PCF case, iterated pre-image extensions $K(f^{-n}(\beta))/K(\beta)$ have ramification uniformly constrained to a finite set of places, and the image of the arboreal Galois representation is topologically finitely generated. There is a growing literature on arboreal representations for PCF maps, including but not limited to [Pil00; Nek05; BN08; AHM05; Ben+17; BEK21; Ada23; Ejd24; Ben+25].

In this paper we focus on unicritical polynomials $f(x) = ax^d + b$ with a generic base point $\beta = t$ where t is transcendental over K. Note that in this case, with d coprime to $\operatorname{char} K$, the extensions $K(f^{-n}(t))/K(t)$ are separable, hence Galois. Let $\operatorname{Arb} f$ denote the image of the arboreal Galois representation of f with transcendental base point and let $\overline{\operatorname{Arb}}\, f \subseteq \operatorname{Arb} f$ denote the arboreal representation of $\operatorname{Gal}(K(t)^{\mathrm{sep}}/K^{\mathrm{sep}}(t))$. The groups $\operatorname{Arb} f$ and $\overline{\operatorname{Arb}}\, f$ are (isomorphic to) the *arithmetic profinite iterated monodromy group* and *geometric profinite iterated monodromy group* of f, respectively. There is a short exact sequence

$$1 \to \overline{\operatorname{Arb}}\, f \to \operatorname{Arb} f \to \operatorname{Gal}(\widehat{K}_f/K) \to 1$$

where $\widehat{K}_f$ is the algebraic closure of K in $K(f^{-\infty}(t))$, the extension formed by adjoining all iterated f-preimages of t. We refer to $\widehat{K}_f$ as the *constant field extension* of f. As with any short exact sequence, there is an outer action of $\operatorname{Gal}(\widehat{K}_f/K)$ on $\overline{\operatorname{Arb}}\, f$ given by lifting to $\operatorname{Arb} f$ and conjugating. To understand $\operatorname{Arb} f$, we study the group $\overline{\operatorname{Arb}}\, f$, the constant field extension $\widehat{K}_f/K$, and the outer action of $\operatorname{Gal}(\widehat{K}_f/K)$ on $\overline{\operatorname{Arb}}\, f$.

1.1. Results.

Our first main result provides an explicit recursive topological presentation of $\overline{\operatorname{Arb}}\, f$ for any unicritical PCF polynomial. First, some notation. If $G \subseteq S_d$ is a subgroup of the symmetric group, then we write $[G]^\infty$ to denote the iterated wreath product of G with itself

$$[G]^\infty = G \ltimes ([G]^\infty)^d,$$

where G acts on d-tuples $(g_1, g_2, \ldots, g_d)$ by permuting indices (see Section 2.2). We write elements of $[G]^\infty$ as $g(g_1, g_2, \ldots, g_d)$ where $g \in G$ and $g_i \in [G]^\infty$. The element $g(g_1, g_2, \ldots, g_d)$ acts on the tree T_d^∞ by g on the first level and g_i on the ith subtree.

Let $\sigma := (123 \cdots d) \in S_d$ be a d-cycle and $C_d = \langle \sigma \rangle$. In Section 3, we show that the tree T_d^∞ may be labeled so that $\overline{\operatorname{Arb}}\, f \subseteq [C_d]^\infty$ when $f(x) = ax^d + b$ is unicritical of degree d prime to $\operatorname{char} K$. In Section 4 we show that the structure of the group $\overline{\operatorname{Arb}}\, f$ is determined by the orbit of the critical point 0 under f, naturally splitting into three cases: post-critically infinite (PCI), periodic, or (strictly) preperiodic. In the first two cases, the combinatorial structure of the orbit alone entirely determines $\overline{\operatorname{Arb}} f$, but in the preperiodic case a small arithmetic input is also required.

Theorem 1.1. *Let $f(x) = ax^d + b \in K[x]$ where d is coprime to* $\operatorname{char} K$*. There exists a labeling of T_d^∞ such that $\overline{\operatorname{Arb}} f \subseteq [C_d]^\infty$, and*

(1) (PCI) If 0 has an infinite orbit under f, then

$$\overline{\operatorname{Arb}} f = [C_d]^\infty.$$

(2) (Periodic) If 0 is periodic with period n under f, then $\overline{\operatorname{Arb}} f = \langle\langle a_1, a_2, \dots, a_n \rangle\rangle$ where

$$a_i = \begin{cases} \sigma(1, \dots, 1, a_n) & \text{if } i = 1, \\ (1, \dots, 1, a_{i-1}) & \text{if } i \neq 1. \end{cases}$$

(3) (Preperiodic) If 0 is strictly preperiodic under f, let $m < n$ be the smallest integers such that $f^{m+1}(0) = f^{n+1}(0)$. Let ζ_d be a choice of primitive dth root of unity in K^{sep}. Since $f^m(0)$ and $f^n(0)$ have the same image under $f(x) = ax^d + b$, there exists some $1 \leq \omega < d$ such that $f^n(0) = \zeta_d^\omega f^m(0)$. Then $\overline{\operatorname{Arb}} f = \langle\langle b_1, b_2, \dots, b_n \rangle\rangle$ where

$$b_i = \begin{cases} \sigma & \text{if } i = 1, \\ (1, \dots, 1, b_n, 1, \dots, 1, b_m) & \text{if } i = m+1, \text{ where } b_n \text{ is in the } \omega\text{th component}, \\ (1, \dots, 1, b_{i-1}) & \text{if } i \neq 1, m+1. \end{cases}$$

The key technical result which allows us to derive these explicit recursive presentations for $\overline{\operatorname{Arb}} f$ in the periodic and preperiodic cases is the following *semirigidity* result, which implies that $\overline{\operatorname{Arb}} f$ is determined by the $[C_d]^\infty$ conjugacy classes of its inertia generators.

Theorem 1.2. *Let $f(x) = ax^d + b \in K[x]$ where d is coprime to* $\operatorname{char} K$*. Suppose f is PCF and that the strict forward orbit of 0 has n elements. Then $\overline{\operatorname{Arb}} f = \langle\langle c_1, c_2, \dots, c_n \rangle\rangle \subseteq [C_d]^\infty$ where each c_i is the image of an inertia generator over $f^i(0)$. If $c_i' \in [C_d]^\infty$ are elements such that c_i is conjugate to c_i' in $[C_d]^\infty$ for each i, then there exists an element $w \in [C_d]^\infty$ and elements $u_i \in \overline{\operatorname{Arb}} f$ such that*

$$w c_i' w^{-1} = u_i c_i u_i^{-1}$$

for each i and

$$w \langle\langle c_1', c_2', \dots, c_n' \rangle\rangle w^{-1} = \overline{\operatorname{Arb}} f.$$

Theorem 1.1 allows us to say a lot about the structure of $\overline{\operatorname{Arb}} f$. For example, we determine the abelianization $\overline{\operatorname{Arb}} f$. Let $\mathbb{Z}_d$ denote the additive group of d-adic integers.

Proposition 1.3. *Let $f(x) = ax^d + b \in K[x]$ where d is coprime to* $\operatorname{char} K$ *and let $\overline{\operatorname{Arb}} f^{\text{ab}}$ denote the abelianization of $\overline{\operatorname{Arb}} f$. If f is PCF, let n denote the length of the strict forward orbit of 0 under f. Then*

$$\overline{\operatorname{Arb}} f^{\text{ab}} \cong \begin{cases} (\mathbb{Z}/d\mathbb{Z})^\infty & \text{(PCI)} \\ \mathbb{Z}_d^n & \text{(Periodic)} \\ (\mathbb{Z}/d\mathbb{Z})^n & \text{(Preperiodic)} \end{cases}$$

Other properties of $\overline{\operatorname{Arb}} f$ require us to further split the preperiodic case into several subcases:

(A_1) $\omega \neq d/2$,
(A_2) $m > 1$ and $(d, n) \neq (2, m+1)$,
(A_3) $d = 2$, $m > 2$, and $n = m + 1$,
(B_1) $d > 2$, $\omega = d/2$, and $m = 1$,
(B_2) $d = 2$, $m = 1$, and $n > 2$,

(*C*) $d = 2$, $m = 2$, and $n = 3$.
(*D*) $d = 2$, $m = 1$, and $n = 2$.

We let (A) refer to the assumption (A_1),(A_2), or (A_3), and let (B) refer to (B_1) or (B_2). Note that these hypotheses exhaust all possible cases with $d \geq 2$, $1 \leq m < n$, and $1 \leq \omega < d$. When $d = 2$, we have $\omega = d/2 = 1$ by default. Case (D) is exceptional throughout; it corresponds, up to conjugacy, to the quadratic Chebyshev polynomial $x^2 - 2$.

We also determine the Hausdorff dimension and orders of finite level truncations of $\overline{\mathrm{Arb}}\, f$. Given $\ell \geq 0$, let $[C_d]^\ell$ denote the ℓ-fold iterated wreath product of C_d and let $\rho_\ell : [C_d]^\infty \to [C_d]^\ell$ denote the truncation map. Let $[\ell]_d := 1 + \ell + \ell^2 + \ldots + \ell^{d-1}$. Given a subgroup $H \subseteq [C_d]^\infty$ we define $\mathrm{ord}_\ell(H) := \mathrm{ord}(\rho_\ell(H))$ to be the order of the level ℓ truncation of H and we define the *Hausdorff dimension* of H to be

$$\mu_{\mathrm{haus}}(H) := \lim_{\ell \to \infty} \frac{\log_d \mathrm{ord}_\ell(H)}{\log_d \mathrm{ord}_\ell([C_d]^\infty)} = \lim_{\ell \to \infty} \frac{\log_d \mathrm{ord}_\ell(H)}{[\ell]_d},$$

provided the limit exists.

Proposition 1.4. *Let $f(x) = ax^d + b \in K[x]$ where d is coprime to $\mathrm{char}\, K$, let n be as in Theorem 1.1, and let q_ℓ and r_ℓ be the unique integers such that $\ell = q_\ell n + r_\ell$ and $0 \leq r_\ell < n$. Then the values of $\mathrm{ord}_\ell(\overline{\mathrm{Arb}}\, f)$ and $\mu_{\mathrm{haus}}(\overline{\mathrm{Arb}}\, f)$ are as listed in the table below.*

	$\log_d \mathrm{ord}_\ell(\overline{\mathrm{Arb}}\, f)$	$\mu_{\mathrm{haus}}(\overline{\mathrm{Arb}}\, f)$
PCI	$[\ell]_d$	1
Periodic	$[\ell]_d - d^{r_\ell}[q_\ell]_{d^n} + q_\ell$	$1 - \frac{d-1}{d^n-1}$
(A)	$[\ell]_d + d[\ell - n]_d - 2[\ell - n + 1]_d + 2$	$1 - \frac{1}{d^{n-1}}$
(B)	$[\ell]_d + \frac{3d}{2}[\ell - n]_d - 3[\ell - n + 1]_d + 3$	$1 - \frac{3}{2d^{n-1}}$
(C)	$11 \cdot 2^{\ell-1} + 2$	$\frac{11}{16}$
(D)	$\ell + 1$	0

If $\ell \geq 0$, let $\widehat{K}_{f,\ell}$ denote the algebraic closure of K in $K(f^{-\ell}(t))$. We establish a general bound on $\widehat{K}_{f,\ell}$ which holds for all polynomials with degree coprime to $\mathrm{char}\, K$ and show that $\widehat{K}_{f,\ell}$ is completely encoded within the structure of $\overline{\mathrm{Arb}}\, f$. Given $g, h \in [C_d]^\infty$ and $\ell \geq 1$ we say that $g \sim_\ell h$ if $\rho_\ell(g)$ is conjugate to $\rho_\ell(h)$.

Theorem 1.5. *Let K be a field and let $f(x) \in K[x]$ be a polynomial with degree d coprime to $\mathrm{char}\, K$. Let $g_\infty \in \overline{\mathrm{Arb}}\, f$ denote the image of an inertia generator over ∞ in $\overline{\mathrm{Arb}}\, f$ and let $\chi_{\mathrm{cyc}} : \mathrm{Gal}(K^{\mathrm{sep}}/K) \to \widehat{\mathbb{Z}}^\times$ denote the cyclotomic character of K. Then for $1 \leq \ell \leq \infty$ we have $\widehat{K}_{f,\ell} \subseteq K(\zeta_{d^\infty})$ and $\tau \in \mathrm{Gal}(K^{\mathrm{sep}}/K)$ fixes $\widehat{K}_{f,\ell}$ if and only if $g_\infty \sim_\ell g_\infty^{\chi_{\mathrm{cyc}}(\tau)}$ in $\overline{\mathrm{Arb}}\, f$.*

Theorem 1.5 reduces the analysis of $\widehat{K}_{f,\ell}$ for polynomials to a purely group theoretic problem about $\overline{\mathrm{Arb}}\, f$. We leverage our explicit recursive presentation of $\overline{\mathrm{Arb}}\, f$ provided by Theorem 1.1 to precisely determine the $\widehat{K}_{f,\ell}$.

Theorem 1.6. *Let $f(x) = ax^d + b \in K[x]$ where d is coprime to $\mathrm{char}\, K$. Let m, n, ω be as in Theorem 1.1 and cases (A) through (D) as described above. Let $1 \leq \ell \leq \infty$.*

(1) (PCI) $\widehat{K}_{f,\ell} = K(\zeta_d)$.
(2) (Periodic) $\widehat{K}_{f,\ell} = K(\zeta_{d^{\lfloor (\ell-1)/n \rfloor + 1}})$. Hence $\widehat{K}_f = K(\zeta_{d^\infty})$.

(3) (Preperiodic) If $\ell \le n$*, then* $\widehat{K}_{f,\ell} = K(\zeta_d)$ *and if* $\ell > n$*, then*

$$\widehat{K}_{f,\ell} = \begin{cases} K(\zeta_{d^2/\gcd(d,\omega)}) & \text{if (A) and either } m > 1 \text{ or } d \text{ odd,} \\ K(\zeta_{d^2/\gcd(d,\omega+d/2)}) & \text{if (A) and } m = 1 \text{ and } d \text{ even,} \\ K(\zeta_{4d}) & \text{if (B) or (C),} \\ K(\zeta_{2^\ell} + \zeta_{2^\ell}^{-1}) & \text{if (D).} \end{cases}$$

In particular, if not (D), then

$$K(\zeta_d) \subseteq \widehat{K}_f \subseteq K(\zeta_{2d^2}).$$

1.2. **Related work.** Nekrashevych [Nek05, Sec. 6.4.2] attributes the introduction of profinite iterated monodromy groups $\mathrm{Arb}\, f$ to private communication with Richard Pink from the year 2000. The first calculation of finite level truncations of $\overline{\mathrm{Arb}}\, f$ were carried out by Pilgrim [Pil00, Thm. 4.2] for a certain subfamily of dynamical Belyi polynomials.

If K has characteristic 0, then one may use topological methods to analyze the group $\overline{\mathrm{Arb}}\, f$. In particular, $\overline{\mathrm{Arb}}\, f$ is the profinite completion of the *discrete iterated monodromy group* of f, denoted IMGf, which is the arboreal representation of the fundamental group of $\mathbb{P}^1(\mathbb{C})$ punctured at each point of the post-critical set of f. Bartholdi and Nekrashevych [BN08] determined IMGf for all PCF quadratic polynomials, showing that these groups are determined by the kneading sequence of the polynomial. Nekrashevych [Nek05, Thm. 5.5.3] showed that the Julia set of f may be recovered from IMGf.

In 2013, Pink posted a series of preprints [Pin13a; Pin13b; Pin13c] analyzing the groups $\mathrm{Arb}\, f$ and $\overline{\mathrm{Arb}}\, f$ for quadratic polynomials $f(x) \in K[x]$ where K has odd characteristic. The first paper [Pin13a] takes an algebro-geometric approach, arguing that quadratic PCF polynomials can be lifted to characteristic 0 maps with the same post-critical combinatorics. This combined with Grothendieck's comparison theorem for the tame étale fundamental group allowed Pink to show that $\overline{\mathrm{Arb}}\, f$ must be the same in both cases.

Our work is inspired by Pink's subsequent papers [Pin13b; Pin13c] in which he takes a purely group theoretic approach to analyzing $\overline{\mathrm{Arb}}\, f$ for quadratic polynomials. One insight gleaned from this perspective is that while the groups IMGf depend on the kneading sequence of f, their closures $\overline{\mathrm{Arb}}\, f$ only depend on the combinatorial structure of the post-critical orbit; hence $\overline{\mathrm{Arb}}\, f$ is a coarser invariant of f than IMGf. At the heart of his strategy is a semirigidity result [Pin13c, Thm. 0.3] which we have generalized to all unicritical polynomials in Theorem 1.2. Pink uses semirigidity to deduce the degree 2 cases of our main results: Compare Theorem 1.1 with [Pin13c, Thm. 2.4.1, Thm. 3.4.1]; Proposition 1.3 with [Pin13c, Thm. 2.2.7, Thm. 3.1.6]; Proposition 1.4 with [Pin13c, Prop. 2.3.1, Prop. 3.3.3]; Theorem 1.6 with [Pin13c, Thm. 2.8.4, Thm. 3.10.5, Cor. 3.10.6].

A number of challenges arise while generalizing Pink's semirigidity results from degree 2 to all $d \ge 2$. For example, Pink leverages the fact that $\overline{\mathrm{Arb}}\, f$ is a pro-p group in the degree 2 case (with $p = 2$) in a crucial way. We circumvent this issue with an intermediate rigidity result which works uniformly for all degrees (Lemma 4.19). Even the identification of $\overline{\mathrm{Arb}}\, f$ and $\mathrm{Arb}\, f$ with subgroups of $\mathrm{Aut}\, T_d^\infty$ requires more care; important "character maps", natural analogues of the sign homomorphisms, are not defined on $\mathrm{Aut}\, T_d^\infty$, nor preserved by conjugation from $\mathrm{Aut}\, T_d^\infty$. When $d > 2$ we are required to make a careful choice of "algebraic paths" to embed $\overline{\mathrm{Arb}}\, f$ into $[C_d]^\infty$ (Proposition 3.13). This did not appear in the degree $d = 2$ case, where $S_2 = C_2$ and the character maps are precisely the usual sign homomorphisms.

Our strategy for analyzing the constant field extensions $\widehat{K}_f/K$ differs from the one taken by Pink. Pink first determines the quotient $N(\overline{\mathrm{Arb}}\, f)/\overline{\mathrm{Arb}}\, f$, where $N(\overline{\mathrm{Arb}}\, f)$ is the normalizer of $\overline{\mathrm{Arb}}\, f$ in $[S_2]^\infty$, and the natural action of this group on the abelianization of $\overline{\mathrm{Arb}}\, f$; then he determines the image of $\mathrm{Gal}(\widehat{K}_f/K)$ in this action. Instead we use the *branch cycle lemma* (attributed to Fried [Fri73]; see Lemma 6.4) to reduce the analysis of $\widehat{K}_f/K$ to the group theoretic problem of determining which powers of an inertia generator at infinity c_∞ are conjugate to c_∞ in $\overline{\mathrm{Arb}}\, f$ (Theorem 6.5). Our study of the normalizer $N(\overline{\mathrm{Arb}}\, f)$ limited to showing it acts transitively on odometers in $\overline{\mathrm{Arb}}\, f$ (Proposition 5.9). Circumventing the analysis of the normalizer quotient provides a more efficient route to the constant field calculations.

Unfortunately, Pink did not publish his results; they are only available as arXiv preprints. While our work draws significant inspiration from Pink, there is no logical dependence on these unpublished papers. We include all the details covering degree 2 for completeness.

The containment $K(\zeta_{d^\infty}) \subseteq \widehat{K}_f$ in the periodic case of Theorem 1.6 was anticipated by a result of Hamblen and Jones [HJ24, Thm. 2.1], building on a construction from Benedetto et al. [Ahm+22, Lem. 1.4]. We refine this containment to an exact determination of $\widehat{K}_{f,\ell}$ using our group theoretic techniques in Proposition 6.7.

1.3. **Directions for future work.** Much of the work on arboreal representations has focused on the extensions $K(f^{-\infty}(\beta))/K(\beta)$ where β is algebraic over K. These may be viewed as specializations of the extensions $K(f^{-\infty}(t))/K(t)$ that we study. We have not considered specializations in this paper, but this is a natural next direction to pursue. In [Ben+25], the authors develop a technique for analyzing these specializations via the Frattini subgroup of $\overline{\mathrm{Arb}}\, f$. They focus on settings where $\overline{\mathrm{Arb}}\, f$ is a pro-nilpotent group and topologically finitely generated, in which case, the Frattini subgroup is especially well-behaved. Their results apply to unicritical polynomials with prime power degree. When d is divisible by at least two primes, the groups $\overline{\mathrm{Arb}}\, f$ will no longer be pro-nilpotent. It is unclear what the Frattini subgroups look like in those case and how their arguments may be adapted.

Of course we would like to understand the groups $\mathrm{Arb}\, f$ for more general PCF rational functions, but it is hard to predict how far our techniques will extend. There are two main obstacles: semirigidity and the branch cycle lemma reduction of the constant field extension problem. Semirigidity seems too strong to generalize in the same form much further beyond the unicritical case. For example, one interesting new phenomenon that arises for $d > 2$ is the parameter ω in the preperiodic case. When $d = 2$, we have $\omega = 1$ by default. The structure of $\overline{\mathrm{Arb}}\, f$ and $\widehat{K}_f$ depends on ω in subtle and interesting ways. Perhaps semirigidity could be extended by accounting for the right invariants? As for the branch cycle lemma reduction, the fact that polynomials have a totally tamely ramified fixed point, and hence a self-centralizing inertia subgroup, significantly constrains the outer action. For example, we use it to show that the outer action is faithful. This is almost certainly true for rational functions, but would seem to require a different approach. Beyond that, this distinguished self-centralizing subgroup is pro-cyclic, which further simplifies the group-theoretic analysis. For rational functions, we cannot expect such straightforward behavior; our single conjugacy problem likely expands to a family of entangled conjugacy problems.

1.4. **Overview.** In Section 2 we review conjugacy in wreath products and systems of recurrences in iterated wreath products. Here we define the notion of a *system of cyclic conjugate recurrences* which are essential for semirigidity.

Section 3 establishes the foundations of iterated preimage extensions and profinite iterated monodromy groups. We formally define self-similar embeddings of iterated monodromy groups using collections of algebraic paths and construct an especially nice collection of paths for any PCF polynomial in Proposition 3.13. The main result of this section is Proposition 3.15, which provides a nice family of conjugate recurrences for topological generators of $\overline{\mathrm{Arb}}\, f$.

Sections 4 and 5 are the technical heart of the paper. In Section 4, we define *model groups* with recursive topological presentations reflecting the conjugate recurrences deduced in the previous section. We analyze the structure of these groups in detail, ultimately leading to the semirigidity results. This allows us to identify $\overline{\mathrm{Arb}}\, f$ with one of these model groups and thereby translate all the results about model groups into results about $\overline{\mathrm{Arb}}\, f$.

Then, in Section 5, we study the normalizer of the model groups, the odometers, and certain distinguished elements called power conjugators. Building on these group-theoretic facts, in Section 6 we prove our results on constant field extensions. Here we obtain the cyclotomic bounds on $\widehat{K}_f$ for any polynomial K, review the branch cycle lemma, and characterize $\widehat{K}_{f,\ell}/K$ in terms of inertia at ∞ for polynomials. We finish with a translation of structural results about $\overline{\mathrm{Arb}}\, f$ into a precise determination of the $\widehat{K}_{f,\ell}/K$ for all unicritical polynomials.

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2. Preliminaries

In this section we review the construction of wreath products, characterize their conjugacy classes, and discuss systems of recursion in iterated wreath products. Throughout this paper we repeatedly leverage the recursive structure of iterated wreath products to make inductive arguments; Proposition 2.8 is the essential tool that makes this work.

2.1. **Conjugacy in wreath products.** Let $d \geq 2$ be an integer. Let G and H be groups and suppose that G acts on the set $\{1, 2, \ldots, d\}$ from the left. The wreath product $G \wr H$ of G and H is the semidirect product $G \ltimes H^d$ associated to the action of G on H^d given by permuting coordinates.

The underlying set is $G \times H^d$, so we write elements of $G \wr H$ in the form $g(h_1, h_2, \ldots, h_d)$ where $g \in G$ and $h_i \in H$ for each i. The natural inclusion of G into $G \wr H$ is given by $g \mapsto g(1, \ldots, 1)$ and conjugation permutes coordinates:

$$(h_1, \ldots, h_d)^g = g^{-1}(h_1, \ldots, h_d)g = (h_{g(1)}, \ldots, h_{g(d)}).$$

The product of two general elements $g(h_1, \ldots, h_d)$ and $g'(h'_1, \ldots, h'_d)$ is given by

$$g(h_1, \ldots, h_d) \cdot g'(h'_1, \ldots, h'_d) = gg'(h_{g'(1)}h'_1, \ldots, h_{g'(d)}h'_d).$$

Definition 2.1. Given $u = g(h_1, \ldots, h_d)$ and $1 \leq i \leq d$ with a g-orbit of length n, let $\pi_{u,i}$ denote the product

$$\pi_{u,i} := h_{g^{n-1}(i)} \cdots h_{g(i)}h_i \in H.$$

Replacing i by $g^j(i)$ for any j cyclically permutes the factors in this product, hence $\pi_{u,i} \sim \pi_{u,g^j(i)}$. Therefore the conjugacy class of $\pi_{u,i}$ only depends on the g-orbit of i.

Proposition 2.2. *If $u = g(h_1, \ldots, h_d)$ and $u' = g'(h'_1, \ldots, h'_d)$ are elements of $G \wr H$, then $u \sim u'$ if and only if there exists an $a \in G$ and $b_i \in H$ for $1 \leq i \leq d$ such that both $g' = g^a$ and $\pi_{u',i} = \pi_{u,a(i)}^{b_i}$ for all $1 \leq i \leq d$.*

Proof. Well, $u \sim u'$ if and only if there is some $v := a(b_1, \ldots, b_d) \in G \wr H$, such that $u' = u^v$, equivalently

$$\begin{aligned} v^{-1}uv &= (b_1^{-1}, \ldots, b_d^{-1})a^{-1}g(h_1, \ldots, h_d)a(b_1, \ldots, b_d) \\ &= g^a(b_{g^a(1)}^{-1}h_{a(1)}b_1, \ldots, b_{g^a(d)}^{-1}h_{a(d)}b_d). \end{aligned}$$

Thus $u \sim u'$ if and only if there is some v such that $g' = g^a$ and $h'_i = b_{g^a(i)}^{-1}h_{a(i)}b_i$ for each i.

If $1 \leq i \leq d$ has a g-orbit of length n, then the $g' = g^a$ orbit of $a(i)$ also has length n, and conversely. Then we calculate

$$\begin{aligned} \pi_{u',i} &= h'_{g'^{n-1}(i)} \cdots h'_i \\ &= (b_i^{-1}h_{a(g^a)^{n-1}(i)}b_{(g^a)^{n-1}(i)}) \cdots (b_{g^a(i)}^{-1}h_{a(i)}b_i) \\ &= b_i^{-1}h_{g^{n-1}a(i)} \cdots h_{ga(i)}h_{a(i)}b_i \\ &= \pi_{u,a(i)}^{b_i}. \end{aligned}$$

Therefore $u \sim u'$ if and only if there exists some $a \in G$ and $b_i \in H$ for $1 \leq i \leq d$ such that $g' = g^a$ and $\pi_{u',i} = \pi_{u,a(i)}^{b_i}$ for each i. □

2.2. **Iterated wreath products.** Here, we define iterated wreath products of finite groups and give topological/inductive criteria equality and conjugacy in these groups.

Let G denote a group acting faithfully on the set $\{1, 2, \ldots, d\}$ on the left. In particular, G is finite.

Definition 2.3. For $\ell \geq 0$, the *ℓth iterated wreath product* of G, denoted $[G]^\ell$, is defined inductively by $[G]^0 := 1$ and $[G]^\ell := G \wr [G]^{\ell-1}$ for $\ell \geq 1$.

Let T_d^ℓ denote the regular rooted d-ary tree of height ℓ, where the children of each node are labeled by the elements of $\{1, 2, \ldots, d\}$. There is a correspondence between the leaves of T_d^ℓ and words of length ℓ in the alphabet $\{1, 2, \ldots, d\}$ given by reading the labels of the nodes along the unique path from the root to a given leaf. Suppose $w = iw'$ is a word where $i \in \{1, \ldots, d\}$ and w' is a word of length $\ell - 1$. If $u := g(g_1, g_2, \ldots, g_d) \in [G]^\ell$ then u acts on w by

$$u(w) = g(i)g_i(w')$$

This action respects the tree structure of T_d^ℓ under the correspondence between words of length ℓ and leaves of T_d^ℓ. Since G acts faithfully on $\{1, 2, \ldots, d\}$, the group $[G]^\ell$ acts faithfully on T_d^ℓ. In particular, an element of $[G]^\ell$ is uniquely determined by its action on T_d^ℓ.

Restricting an element of $[G]^\ell$ to words of length $\ell - 1$ gives an element of $[G]^{\ell-1}$ and defines a surjective homomorphism $[G]^\ell \twoheadrightarrow [G]^{\ell-1}$. Hence the groups $[G]^\ell$ form an inverse system

$$1 = [G]^0 \twoheadleftarrow [G]^1 \twoheadleftarrow [G]^2 \twoheadleftarrow [G]^3 \twoheadleftarrow \ldots. \tag{1}$$

Definition 2.4. The *iterated wreath product* of G, denoted $[G]^\infty$, is the inverse limit of (1).

The trees T_d^ℓ likewise form an inverse system whose limit we denote T_d^∞. Since G is finite, so too are the groups $[G]^\ell$ and the trees T_d^ℓ. Hence $[G]^\infty$ and T_d^∞ are both naturally endowed with a profinite topology and the group $[G]^\infty$ acts continuously on T_d^∞, or equivalently on right-infinite words $i_1 i_2 i_3 \ldots$ in the alphabet $\{1, 2, \ldots, d\}$.

We require a little more language to discuss iterated wreath products and their subgroups effectively:

Definition 2.5. Let $\rho_\ell : [G]^\infty \twoheadrightarrow [G]^\ell$ denote the restriction to words of length ℓ. The kernel of ρ_ℓ, called the *level ℓ stabilizer* and denoted $\mathrm{St}_\ell[G]^\infty$, is the subgroup of all elements of $[G]^\infty$ which stabilize the first ℓ levels of the tree.

Given two elements $u, v \in [G]^\infty$ and an integer $\ell \geq 0$ we write $u =_\ell v$ as a shorthand for $\rho_\ell(u) = \rho_\ell(v)$. We write $u \sim_\ell v$ as a shorthand for $\rho_\ell(u) \sim \rho_\ell(v)$ in $[G]^\ell$. If $U, V \subseteq [G]^\infty$ are subgroups, then we define $U =_\ell V$ and $U \sim_\ell V$ analogously. Given an integer $\ell \geq 0$ and an element $u \in [G]^\infty$, let $\mathrm{ord}_\ell(u)$ denote the order of $\rho_\ell(a)$ in the group $[G]^\ell$. If $K \subseteq H \subseteq [G]^\infty$, then we define

$$[H : K]_\ell := [\rho_\ell(H) : \rho_\ell(K)] = [H\,\mathrm{St}_\ell[G]^\infty : K\,\mathrm{St}_\ell[G]^\infty].$$

The recursive structure of the iterated wreath product makes it amenable to inductive arguments. Some subgroups have a similar property, including those we will study in subsequent sections,

Definition 2.6. A subgroup $H \subseteq [G]^\infty$ is said to be *self-similar* if $\mathrm{St}_1 H \subseteq H^d$. Note that self-similarity is not generally stable under conjugation, which is to say that self-similarity depends on the labeling of the tree.

By construction, the stabilizers $\mathrm{St}_\ell[G]^\infty$ form a neighborhood basis of the identity. With the associated truncatiosn ρ_ℓ, they furnish a convenient criterion for equality of closed sets:

Lemma 2.7. *Suppose that $U, V \subseteq [G]^\infty$ are closed subsets. Then*

(1) $U = V$ *if and only if* $U =_\ell V$ *for all* $\ell \geq 0$,
(2) $U \sim V$ *if and only if* $U \sim_\ell V$ *for all* $\ell \geq 0$.

Proof. The forward implications of (1) and (2) follow immediately by taking quotients. We now consider the reverse implications.

(1) If $\rho_\ell(U) = \rho_\ell(V)$ for all $\ell \geq 0$, then the definition of the profinite topology implies that $\overline{U} = \overline{V}$. Since U and V are closed by assumption, we conclude that $U = V$.

(2) Let $\widetilde{H}_\ell \subseteq [G]^\ell$ be the stabilizer of $\rho_\ell(U)$ under the conjugation action, and let

$$H_\ell := \rho_\ell^{-1}(\widetilde{H}_\ell) \subseteq [G]^\infty.$$

Note that $[G]^\ell$ finite implies that $\widetilde{H}_\ell$ is closed, hence compact, and of finite index.

Let $M_\ell \subseteq [G]^\infty$ be the set

$$M_\ell := \{m \in [G]^\infty : m^{-1}Um =_\ell V\}.$$

Each M_ℓ is a union of cosets of H_ℓ, necessarily finite, and hence compact. The M_ℓ are nested

$$M_0 \supseteq M_1 \supseteq M_2 \supseteq \ldots,$$

and the assumption $U \sim_\ell V$ implies that M_ℓ is nonempty for each $\ell \geq 0$.

Therefore, the intersection $\bigcap_{\ell \geq 0} M_\ell$ is nonempty and for any $m \in \bigcap_{\ell\geq 0} M_\ell$, we have $m^{-1}Um =_\ell V$ for all $\ell \geq 0$. Thus by (1) we have $m^{-1}Um = V$. □

The following proposition provides inductive criteria for checking equality and conjugacy in iterated wreath products. We make frequent use of these criteria throughout the paper.

Proposition 2.8. *If* $u, v \in [G]^\infty$ *are elements and* $U, V \subseteq [G]^\infty$ *are closed subgroups, then*

(1) $u = v$ *if and only if* $u =_\ell v$ *for all* $\ell \geq 0$,
(2) $U = V$ *if and only if* $U =_\ell V$ *for all* $\ell \geq 0$,
(3) $u \sim v$ *if and only if* $u \sim_\ell v$ *for all* $\ell \geq 0$,
(4) $U \sim V$ *if and only if* $U \sim_\ell V$ *for all* $\ell \geq 0$.

Proof. This is an immediate consequence of Lemma 2.7; note singleton subsets are closed in the Hausdorff group $[G]^\infty$. □

Remark. In Proposition 2.8, the assumption that the subgroups U and V are closed is essential. For example, in Bartholdi and Nekrashevych's resolution of the twisted rabbit problem [BN06], the authors identify three distinct subgroups of $[S_2]^\infty$ which nevertheless coincide at every finite level truncation. These are the iterated monodromy groups of the rabbit, co-rabbit, and airplane, and Proposition 2.8 implies that they have the same closure.

2.3. **Systems of recurrences.** One simple way to construct elements of $[G]^\infty$ is via recursive relations. Suppose $x_1, x_2, \ldots, x_n$ is a list of n indeterminates, where n may be infinite, and let $[G]^\infty(x_1, x_2, \ldots, x_n)$ denote the group formed by freely adjoining the x_i to $[G]^\infty$.

Definition 2.9. A *system of recurrences* in $[G]^\infty$ is a list of n equations

$$x_i = g_i(h_{i,1}, \ldots, h_{i,d}),$$

where $g_i \in G$ and $h_{i,j} \in [G]^\infty(x_1, x_2, \ldots, x_n)$.

Example 2.10. Let $G = S_3$ and $n = 2$. Let $\sigma := (123)$ and $\tau := (23)$ be elements of S_3. Then

$$x_1 = \sigma(1, x_1, x_2)$$
$$x_2 = \tau(x_1x_2, 1, 1)$$

is an example of a system of recurrences with two equations. The system of recurrences completely determines how a solution $(x_1, x_2) = (a_1, a_2)$ acts on words. For example, we calculate

$$\begin{aligned} a_1(21131) &= 3a_2(1131) \\ &= 31a_1a_2(131) \\ &= 31a_1(1a_1a_2(31)) \\ &= 312a_1a_1a_2(31) \\ &= 312a_1a_1(21) \\ &= 312a_1(3a_21) \\ &= 3121a_2(1) \\ &= 31211. \end{aligned}$$

Since elements of $[S_3]^\infty$ are completely determined by how they act on right-infinite words, it follows that the system of congruences has a unique solution. Lemma 2.11 generalizes this observation. □

Lemma 2.11. *Any system of recurrences* $x_i = g_i(h_{i,1}, h_{i,2}, \ldots, h_{i,d})$ *in* $[G]^\infty$ *has a unique solution* $x_i = a_i \in [G]^\infty$ *for* $1 \leq i \leq n$.

Proof. For each $\ell \geq 0$, let $A_\ell \subseteq ([G]^\infty)^n$ denote the set of all n-tuples $(a_1, \ldots, a_n)$ such that $x_i =_\ell a_i$ satisfies the system of recursions in $[G]^\ell$. The A_ℓ are compact and nested

$$A_0 \supseteq A_1 \supseteq A_2 \supseteq \ldots.$$

We prove by induction on ℓ that A_ℓ is non-empty and that if $(a_1, a_2, \ldots, a_n), (a_1', a_2', \ldots, a_n') \in A_\ell$ are two elements, then $a_i =_\ell a_i'$ for $1 \leq i \leq n$.

Since $g =_0 1$ for all $g \in [G]^\infty$, it follows that $(1, \ldots, 1) \in A_0$ and that any $(a_1, \ldots, a_n) \in A_0$ satisfies $a_i =_0 1$. Now suppose $\ell \geq 1$ and that our assertion holds for $\ell - 1$. Let $(\tilde{a}_1, \ldots, \tilde{a}_n) \in A_{\ell-1}$ and define $(a_1, \ldots, a_n)$ by

$$a_i := g_i(h_{i,1}(\tilde{a}), \ldots, h_{i,d}(\tilde{a})),$$

where $h_{i,j}(\tilde{a}) \in [G]^\infty$ is the element we get by substituting $x_k \mapsto \tilde{a}_k$ into $h_{i,j}$ for each k. Define $h_{i,j}(a)$ analogously via the substitution $x_k \mapsto a_k$. Since $(\tilde{a}_1, \ldots, \tilde{a}_n) \in A_{\ell-1}$, we have

$$\tilde{a}_i =_{\ell-1} a_i = g_i(h_{i,1}(\tilde{a}), \ldots, h_{i,d}(\tilde{a})).$$

Thus, $h_{i,j}(a) =_{\ell-1} h_{i,j}(\tilde{a})$ for all i and j, implying that

$$a_i = g_i(h_{i,1}(\tilde{a}), \ldots, h_{i,d}(\tilde{a})) =_\ell g_i(h_{i,1}(a), \ldots, h_{i,d}(a)).$$

Hence, $(a_1, \ldots, a_n) \in A_\ell$. If $(a_1', \ldots, a_n') \in A_\ell$ is another element, then our inductive hypothesis implies that $a_i =_{\ell-1} a_i'$. Thus $h_{i,j}(a) =_{\ell-1} h_{i,j}(a')$ for each i and j, and

$$a_i =_\ell g_i(h_{i,1}(a), \ldots, h_{i,d}(a)) =_\ell g_i(h_{i,1}(a'), \ldots, h_{i,d}(a')) =_\ell a_i'.$$

This completes our inductive step and hence our induction. Therefore, $A := \bigcap_{\ell \geq 0} A_\ell$ is nonempty and Proposition 2.8 implies that A contains a unique solution of the system of recursions. □

In the application we are working towards, we encounter elements of an iterated wreath product which satisfy a system of recurrences up to an unknown conjugacy.

Definition 2.12. A *system of n conjugate recurrences* in $[G]^\infty$, where $0 \leq n \leq \infty$, is a list of conjugation identities

$$x_i \sim g_i(h_{i,1}, \ldots, h_{i,d}),$$

one for each $1 \leq i \leq n$, where $g_i \in G$ and $h_{i,j} \in [G]^\infty(x_1, \ldots, x_n)$. We say a system of conjugate recurrences is *cyclic* if for each $1 \leq i \leq n$ and each $1 \leq j \leq d$ there exists integers k, m_k with $1 \leq k \leq n$ such that $\pi_{x_i,j} \sim x_k^{m_k}$ in $[G]^\infty(x_1, \ldots, x_n)$.

A solution to a system of conjugate recurrences is a solution to any particular system of recursions obtained from a choice of conjugating elements for each i. Once conjugating elements are chosen, we get a system of recurrences which has a unique solution by Lemma 2.11. When the system of conjugate recurrences is cyclic, the following proposition shows that all solutions of the system are themselves conjugate in $[G]^\infty$.

Proposition 2.13. *Let $x_i \sim g_i(h_{i,1}, \ldots, h_{i,d})$ be a system of n conjugate cyclic recurrences in $[G]^\infty$. If $x_i = a_i$ for $1 \le i \le n$ is one solution of this system, then $x_i = a_i'$ is another solution if and only if $a_i \sim a_i'$ in $[G]^\infty$ for $1 \le i \le n$. In other words, conjugate cyclic recurrences have unique solutions up to conjugacy.*

Remark. It is not hard to see that if $(a_1, \ldots, a_n)$ is a solution of a system of conjugate recurrences, then so is $(ua_1u^{-1}, \ldots, ua_nu^{-1})$ for any $u \in [G]^\infty$. Proposition 2.13 shows something stronger: If the system of conjugate recurrences is cyclic, then we can conjugate the a_i independently for each i and still get a solution of the system.

Proof. First suppose that $x_i = a_i$ for $1 \le i \le n$ is a solution of the system of conjugate cyclic recurrences and that a_i' are elements such that $a_i \sim a_i'$ in $[G]^\infty$ for each i. Cyclicity implies that for each $1 \le i \le n$ and $1 \le j \le d$ there exists integers k, m_k such that $1 \le k \le n$ and $\pi_{a_i,j} \sim a_k^{m_k}$ in $[G]^\infty$. Then $a_k \sim a_k'$ for all k implies that $a_k^{m_k} \sim a_k'^{m_k}$. Thus Proposition 2.2 implies

$$a_i' \sim a_i \sim g_i(h_{i,1}(a), \ldots, h_{i,d}(a)) \sim g_i(h_{i,1}(a'), \ldots, h_{i,d}(a')).$$

Hence $x_i = a_i'$ for $1 \le i \le n$ is a solution of the system of conjugate recurrences.

Next suppose that $x_i = a_i$ and $x_i = a_i'$ are two solutions of the system of cyclic conjugate recurrences. We wish to show that $a_i \sim a_i'$ in $[G]^\infty$ for all $1 \le i \le n$. By Proposition 2.8 it suffices to prove by induction on ℓ that $a_i \sim_\ell a_i'$ for each $1 \le i \le n$ and for all $\ell \ge 0$. First note that $a_i =_0 a_i' =_0 1$ for each i, which establishes the base case. Next suppose that $\ell \ge 1$ and that for each i we have that $a_i \sim_{\ell-1} a_i'$. Cyclicity implies that for each $1 \le i \le n$ and $1 \le j \le d$,

$$\pi_{a_i,j} \sim a_k^{m_k} \sim_{\ell-1} a_k'^{m_k} \sim \pi_{a_i',j}.$$

Thus Proposition 2.2 implies that

$$a_i \sim g_i(h_{i,1}(a), \ldots, h_{i,d}(a)) \sim_\ell g_i(h_{i,1}(a'), \ldots, h_{i,d}(a')) \sim a_i',$$

which completes our induction. □

2.4. **Iterated wreath products of cyclic groups.** For the remainder of this paper, we let $\sigma = (123 \cdots d) \in S_d$ denote the standard d-cycle and let $C_d := \langle \sigma \rangle$ denote the cyclic subgroup generated by σ. We construct the iterated wreath product $[C_d]^\infty$ with respect to the natural action of C_d on $\{1, 2, \ldots, d\}$. Note that for each $\ell \ge 0$ the order of the group $[C_d]^\ell$ is $d^{[\ell]_d}$, where $[\ell]_d := \frac{d^\ell - 1}{d-1}$. Hence $\mathrm{ord}_\ell(g)$ divides a power of d for every $g \in [C_d]^\infty$ and every $\ell \ge 0$, which implies that g^m is well-defined for every d-adic integer $m \in \mathbb{Z}_d$. The groups we study in this paper are all closed subgroups of $[C_d]^\infty$, hence will contain these powers.

Lemma 2.14 highlights a special case of Proposition 2.2 which is particularly useful for checking conjugacy in $[C_d]^\infty$.

Lemma 2.14. *If $\sigma(h_1, \ldots, h_d) \in C_d \wr H$, then*

$$\sigma(h_1, \ldots, h_d) \sim \sigma(1, \ldots, 1, h_d h_{d-1} \cdots h_1).$$

The following lemma is used in the proof of Proposition 3.15.

Lemma 2.15. *If $g \in [C_d]^\infty$ and $\varepsilon \in \mathbb{Z}_d^\times$ satisfies $\varepsilon \equiv 1 \bmod d$, then $g \sim g^\varepsilon$ in $[C_d]^\infty$.*

Proof. By Proposition 2.8, it suffices to prove that $g \sim_\ell g^\varepsilon$ for all $g \in [C_d]^\infty$ and all $\ell \geq 0$. We proceed by induction on ℓ. The base case is trivial since $g =_0 g^\varepsilon =_0 1$. Now suppose $\ell \geq 1$ and $g \sim_{\ell-1} g^\varepsilon$.

If $g =_1 \sigma^j$, let k be the number of σ^j-orbits in $\{1, 2, \ldots, d\}$. Then Proposition 2.2 implies that g is conjugate to

$$g \sim g' := \sigma^j(g_1, \ldots, g_k, 1, \ldots, 1)$$

for some $g_i \in [C_d]^\infty$. Note that

$$g'^d = (g_1, \ldots, g_k, g_1, \ldots, g_k, \ldots, g_1, \ldots, g_k).$$

By assumption we may write $\varepsilon = 1 + d\varepsilon'$ for some $\varepsilon' \in \mathbb{Z}_d$. Then

$$\begin{aligned} g^\varepsilon &\sim g'^\varepsilon \\ &= g'(g'^d)^{\varepsilon'} \\ &= \sigma^j(g_1^{1+\varepsilon'}, \ldots, g_k^{1+\varepsilon'}, g_1^{\varepsilon'}, \ldots, g_k^{\varepsilon'}, \ldots, g_1^{\varepsilon'}, \ldots, g_k^{\varepsilon'}, g_1^{\varepsilon'}, \ldots, g_k^{\varepsilon'}) \\ &\sim \sigma^j(g_1^{1+d\varepsilon'}, \ldots, g_k^{1+d\varepsilon'}, 1, \ldots, 1) \\ &= \sigma^j(g_1^\varepsilon, \ldots, g_k^\varepsilon, 1, \ldots, 1) \end{aligned}$$

where the second conjugacy follows from Proposition 2.2. Our inductive hypothesis implies that $g_i^\varepsilon \sim_{\ell-1} g_i$ for each $1 \leq i \leq k$. Appealing again to Proposition 2.2 we have

$$g^\varepsilon \sim \sigma^j(g_1^\varepsilon, g_2^\varepsilon, \ldots, g_k^\varepsilon, 1, \ldots, 1) \sim_\ell \sigma^j(g_1, \ldots, g_k, 1, \ldots, 1) \ = g' \sim g.$$

This completes the induction. □

The abelianization of a wreath product $G \wr H$ is isomorphic to $G^{\mathrm{ab}} \times H^{\mathrm{ab}}$ with $g(h_1, \ldots, h_d)$ mapping to $(g, h_1 \cdots h_d)$. It follows that the abelianization of $[C_d]^\infty$ is isomorphic to the infinite direct product C_d^∞.

Definition 2.16. Given an integer $\ell \geq 1$, the (additive) *level ℓ character*, denoted χ_ℓ, is the function $\chi_\ell : [C_d]^\infty \to \mathbb{Z}/d\mathbb{Z}$ given by the composition of the abelianization map, projection onto the ℓth coordinate, and finally the isomorphism sending $\sigma^j \mapsto j \bmod d$.

The functions χ_ℓ may be calculated recursively by $\chi_1(\sigma^j(g_1, \ldots, g_d)) = j$ and for $\ell \geq 1$,

$$\chi_\ell(\sigma^j(g_1, \ldots, g_d)) = \sum_{i=1}^d \chi_{\ell-1}(g_i).$$

Remark. When $d = 2$, we have $S_2 = C_2$, hence $[S_2]^\infty = [C_2]^\infty$. In that case the functions χ_ℓ are equivalent to the ℓth level sign functions sgn_ℓ, differing only by an isomorphism of their codomains. The sign functions play an important role in analyzing $[S_d]^\infty$ and its subgroups. When $d > 2$, there are also sign functions defined on $[C_d]^\infty$ by restriction from $[S_d]^\infty$, but they are less useful. For example, when d is odd, all the sign functions are trivial on $[C_d]^\infty$, and when d is even, they are $d/2$ powers of our sign functions. The characters χ_ℓ are a finer invariant, and the appropriate generalization to this family of groups.

Definition 2.17. The *standard odometer* is the element $c_\infty \in [C_d]^\infty$ defined recursively by

$$c_\infty = \sigma(1, \ldots, 1, c_\infty).$$

An *odometer* is any element of $[S_d]^\infty$ that is conjugate to c_∞ in $[S_d]^\infty$. A *strict odometer* is any element of $[C_d]^\infty$ that is conjugate to c_∞ in $[C_d]^\infty$.

Since $\chi_1(c_\infty) = 1$ and $\chi_\ell(c_\infty) = \chi_{\ell-1}(c_\infty)$ for $\ell \geq 1$, it follows that $\chi_\ell(c_\infty) = 1$ for all $\ell \geq 1$. Hence if $c \in [C_d]^\infty$ is any strict odometer, then we also have $\chi_\ell(c) = 1$ for all $\ell \geq 1$. The following lemma shows that this is a sufficient condition to be an odometer.

Lemma 2.18. *If $c \in [C_d]^\infty$, then c is a strict odometer if and only if $\chi_\ell(c) = 1$ for all $\ell \geq 1$.*

Proof. We prove by induction on ℓ that if $c \in [C_d]^\infty$ is an element such that $\chi_k(c) = 1$ for all $1 \leq k \leq \ell$, then $c \sim_\ell c_\infty$ in $[C_d]^\infty$. Note that if this holds for all $\ell \geq 0$, then Proposition 2.8 implies that $c \sim c_\infty$ in $[C_d]^\infty$. If $\ell = 1$, then $\chi_1(c) = 1$ implies that $c =_1 \sigma =_1 c_\infty$, and the conclusion is immediate.

Now suppose that $\ell > 1$ and that our hypothesis holds for $\ell - 1$. Thus $c =_1 \sigma$ and Lemma 2.14 implies that c is conjugate to $\sigma(1, \ldots, 1, c')$ for some $c' \in [C_d]^\infty$. If $1 < k \leq \ell$, then $1 = \chi_k(c) = \chi_{k-1}(c')$. Our inductive hypothesis implies that $c' \sim_{\ell-1} c_\infty$. Thus by Lemma 2.14,

$$c \sim_\ell \sigma(1, \ldots, 1, c') \sim_\ell \sigma(1, \ldots, 1, c_\infty) = c_\infty,$$

where both the conjugacies are in $[C_d]^\infty$. This completes our induction. □

Lemma 2.19. *If $c \in [S_d]^\infty$ is an odometer, then c acts via a d^ℓ-cycle on words of length ℓ, and in particular* $\mathrm{ord}_\ell(c) = d^\ell$.

Proof. Identifying our alphabet $\{1, 2, \ldots, d\}$ with $\{0, 1, \ldots, d-1\}$ via $i \mapsto i - 1$, there is a natural bijection between right-infinite words and elements of $\mathbb{Z}_d$. Let $\tau : \mathbb{Z}_d \to \mathbb{Z}_d$ be the translation by 1 function, $\tau(x) := x + 1$. Then τ satisfies the recursion

$$\tau = \sigma(1, \ldots, 1, \tau),$$

where the nontrivial restriction comes from carrying. Clearly τ acts via a d^ℓ-cycle on words of length ℓ. Since c_∞ and τ only differ by a relabeling of the alphabet, the same must be true for c_∞. Since all odometers are conjugate to c_∞ we conclude that every odometer c acts via a d^ℓ-cycle on words of length ℓ and thus, $\mathrm{ord}_\ell(c) = d^\ell$. □

The essential property of odometers that we use in our analysis is that they are self-centralizing within the full tree automorphism group $[S_d]^\infty$.

Notation 2.20. If G is a topological group and $g_1, \ldots, g_n \in G$, then we write $\langle\!\langle g_1, \ldots, g_n \rangle\!\rangle$ to denote the subgroup of G topologically generated by the g_i.

Proposition 2.21. *If $c \in [S_d]^\infty$ is an odometer, $0 \leq \ell \leq \infty$, and $g \in [S_d]^\infty$ is an element such that $[g, c] =_\ell 1$, then $g \in_\ell \langle\!\langle c \rangle\!\rangle$.*

Proof. It suffices to prove this for $c = c_\infty$. Suppose $g \in [S_d]^\infty$ and $[g, c] =_\ell 1$. Lemma 2.19 implies that c acts on words of length ℓ via a d^ℓ-cycle. Since d^ℓ-cycles are self-centralizing in S_{d^ℓ}, we conclude that $g \in_\ell \langle\!\langle c \rangle\!\rangle$. The $\ell = \infty$ case then follows from Proposition 2.8. □

2.5. **Heisenberg group.** The *Heisenberg group* $\mathcal{H}_d$ arises naturally as a quotient of the iterated monodromy groups we study in Section 4.

Definition 2.22. Let $\mathcal{H}_d := \langle \sigma \rangle \ltimes (\mathbb{Z}/d\mathbb{Z}))^2$ where σ acts on $(\mathbb{Z}/d\mathbb{Z})^2$ via $\sigma^{-1}(i, j)\sigma = (i + j, j)$.

Lemma 2.23 provides a convenient presentation for $\mathcal{H}_d$.

Lemma 2.23. *For all $d \geq 1$,*

$$\langle g_1, g_2 : g_1^d = g_2^d = [g_1, [g_1, g_2]] = [g_2, [g_1, g_2]] = 1 \rangle \cong \mathcal{H}_d$$

where the isomorphism sends $g_1 \mapsto \sigma$ and $g_2 \mapsto (0, 1)$.

Proof. Let $G := \langle g_1, g_2 : g_1^d = g_2^d = [g_1, [g_1, g_2]] = [g_2, [g_1, g_2]] = 1 \rangle$. Consider the map from the free group generated by g_1, g_2 to $\mathcal{H}_d$ which sends $g_1 \mapsto (0, 1)$ and $g_2 \mapsto \sigma$. Observe that $[g_1, g_2] \mapsto (1, 0)$, which by construction commutes with g_1 and g_2. Since $(0, 1)$ and σ both have order d, this map factors through G. Thus it suffices to show that G has order at most d^3.

Since g_1 and g_2 generate G, the relations imply $[g_1, g_2]$ is in the center. Therefore $g \in G$ may be written in the form

$$g = g_1^i g_2^j [g_1, g_2]^k$$

by writing g as a word in g_1 and g_2 and iteratively applying $g_2 g_1 = g_1 g_2 [g_1, g_2]^{-1}$. It suffices to show that $[g_1, g_2]$ has order at most d.

Every commutator in G may be expressed as a word in conjugates of $[g_1, g_2]$, which lies in the center of G, hence belongs to the center of G. Therefore if $h_1, h_2, h_3 \in G$, then

$$[h_1, h_2 h_3] = [h_1, h_3] h_3^{-1} [h_1, h_2] h_3 = [h_1, h_2][h_1, h_3].$$

It follows that

$$[g_1, g_2]^d = [g_1, g_2^d] = [g_1, 1] = 1. \qquad \square$$

The group $\mathcal{H}_d$ has a distinguished involution which plays an important role later.

Lemma 2.24. *With respect to the presentation in Lemma 2.23, there is a unique involution τ of $\mathcal{H}_d$ such that $\tau(g_1) = g_2$. Explicitly, in terms of $\langle \sigma \rangle \ltimes (\mathbb{Z}/d\mathbb{Z})^2$, it is given by*

$$\tau(\sigma^r (s, t)) = \sigma^t (rt - s, r).$$

Proof. Exchanging g_1 and g_2 in the presentation for $\mathcal{H}_d$ provided by Lemma 2.23 yields an isomorphic group. Hence there is an involution $\tau : \mathcal{H}_d \to \mathcal{H}_d$ which satisfies $\tau(\sigma) = (0, 1)$ and $\tau(0, 1) = \sigma$; this involution is unique since σ and $(0, 1)$ generate $\mathcal{H}_d$. Note that $\sigma^r (s, t) = \sigma^r (0, 1)^t [(0, 1), \sigma]^s$, hence

$$\tau(\sigma^r (s, t) = (0, 1)^r \sigma^t [(0, 1), \sigma]^{-s} = (0, r) \sigma^t (-s, 0) = \sigma^t (rt - s, r). \qquad \square$$

3. Profinite iterated monodromy groups

Let K be a field and f a rational function over K. In this section, we construct the geometric and arithmetic profinite iterated monodromy groups of f over K, relate them to the constant field extension, and produce embeddings into iterated wreath products which realize them as self-similar groups in a Galois-equivariant fashion. For polynomials, we produce embeddings that are especially well-behaved with respect to the ramification at infinity.

3.1. **Monodromy groups.** Let K be a field, f rational function over K, and t some transcendental over K. Then f determines a branched self-cover of $\mathbb{P}^1_K$, which, in terms of function fields, corresponds to the extension extension $K(x)/K(t)$ defined by clearing denominators in $f(x) = t$. The resulting polynomial has degree d, and is irreducible by Gauss's lemma. We will further assume that this extension is separable; for a polynomial, d coprime to $\operatorname{char} K$ is sufficient.

Fix a separable closure $(K(t))^{\mathrm{sep}}$ and let K^{sep} be the separable closure of K within $(K(t))^{\mathrm{sep}}$.

Three natural field extensions and their associated Galois groups appear:

Definition 3.1. The *arithmetic monodromy group of f*, denoted $\operatorname{Mon} f$ is

$$\operatorname{Mon} f := \operatorname{Gal}(K(f^{-1}(t))/K(t)).$$

The *geometric monodromy group of f*, denoted $\overline{\operatorname{Mon}}\, f$ is the subgroup

$$\overline{\operatorname{Mon}}\, f := \operatorname{Gal}(K^{\mathrm{sep}}(f^{-1}(t))/K^{\mathrm{sep}}(t)).$$

Let $\widehat{K}_f$ be the algebraic closure of K inside $K(f^{-1}(t))$, or equivalently $K^{\mathrm{sep}} \cap K(f^{-1}(t))$. We call $\widehat{K}$ the *constant field extension* associated to f. The last group is simply $\operatorname{Gal}(\widehat{K}_f/K)$, the *constant field Galois group.*

Restriction induces a natural inclusion of $\overline{\mathrm{Mon}}\, f$ into $\mathrm{Mon}\, f$. There is a further restriction from $\mathrm{Mon}\, f$ to $\mathrm{Gal}(\widehat{K}_f/K)$. It is clear that $\overline{\mathrm{Mon}}\, f$ is in the kernel of the latter restriction. In fact, it can be shown that the following sequence of restrictions is exact:

$$0 \to \overline{\mathrm{Mon}}\, f \to \mathrm{Mon}\, f \to \mathrm{Gal}(\widehat{K}_f/K) \to 0.$$

Recall that finite extensions of $\mathbb{C}(t)$ correspond to finite branched covers of $\mathbb{P}^1_{\mathbb{C}}$. In this case, the arithmetic and geometric monodromy groups of f coincide and are isomorphic to the group of deck transformations of the branched cover associated to the Galois closure of f. For more general fields K, the arithmetic monodromy group of f incorporates more delicate information about the interaction between the arithmetic Galois theory of K^{sep}/K and the geometric deck transformations of the branched cover associated to f.

3.2. **Iterated preimage extensions.** Let f^n denote the n-fold composition of f with itself. The iterated preimage extensions of $K(t)$ naturally form a tower

$$K(t) \subseteq K(f^{-1}(t)) \subseteq K(f^{-2}(t)) \subseteq \ldots \subseteq K(f^{-\infty}(t)) := \bigcup_{\ell \geq 0} K(f^{-\ell}(t)),$$

and similarly with K replaced by K^{sep}. With f understood, let $\widehat{K}_\ell$ be a shorthand for $\widehat{K}_{f^\ell}$, and let $\widehat{K}_\infty$ be the field such that $K(f^{-\infty}(t)) \cap K^{\mathrm{sep}}(t) = \widehat{K}_\infty(t)$. Equivalently, $\widehat{K}_\infty = \bigcup_{\ell \geq 0} \widehat{K}_\ell$. The arithmetic and geometric monodromy groups for each iterate f^n and the associated constant field extensions are all compatible, and passing to the limit gives rise to the profinite iterated monodromy groups we study:

Definition 3.2. The *arithmetic profinite iterated monodromy group of* f is the Galois group

$$\mathrm{pIMG}(f) := \mathrm{Gal}(K(f^{-\infty}(t))/K(t)) \cong \varprojlim \mathrm{Mon}\, f^n.$$

The *geometric profinite iterated monodromy group of* f is the Galois group

$$\overline{\mathrm{pIMG}}(f) := \mathrm{Gal}(K(f^{-\infty}(t)/\widehat{K}_f(t)) \cong \mathrm{Gal}(K^{\mathrm{sep}}(f^{-\infty}(t))/K^{\mathrm{sep}}(t)) \cong \varprojlim \overline{\mathrm{Mon}}\, f^n.$$

The “profinite” in the names of these groups refers to the fact that both groups, being Galois groups, are profinite. This is in contrast with the *discrete* iterated monodromy group associated to a rational function defined over the complex numbers. The discrete iterated monodromy group of $f(x) \in \mathbb{C}(x)$ is constructed topologically and the geometric profinite iterated monodromy group is isomorphic to its profinite completion [Nek05, Ch. 5].

3.3. **Self-Similarity.** The iterated f-preimages of t naturally form a regular rooted d-ary tree. Since f has coefficients in K, the group $\mathrm{pIMG}\, f$ acts via tree automorphisms on these iterated preimages. This is an example of an *arboreal representation* of the Galois group $\mathrm{pIMG}\, f$. We will construct labelings of this tree which give this arboreal representation several useful properties.

Definition 3.3. If t' is transcendental over K, then a *path* from t to t' is a K^{sep}-isomorphism $\lambda : K(t)^{\mathrm{sep}} \to K(t')^{\mathrm{sep}}$ such that $\lambda(t) = t'$.

Note that a path from t to t' induces a K^{sep}-isomorphism between $K(f^{-\infty}(t))$ and $K(f^{-\infty}(t'))$. Paths exist between any two elements transcendental over K, but there is typically no natural way to choose or construct an explicit path. If $K = \mathbb{C}$ and $a, b \in \mathbb{C}$, then a path from a to b in the sense of topology induces a path from $t - a$ to $t - b$ in the algebraic sense defined above, but not all algebraic paths from $t - a$ to $t - b$ arise in this way: there are far more algebraic paths, in the same way that profinite iterated monodromy groups have far more elements than discrete iterated monodromy groups.

Since t is transcendental over K, each $t' \in f^{-1}(t)$ is also transcendental over K. If λ is a path from t to t', then $f(\lambda(t)) = t$ and it follows that for all $\ell \geq 0$ and $t'' \in f^{-\ell}(t)$ we have

$f^\ell(\lambda(t'')) = t''$. Also, because $t' \in K(t)^{\mathrm{sep}}$ we may suppose that $K(t')^{\mathrm{sep}} \subseteq K(t)^{\mathrm{sep}}$. Hence λ is a K^{sep}-endomorphism of $K(t)^{\mathrm{sep}}$.

Definition 3.4. A *preimage labeling or ordering* (for f and t) is a bijection $i \mapsto t_i$, where the t_i are the distinct solutions of $f(x) = t$ in $K(t)^{\mathrm{sep}}$. Given a preimage labeling, a *choice of paths* Λ for f and t consists of a collection of paths λ_i from t to t_i for each i.

Remark. When $K^{\mathrm{sep}} = \mathbb{C}$, one may use topology to construct a convenient choices of paths for analyzing $\overline{\mathrm{pIMG}}\, f$. See [BN08] for an example in the quadratic case. Since we are working with an arbitrary field K, we cannot as directly rely on topology to help us choose paths. We instead take a purely algebraic approach.

A preimage labeling determines an identification of $\mathrm{Mon}\, f$ and $\overline{\mathrm{Mon}}\, f$ with subgroups of S_d with its usual permutation action. We use the choice of paths Λ to propagate this upward, and label the tree of iterated f-preimages of t by words in the alphabet $\{1, 2, \ldots, d\}$ as follows: If $w = i_1 i_2 \ldots i_\ell$ is a word in the alphabet $\{1, 2, \ldots, d\}$, let $\lambda_w := \lambda_{i_1}\lambda_{i_2}\cdots\lambda_{i_\ell}$. We then define $t_w := \lambda_w(t)$. Observe that $f^\ell(t_w) = t$, hence $t_w \in f^{-\ell}(t)$.

Suppose we have fixed a preimage labeling $i \mapsto t_i$. Let $\gamma \mapsto \tilde{\gamma}$ denote the composition of the natural restriction map $\mathrm{pIMG}\, f \to \mathrm{Mon}\, f$ with the induced inclusion into S_d. If $1 \leq i, j \leq d$, then $\tilde{\gamma}(i) = j$ if and only if $\gamma(t_i) = t_j$. In this case, γ induces a $K(t)$-isomorphism

$$\gamma|_i : K(f^{-\infty}(t_i)) \to K(f^{-\infty}(t_j)).$$

If $\delta \in \mathrm{pIMG}\, f$, then $(\gamma\delta)|_i = \gamma|_{\tilde{\delta}(i)}\delta|_i$.

Lemma 3.5. *Let Λ be a choice of paths. If $\gamma \in \mathrm{pIMG}\, f$ and $1 \leq i \leq d$, let $\gamma_{\Lambda,i} = \lambda_{\tilde{\gamma}(i)}^{-1}\gamma|_i\lambda_i$. Then the function $\rho'_\Lambda : \mathrm{pIMG}\, f \to \mathrm{Mon}\, f \ltimes (\mathrm{pIMG}\, f)^d$ defined by*

$$\rho'_\Lambda(\gamma) := \tilde{\gamma}(\gamma_{\Lambda,1}, \ldots, \gamma_{\Lambda,d}),$$

is an injective homomorphism.

Proof. Observe that

$$\begin{aligned}\rho'_\Lambda(\gamma)\rho'_\Lambda(\delta) &= \tilde{\gamma}(\gamma_{\Lambda,1}, \ldots, \gamma_{\Lambda,d})\tilde{\delta}(\delta_{\Lambda,1}, \ldots, \delta_{\Lambda,d})\\ &= \tilde{\gamma}\tilde{\delta}(\gamma_{\Lambda,\tilde{\delta}(1)}\delta_{\Lambda,1}, \ldots, \gamma_{\Lambda,\tilde{\delta}(d)}\delta_{\Lambda,d}).\end{aligned}$$

On the other hand, we have $\widetilde{\gamma\delta} = \tilde{\gamma}\tilde{\delta}$ and for each i,

$$\gamma_{\Lambda,\tilde{\delta}(i)}\delta_{\Lambda,i} = (\lambda^{-1}_{\tilde{\gamma}\tilde{\delta}(i)}\gamma|_{\tilde{\delta}(i)}\lambda_{\tilde{\delta}(i)})(\lambda^{-1}_{\tilde{\delta}(i)}\delta|_i\lambda_i) = \lambda^{-1}_{\tilde{\gamma}\tilde{\delta}(i)}\gamma|_{\tilde{\delta}(i)}\delta|_i\lambda_i = \lambda^{-1}_{\widetilde{\gamma\delta}(i)}(\gamma\delta)|_i\lambda_i = (\gamma\delta)_{\Lambda,i}.$$

Therefore $\rho'_\Lambda(\gamma\delta) = \rho'_\Lambda(\gamma)\rho'_\Lambda(\delta)$. If $\gamma \in \ker \rho'_\Lambda$, then $\tilde{\gamma}(i) = i$ for each i and $\gamma|_i$ is the identity on $K(f^{-\infty}(t_i))$. Hence γ acts trivially on the set $\bigcup_{\ell \geq 0} f^{-\ell}(t)$, implying $\gamma = 1$ in $\mathrm{pIMG}\, f$. Thus ρ'_Λ is injective. □

Iterating ρ'_Λ gives a homomorphism $\rho_\Lambda : \mathrm{pIMG}\, f \to [\mathrm{Mon}\, f]^\infty$ defined recursively by

$$\rho_\Lambda(\gamma) := \tilde{\gamma}(\rho_\Lambda(\gamma_{\Lambda,1}), \ldots, \rho_\Lambda(\gamma_{\Lambda,d})).$$

The following proposition shows how $\rho_\Lambda(\gamma)$ acts with respect to the labeling of the iterated preimage tree corresponding to Λ.

Lemma 3.6. *If w is a word in the alphabet $\{1, 2, \ldots, d\}$ and $\gamma \in \mathrm{pIMG}\, f$, then*

$$\gamma(t_w) = t_{\rho_\Lambda(\gamma)(w)}.$$

Proof. We proceed by induction on the length ℓ of w. The claim is trivially true for words of length 0. Now suppose that $\ell \geq 1$ and that the assertion holds for all words of length $\ell - 1$ and all $\gamma \in \mathrm{pIMG}\, f$.

Every word of length ℓ may be expressed as iw for some $1 \leq i \leq d$ and some word w of length $\ell - 1$. If $\gamma \in \mathrm{pIMG}\, f$, then there is some word w' with the same length as w such that $\gamma(t_{iw}) = t_{\tilde{\gamma}(i)w'}$. Since $t_{iw} = \lambda_i(t_w)$ and $t_{\tilde{\gamma}(i)w'} = \lambda_{\tilde{\gamma}(i)}(t_{w'})$, it follows that

$$\gamma_{\Lambda,i}(t_w) = \lambda_{\tilde{\gamma}(i)}^{-1}\gamma|_i\lambda_i(t_w) = t_{w'}.$$

On the other hand, our inductive hypothesis implies that $\gamma_{\Lambda,i}(t_w) = t_{\rho_\Lambda(\gamma_{\Lambda,i})}$. Therefore $w' = \rho_\Lambda(\gamma_{\Lambda,i})$. Hence

$$\gamma(t_{iw}) = t_{\tilde{\gamma}(i)\rho_\Lambda(\gamma_{\Lambda,i})(w)} = t_{\rho_\Lambda(\gamma)(iw)},$$

which completes our induction. □

Lemma 3.7. *Fix a preimage labeling and suppose Λ_1 and Λ_2 are two choices of paths. There exists an element $w \in \mathrm{St}_1[\mathrm{Mon}\, f]^\infty = ([\mathrm{Mon}\, f]^\infty)^d$ such that $w^{-1}\rho_{\Lambda_1}w = \rho_{\Lambda_2}$.*

Proof. Proposition 2.8 implies that it suffices to prove that for all $\ell \geq 0$ there exists a $w \in \mathrm{St}_1[\mathrm{Mon}\, f]^\infty$ such that $w^{-1}\rho_{\Lambda_1}w =_\ell \rho_{\Lambda_2}$. We proceed by induction on ℓ; the $\ell = 0$ case is immediate.

Suppose that $\ell \geq 1$ and that the claim is true for $\ell - 1$. Let $w \in \mathrm{St}_1[\mathrm{Mon}\, f]^\infty$ be an element such that $w^{-1}\rho_{\Lambda_1}w =_{\ell-1} \rho_{\Lambda_2}$. Let $w_1 := (w, \ldots, w) \in \mathrm{St}_1[\mathrm{Mon}\, f]^\infty$. Then for $\gamma \in \mathrm{pIMG}\, f$ we have

$$\begin{aligned} w_1^{-1}\rho_{\Lambda_1}(\gamma)w_1 &= \tilde{\gamma}(w^{-1}\rho_{\Lambda_1}w(\gamma_{\Lambda_1,1}), \ldots, w^{-1}\rho_{\Lambda_1}w(\gamma_{\Lambda_1,d})) \\ &=_\ell \tilde{\gamma}(\rho_{\Lambda_2}(\gamma_{\Lambda_1,1}), \ldots, \rho_{\Lambda_2}(\gamma_{\Lambda_1,d})). \end{aligned}$$

Let $\lambda_{1,i}$ and $\lambda_{2,i}$ denote the paths associated to Λ_1 and Λ_2 respectively. Define

$$w_2 := (\rho_{\Lambda_2}(\lambda_{1,1}\lambda_{2,1}^{-1}), \ldots, \rho_{\Lambda_2}(\lambda_{1,d}\lambda_{2,d}^{-1})) \in \mathrm{St}_1[\mathrm{Mon}\, f]^\infty,$$

and set $w' := w_1w_2 \in \mathrm{St}_1[\mathrm{Mon}\, f]^\infty$. Then

$$\begin{aligned} {w'}^{-1}\rho_{\Lambda_1}(\gamma)w' &=_\ell w_2^{-1}\tilde{\gamma}(\rho_{\Lambda_2}(\gamma_{\Lambda_1,1}), \ldots, \rho_{\Lambda_2}(\gamma_{\Lambda_1,d}))w_2 \\ &= \tilde{\gamma}(\rho_{\Lambda_2}(\lambda_{2,\tilde{\gamma}(1)}\lambda_{1,\tilde{\gamma}(1)}^{-1}\gamma_{\Lambda_1,d}\lambda_{1,1}\lambda_{2,1}^{-1}), \ldots, \rho_{\Lambda_2}(\lambda_{2,\tilde{\gamma}(d)}\lambda_{1,\tilde{\gamma}(d)}^{-1}\gamma_{\Lambda_1,d}\lambda_{1,d}\lambda_{2,d}^{-1})) \\ &= \tilde{\gamma}(\rho_{\Lambda_2}(\gamma_{\Lambda_2,1}), \ldots, \rho_{\Lambda_2}(\gamma_{\Lambda_2,d})) \\ &= \rho_{\Lambda_2}(\gamma). \end{aligned}$$

This completes our inductive step, hence our proof. □

A choice of paths Λ for f and t also gives an embedding

$$\bar{\rho}_\Lambda : \overline{\mathrm{pIMG}}\, f \to [\overline{\mathrm{Mon}}\, f]^\infty \subseteq [\mathrm{Mon}\, f]^\infty.$$

Definition 3.8. Let Λ be a choice of paths for f and t. Then we define

$$\begin{aligned} \mathrm{Arb}\, f &:= \rho_\Lambda(\mathrm{pIMG}\, f) \subseteq [\mathrm{Mon}\, f]^\infty, \\ \overline{\mathrm{Arb}}\, f &:= \bar{\rho}_\Lambda(\overline{\mathrm{pIMG}}\, f) \subseteq [\overline{\mathrm{Mon}}\, f]^\infty. \end{aligned}$$

The dependence on the paths is suppressed in the notation.

The homomorphisms ρ_Λ and $\bar{\rho}_\Lambda$ are examples of *arboreal Galois representations*, hence the notation $\mathrm{Arb}\, f$ and $\overline{\mathrm{Arb}}\, f$ for their images. With a fixed preimage labeling, Lemma 3.7 implies that $\mathrm{Arb}\, f$ and $\overline{\mathrm{Arb}}\, f$ are well-defined up to conjugation by an element of $\mathrm{St}_1[\overline{\mathrm{Mon}}\, f]^\infty$.

Lemma 3.9. *The groups $\mathrm{Arb}\, f$ and $\overline{\mathrm{Arb}}\, f$ are self-similar with respect to any choice of paths Λ.*

Proof. This follows immediately from the definition of ρ_Λ and the fact that $\gamma_{\Lambda,i} := \lambda_{\tilde{\gamma}i}^{-1}\gamma|_i\lambda_i \in \mathrm{pIMG}\, f$ for every $\gamma \in \mathrm{pIMG}\, f$. □

3.4. **Post-critically finite rational functions.** Let $C_f \subseteq \mathbb{P}^1_{K^{\rm sep}}$ denote the set of critical points of $f(x)$. The *post-critical set* of f, denoted P_f is the strict forward f-orbit of C_f,

$$P_f := \bigcup_{n\geq 1} f^n(C_f).$$

Definition 3.10. We say f is *post-critically finite* or *PCF* if P_f is finite.

Recall that there is a natural correspondence between places in $K^{\rm sep}(t)$ and points in $\mathbb{P}^1_{K^{\rm sep}}$. Hence when we talk about ramification or inertia groups over a point $p \in \mathbb{P}^1_{K^{\rm sep}}$, we mean the ramification or inertia groups over the place corresponding to p. The points which ramify in $K^{\rm sep}(f^{-\infty}(t))/K^{\rm sep}(t)$ correspond to the critical values of iterates of f, often called branch points. The chain rule implies that P_f contains the critical values of all iterates of f. Thus the extension $K^{\rm sep}(f^{-\infty}(t))/K^{\rm sep}(t)$ is only ramified at finitely many points when f is PCF. If $\operatorname{char} K$ does not divide the ramification index of any critical point of f, then $K^{\rm sep}(f^{-\infty}(t))/K^{\rm sep}(t)$ is at most tamely ramified over these points, hence the inertia groups over all points are topologically cyclic.

Lemma 3.11. *Suppose that f is PCF and that P_f contains $n+1$ points $p_1, p_2, \ldots, p_{n+1}$. Further assume that $\operatorname{char} K$ does not divide the ramification index of any critical point of f. Then*

$$\overline{\mathrm{pIMG}}\, f = \langle\!\langle \gamma_1, \gamma_2, \ldots, \gamma_n, \gamma_{n+1} \rangle\!\rangle,$$

where each γ_i is a topological generator of an inertia subgroup of a (pro-)place in $K^{\rm sep}(f^{-\infty}(t))/K^{\rm sep}(t)$ over the point p_i, and $\gamma_{n+1} = \gamma_1 \cdots \gamma_n$.

Proof. Let $K^{\rm sep}(t)_{P_f}/K^{\rm sep}(t)$ denote the maximal tamely ramified extension of $K^{\rm sep}(t)$ which is only ramified over the points in P_f. Grothendieck proved that

$$\mathrm{Gal}(K^{\rm sep}(t)_{P_f}/K^{\rm sep}(t)) = \langle\!\langle \tau_1, \tau_2, \ldots, \tau_n, \tau_{n+1} : \tau_{n+1} = \tau_1 \cdots \tau_n \rangle\!\rangle, \tag{2}$$

where each τ_i is a topological generator of an inertia subgroup over the point p_i (see [GR71, Exposé X Corollaire 3.9 and Exposé XII, Corollaire 5.2], or [Sza09, Thm. 4.9.1] for a precise statement in English). Since $K^{\rm sep}(f^{-\infty}(t))/K^{\rm sep}(t)$ is only tamely ramified over P_f, it follows that $\overline{\mathrm{pIMG}}\, f$ is a quotient of $\mathrm{Gal}(K^{\rm sep}(t)_{P_f}/K^{\rm sep}(t))$, hence that

$$\overline{\mathrm{pIMG}}\, f = \langle\!\langle \gamma_1, \gamma_2, \ldots, \gamma_n, \gamma_{n+1} \rangle\!\rangle,$$

where each γ_i topologically generates an inertia subgroup over p_i and $\gamma_{n+1} = \gamma_1 \cdots \gamma_n$. □

Remark. The Galois group $\mathrm{Gal}(K^{\rm sep}(t)_{P_f}/K^{\rm sep}(t))$, the maximal extension of $K(t)$ unramified outside of P_f may be interpreted as $\pi_1^{\text{ét,tame}}(\mathbb{P}^1_{K^{\rm sep}} \setminus P_f, p)$, the tame étale fundamental group of $\mathbb{P}^1_{K^{\rm sep}} \setminus P_f$ with respect to the geometric point p corresponding to our choice of separable closure of $K^{\rm sep}(t)$ in which $K^{\rm sep}(t)_{P_f}$ lives. This presentation of $\pi_1^{\text{ét,tame}}(\mathbb{P}^1_{K^{\rm sep}} \setminus P_f, p)$ follows from the fact that tamely ramified extensions can be lifted to characteristic 0 where the étale fundamental group is known to be the profinite completion of the topological fundamental group of $\mathbb{P}^1_{\mathbb{C}} \setminus \tilde{P}_f$, where $\tilde{P}_f$ is a lift of the set P_f. This topological fundamental group is classically known to have a presentation matching (2), where the generators correspond to loops winding once around each of the corresponding punctures.

Remark. Note that Lemma 3.11 only tells us that $\overline{\mathrm{pIMG}}\, f$ is topologically finitely generated by inertia generators. This choice of inertia generators is not unique, and not all choices necessarily generate the group. There will typically be many intricate relations among these generators which depend on dynamical properties of f.

The following Lemma is useful for determining the conjugacy class of $\rho_\Lambda(\gamma)$ when γ is an inertia generator.

Lemma 3.12. *Fix a preimage labeling and a choice of paths Λ for f and t. Let $(t-p)$ be a prime in $K^{\rm sep}(t)$ and let P be a prime in $K^{\rm sep}(f^{-\infty}(t))$ over $(t-p)$. Let $\gamma \in \overline{\rm pIMG}\, f$ be a topological generator for the inertia group of P over $(t-p)$.*

For each $1 \le i \le d$, we have $P \cap K^{\rm sep}(t_i) = (t_i - q_i)$ for some $q_i \in f^{-1}(p)$. Let e_i denote the ramification index of f at q_i; note that e_i is also the length of the orbit of i under $\tilde\gamma$. Then

$$\gamma_{\Lambda,\tilde\gamma^{e_i-1}(i)} \cdots \gamma_{\Lambda,i} = \lambda_i^{-1}\gamma^{e_i}|_i\lambda_i \tag{3}$$

is a topological generator for the inertia group of $P_i := \lambda_i^{-1}(P \cap K^{\rm sep}(f^{-\infty}(t_i)))$ over $(t-q_i)$.

Proof. The identity (3) follows from the definition of $\gamma_{\Lambda,i}$ and the assumption that e_i is the length of the $\tilde\gamma$ orbit of i. Since e_i is the ramification index of f at q_i, it follows that γ^{e_i} topologically generates the inertia group for $\lambda_i(P_i) := P \cap K^{\rm sep}(f^{-\infty}(t_i))$ over $(t_i - q_i)$. Therefore $\lambda_i^{-1}\gamma^{e_i}|_i\lambda_i$ topologically generates the inertia group of P_i over $\lambda_i^{-1}(t_i - q_i) = (t-q_i)$. ☐

3.5. **Choosing paths for polynomials.** Suppose now that $f(x) \in K[x]$ is a polynomial of degree d prime to $\operatorname{char} K$. It follows that ∞ is a totally tamely ramified fixed point for f, and hence of any extension $K(x)/K(t)$ defined by $f^n(x) = t$), and so an inertia subgroup $\langle\langle\gamma_\infty\rangle\rangle$ over infinity in the Galois closure $K^{\rm sep}(f^{-\infty}(t))/K^{\rm sep}(t)$ is isomorphic to $\mathbb{Z}_d$.

Recall the standard odometer c_∞ in $[C_d]^\infty$ defined recursively by $c_\infty = \sigma(1,\ldots,1,c_\infty)$. If one views the tree labeling as encoding digits of d-adic integers, the standard odometer represents addition by 1 in $\mathbb{Z}_d$, so $\langle\langle c_\infty\rangle\rangle \cong \mathbb{Z}_d$.

Abstractly, then, inertia subgroups are isomorphic to subgroups generated by an odometer. It turns out this isomorphisms can be realized Galois-theoretically: Proposition 3.13 shows that for each choice of topological generator $\gamma_\infty \in \overline{\rm pIMG}\, f$ of an inertia subgroup at ∞, there exists a choice of paths with respect to which γ_∞ acts via the standard odometer.

Proposition 3.13. *For each topological generator $\gamma_\infty \in \overline{\rm pIMG}\, f$ of an inertia group over ∞, there exists a preimage labeling $i \to t_i$ and a path λ from t to t_1 such that the collection of paths Λ defined by $\lambda_i := \gamma_\infty^{i-1}|_1\lambda$ from t to t_i satisfies $\rho_\Lambda(\gamma_\infty) = c_\infty$.*

Proof. Up to a linear change of coordinate defined over K, we may write $f(x) = ax^d + b$. Let $K^{\rm sep}((1/t)) \supseteq K^{\rm sep}(t)$ be the field of formal Laurent series in $1/t$, which may be interpreted as the completion of $K^{\rm sep}(t)$ at ∞. Let $K((1/t))^{\rm sep}$ be a choice of separable closure; note that $K((1/t))^{\rm sep}$ contains a separable closure of $K(t)$. Since d is coprime to $\operatorname{char} K$, the extension $K^{\rm sep}((1/t))(f^{-\infty}(t))/K^{\rm sep}((1/t))$ is a pro-d cyclic extension with Galois group canonically isomorphic to the inertia group over ∞ of $K^{\rm sep}(f^{-\infty}(t))/K^{\rm sep}(t)$. Hence we may extend γ_∞ to a topological generator of $\operatorname{Gal}(K^{\rm sep}((1/t))(f^{-\infty}(t))/K^{\rm sep}((1/t)))$. Note that $\langle\langle\gamma_\infty\rangle\rangle$ is isomorphic to $\mathbb{Z}_d$, the additive group of d-adic integers.

Let $t_1 \in f^{-1}(t) \subseteq K((1/t))^{\rm sep}$ and let $\lambda : K((1/t))^{\rm sep} \to K((1/t_1))^{\rm sep}$ be a $K^{\rm sep}$-isomorphism such that $\lambda(t) = t_1$; note that λ induces a path from t to t_1. Furthermore, $|g| = |\lambda(g)|_1$ for all $g \in K((1/t))^{\rm sep}$ where $|\cdot|$ and $|\cdot|_1$ are the standard normalized absolute values on $K((1/t))^{\rm sep}$ and $K((1/t_1))^{\rm sep}$ respectively. Let $t_{0,\ell} := \lambda^\ell(t)$. Then $(t_{0,\ell})$ is a tower of iterated f-preimages of t.

Kummer theory implies that for each $\ell \ge 0$,

$$K^{\rm sep}((1/t))(f^{-\ell}(t)) = K^{\rm sep}((1/t^{1/d^\ell})).$$

Considering the Newton polygon of $f^\ell(x) - t$ with respect to the $1/t$-adic valuation, it follows that each element of $f^{-\ell}(t)$ has valuation $-1/d^\ell$. This implies that for each element $t' \in f^{-\ell}(t)$ there is a unique root of $g_\ell(x) := a^{d^{\ell-1}}x^{d^\ell} - t$ which is closest to t' with respect to the $1/t$-adic metric. Let τ_ℓ be the root of $g_\ell(x)$ closest to $t_{0,\ell}$. Note that the these roots are compatible in the sense that $a\tau_{\ell+1}^d = \tau_\ell$.

Let (ζ_{d^ℓ}) be the sequence of primitive roots of unity in $K^{\rm sep}$ such that

$$\gamma_\infty(\tau_\ell) = \zeta_{d^\ell}\tau_\ell.$$

Compatibility of the τ_ℓ implies that $\zeta_{d^{\ell+1}}^d = \zeta_{d^\ell}$. By construction of τ_ℓ, we may express $t_{0,\ell}$ as a Laurent series in τ_ℓ with coefficients in K^{sep} of the form

$$t_{0,\ell} = \tau_\ell + \sum_{i=1-d}^{\infty} a_{i,\ell}\tau_\ell^{-i}.$$

For $\varepsilon \in \mathbb{Z}/d^\ell\mathbb{Z}$, define $t_{\varepsilon,\ell} := \gamma_\infty^\varepsilon t_{0,\ell}$. Thus the leading term of $t_{\varepsilon,\ell}$ expanded as a Laurent series in τ_ℓ is $\zeta_{d^\ell}^\varepsilon \tau_\ell$. Since the leading term of

$$\lambda(t_{\varepsilon,\ell}) = \zeta_{d^\ell}^\varepsilon \lambda(t_{0,\ell}) = \zeta_{d^\ell}^\varepsilon t_{0,\ell+1}$$

is $\zeta_{d^\ell}^\varepsilon \tau_{\ell+1} = \zeta_{d^{\ell+1}}^{d\varepsilon}\tau_{\ell+1}$, it follows that

$$\lambda(t_{\varepsilon,\ell}) = t_{d\varepsilon,\ell+1}.$$

Define Λ to be the collection of paths λ_i from t to t_i defined by $\lambda_i := \gamma_\infty^{i-1}|_1\lambda$. If $1 \leq i < d$, then

$$(\gamma_\infty)_{\Lambda,i} = \lambda_{i+1}^{-1}\gamma_\infty|_i\lambda_i = \lambda^{-1}(\gamma_\infty^i|_1)^{-1}\gamma_\infty|_i\gamma_\infty^{i-1}|_1\lambda = 1,$$

and if $i = d$, then

$$(\gamma_\infty)_{\Lambda,d} = \lambda_1^{-1}\gamma_\infty|_d\lambda_d = \lambda^{-1}\gamma_\infty|_d\gamma_\infty^{d-1}|_1\lambda = \lambda^{-1}\gamma_\infty^d|_1\lambda.$$

To complete the proof it suffices to show that $(\gamma_\infty)_{\Lambda,d}(t_{\varepsilon,\ell}) = \gamma_\infty(t_{\varepsilon,\ell})$ for all $\ell \geq 0$ and $\varepsilon \in \mathbb{Z}/d^\ell\mathbb{Z}$. By construction we have $\gamma_\infty(t_{\varepsilon,\ell}) = t_{\varepsilon+1,\ell}$. On the other hand,

$$\lambda^{-1}\gamma_\infty^d|_1\lambda(t_{\varepsilon,\ell}) = \lambda^{-1}\gamma_\infty^d(t_{d\varepsilon,\ell+1}) = \lambda^{-1}(t_{d\varepsilon+d,\ell+1}) = t_{\varepsilon+1,\ell}. \qquad \square$$

3.6. **Unicritical polynomials.** A polynomial $f(x) \in K[x]$ is said to be *unicritical* if f has a unique finite critical point. Since $\mathrm{Gal}(K^{\mathrm{sep}}/K)$ permutes critical points of f, the unique finite critical point must belong to K. Changing coordinates over K we may assume that $f(x) = ax^d + b$ for some $a, b \in K$.

The post-critical set P_f consists of ∞ and the forward orbit of 0. Let $p_i := f^i(0)$. We say f is *post-critically infinite* if 0 has an infinite orbit under f; otherwise f is PCF. Let γ_i for $1 \leq i \leq n$ and γ_∞ be topological generators for inertia groups over each p_i. Since f is unicritical, we have identifications $\overline{\mathrm{Mon}}\, f = C_d$ and $\mathrm{Mon}\, f \subseteq \mathrm{Aff}_{1,d}$, where

$$\mathrm{Aff}_{1,d} \cong \{rx + s : r \in \mathbb{Z}/d\mathbb{Z}^\times, s \in \mathbb{Z}/d\mathbb{Z}\}$$

is the one dimensional affine group modulo d. Note that p_1 is the only finite critical value of f, hence $\gamma_i =_1 1$ for $i > 1$. The group $\overline{\mathrm{Arb}}\, f$ is as large as possible in the post-critically infinite case.

Proposition 3.14. *Suppose $f(x) = ax^d + b$ is post-critically infinite. Then $\overline{\mathrm{Arb}}\, f = [C_d]^\infty$.*

Proof. The element γ_i acts trivially on all levels $j < i$, acts (without loss of generality) by σ on a single branch at level i, and acts trivially on each of the sub-trees above that branch. Such elements generate the group $[C_d]^\infty$. Therefore $\overline{\mathrm{Arb}}\, f = [C_d]^\infty$. $\square$

We now turn to the case of a PCF unicritical polynomial. Suppose that f has exactly n finite post-critical points. Proposition 3.13 provides us with a preimage labeling and a choice of paths Λ for f and t such that $\rho_\Lambda(\gamma_\infty) = c_\infty$ is the standard odometer. Let $c_i := \rho_\Lambda(\gamma_i) \in [C_d]^\infty$. Thus $\overline{\mathrm{Arb}}\, f = \langle\langle c_1, c_2, \ldots, c_n \rangle\rangle$ and $c_\infty = c_1 \cdots c_n$.

The identity $c_\infty = c_1 \cdots c_n$ implies that $c_1 =_1 c_\infty =_1 \sigma$. If $t' = \sqrt[d]{\frac{t-b}{a}} \in f^{-1}(t)$, then there exists a primitive dth root of unity $\zeta_d \in K^{\mathrm{sep}}$ such that $\gamma_1(t') = \gamma_\infty(t') = \zeta_d t'$.

The qualitative structure of $\overline{\mathrm{Arb}}\, f$ depends fundamentally on whether the finite critical point of f is periodic or (strictly) preperiodic. We refer to these as the *periodic* and *preperiodic* cases, respectively. In the periodic case, p_1 is periodic with period n. In the preperiodic case, there is an integer $1 \leq m < n$ such that $f^m(p_1) = f^n(p_1)$. This is equivalent to saying that $f(p_m) = f(p_n) = p_{m+1}$. Since f is unicritical, there must be some integer $1 \leq \omega < d$ such that $p_n = \zeta_d^\omega p_m$.

Proposition 3.15 shows that the c_i satisfy a system of cyclic conjugate recurrences in $[C_d]^\infty$ which depend on d and n in the periodic case, and d, m, n, and ω in the preperiodic case.

Proposition 3.15. *Let $f(x) = ax^d + b$ be PCF. The topological generators $c_1, c_2, \ldots, c_n$ of $\overline{\mathrm{Arb}}\, f$ satisfy the following system of conjugate recurrences in $[C_d]^\infty$,*

(1) (Periodic Case)

$$c_i \sim \begin{cases} \sigma(1,\ldots,1,c_n) & \text{if } i=1, \\ (1,\ldots,1,c_{i-1}) & \text{if } i \neq 1 \end{cases}$$

(2) (Preperiodic Case)

$$c_i \sim \begin{cases} \sigma & \text{if } i=1 \\ (1,\ldots,1,c_n,1,\ldots,1,c_m) & \text{if } i=m+1, \text{where } c_n \text{ is in the } \omega\text{th component} \\ (1,\ldots,1,c_{i-1}) & \text{if } i \neq 1, m+1. \end{cases}$$

Proof. (1) If $i \neq 1$, then p_i is not a critical value of f. Hence $c_i =_1 1$. Thus

$$\rho_\Lambda(\gamma_i) = (\rho_\Lambda(\gamma_{\Lambda,1}), \ldots, \rho_\Lambda(\gamma_{\Lambda,d}))$$

where Lemma 3.12 implies that each $\rho_\Lambda(\gamma_{\Lambda,j})$ is the image of an inertia generator over $(t - q_{i,j})$ where $q_{i,j} \in f^{-1}(p_i)$. Our assumption that the unique finite critical point of f is periodic implies that there is a unique j such that $q_{i,j} = p_{i-1}$ and for all other j' we have $q_{i,j'} \notin P_f$. Hence $\rho_\Lambda(\gamma_{\Lambda,j}) \sim c_{i-1}^{\varepsilon_{i-1}}$ in $\overline{\mathrm{Arb}}\, f$ for some d-adic unit ε_{i-1} and $\rho_\Lambda(\gamma_{\Lambda,j'}) = 1$. Therefore, conjugating by a power of σ we have the following conjugacy in $[C_d]^\infty$,

$$c_i \sim (1,\ldots,1,c_{i-1}^{\varepsilon_{i-1}}).$$

Next suppose $i = 1$. Then p_1 is a critical value of f with unique preimage $p_n = 0$. Thus $c_1 =_1 \sigma$. To ease notation, let us write $\gamma := \gamma_1$. Lemma 3.12 implies that $\rho_\Lambda(\gamma_{\Lambda,d}) \cdots \rho_\Lambda(\gamma_{\Lambda,1})$ is a topological generator of an inertia group over p_n. Therefore Proposition 2.2 implies that

$$c_1 \sim \sigma(1,\ldots,1,c_n^{\varepsilon_n})$$

in $[C_d]^\infty$ for some d-adic unit ε_n.

Since c_∞ is the standard odometer, we have $\chi_\ell(c_\infty) = 1$ for all $\ell \geq 1$. Since $c_i =_\ell 1$ for all $\ell < i$, we have $\chi_\ell(c_i) = 0$ for $\ell < i$. The conjugation relations imply that $\chi_1(c_1) = 1$ and that $\chi_\ell(c_i) = \varepsilon_{i-1}\chi_{\ell-1}(c_{i-1})$ for $\ell > 1$ with the $i-1$ subscripts interpreted modulo n. These recursive relations combined with the identity $c_\infty = c_n \cdots c_1$ imply that for $1 \leq \ell \leq n$,

$$\chi_\ell(c_i) = \begin{cases} 1 & i = \ell \\ 0 & i \neq \ell. \end{cases}$$

Thus for $1 \leq i < n$,

$$1 = \chi_i(c_i) = \varepsilon_{i-1}\chi_{i-1}(c_{i-1}) = \varepsilon_{i-1}.$$

If $1 < i \leq n$, then $\chi_{n+1}(c_i) = \chi_n(c_{i-1}) = 0$. Hence

$$1 = \chi_{n+1}(c_\infty) = \chi_{n+1}(c_1) = \varepsilon_n\chi_n(c_n) = \varepsilon_n.$$

Therefore $\varepsilon_i \equiv 1 \bmod d$ for all $1 \leq i \leq n$. Lemma 2.15 implies that $c_i^{\varepsilon_i} \sim c_i$ in $[C_d]^\infty$, and we conclude that the following conjugacies hold in $[C_d]^\infty$,

$$c_i \sim \begin{cases} \sigma(1,\ldots,1,c_n) & \text{if } i=1, \\ (1,\ldots,1,c_{i-1}) & \text{if } i \neq 1. \end{cases}$$

(2) If $i \neq 1, m+1$, then we argue as in the periodic case that there exists d-adic units ε_{i-1} such that

$$c_i \sim (1,\ldots,1,c_{i-1}^{\varepsilon_{i-1}})$$

in $[C_d]^\infty$. If $i = 1$, then $c_1 =_1 \sigma$ and Lemma 3.12 implies that $\rho_\Lambda(\gamma_{\Lambda,d}) \cdots \rho_\Lambda(\gamma_{\Lambda,1})$ is a topological generator of an inertia group over 0. In the preperiodic case, 0 is not an element of P_f, hence

$$\rho_\Lambda(\gamma_{\Lambda,d}) \cdots \rho_\Lambda(\gamma_{\Lambda,1}) = 1.$$

Therefore Proposition 2.2 implies that $c_1 \sim \sigma$ in $[C_d]^\infty$. If $i = m+1$, then $c_{m+1} =_1 1$. Suppose P is a prime in $K^{\text{sep}}(f^{-\infty}(t))$ over $(t - p_{m+1})$ such that γ_{m+1} topologically generates the inertia group for P. Let $q_j \in f^{-1}(p_{m+1})$ be such that $P \cap K^{\text{sep}}(t_j) = (t_j - q_j)$. Replacing γ_{m+1} with a conjugate by a power of γ_1, we may suppose that $q_d = p_m$. Since f is unicritical, we have $K^{\text{sep}}(f^{-1}(t)) = K^{\text{sep}}(t_j)$ and $t_j = \zeta_d^j t_d$ for each j. Thus for each j,

$$(t_d - p_m) = (t_d - q_d) = (t_j - q_j) = (\zeta_d^j t_d - q_j) = (t_d - \zeta_d^{-j} q_j).$$

Hence $q_j = \zeta_d^j p_m$. Therefore $q_\omega = \zeta_d^\omega p_m = p_n$. If $j \neq \omega, d$, then $q_j \notin P_f$. Thus there exists d-adic units ε_m and ε_n such that

$$c_{m+1} \sim (1, \ldots, 1, c_n^{\varepsilon_n}, 1, \ldots, c_m^{\varepsilon_m}),$$

in $[C_d]^\infty$ where the $c_n^{\varepsilon_n}$ is in the ωth component.

The conjugation identities imply that

$$\chi_\ell(c_1) = \begin{cases} 1 & \ell = 1 \\ 0 & \ell \neq 1, \end{cases}$$

and $\chi_1(c_i) = 0$ for $i > 1$; if $i \neq 1, m+1$, then $\chi_\ell(c_i) = \varepsilon_{i-1}\chi_{\ell-1}(c_{i-1})$ for all $\ell > 1$; and $\chi_\ell(c_{m+1}) = \varepsilon_m \chi_{\ell-1}(c_m) + \varepsilon_n \chi_{\ell-1}(c_n)$. These recursive relations combined with $\chi_\ell(c_i) = 0$ for $\ell < i$ and $c_\infty = c_1 \cdots c_n$ imply that for $1 \leq \ell \leq n$,

$$\chi_\ell(c_i) = \begin{cases} 1 & i = \ell \\ 0 & i \neq \ell. \end{cases}$$

Hence for $i \neq 1, m+1$,

$$1 = \chi_i(c_i) = \varepsilon_{i-1}\chi_{i-1}(c_{i-1}) = \varepsilon_{i-1},$$

and

$$1 = \chi_{m+1}(c_{m+1}) = \varepsilon_m \chi_m(c_m) + \varepsilon_n \chi_m(c_n) = \varepsilon_m.$$

If $i \neq 1, m+1$, then

$$\chi_{n+1}(c_i) = \chi_n(c_{i-1}) = 0$$

Hence, since $\chi_{n+1}(c_1) = 0$,

$$1 = \chi_{n+1}(c_\infty) = \chi_{n+1}(c_{m+1}) = \chi_n(c_m) + \varepsilon_n \chi_n(c_n) = \varepsilon_n.$$

Therefore $\varepsilon_i \equiv 1 \bmod d$ for all i.

Since the unique finite critical point is strictly preperiodic and f is unicritical, it follows that each c_i has order d for $1 \leq i \leq n$. Thus

$$c_i \sim \begin{cases} \sigma & \text{if } i = 1 \\ (1, \ldots, 1, c_n, 1, \ldots, 1, c_m) & \text{if } i = m+1, \text{where } c_n \text{ is in the } \omega\text{th component} \\ (1, \ldots, 1, c_{i-1}) & \text{if } i \neq 1, m+1. \end{cases}$$ □

4. Model groups and semirigidity

In Proposition 3.15 we showed that for a unicritical PCF polynomial $f(x) \in K[x]$, the group $\overline{\operatorname{Arb}} f$ is topologically generated by a collection of elements in $[C_d]^\infty$ satisfying a system of *conjugate* recurrences which is determined entirely by the combinatorial structure of the post-critical orbit in the periodic case, and in the preperiodic case further requires only a small piece of arithmetic information, the parameter ω. In this section we study two families of *model groups* $\mathcal{A}(d,n)$ and $\mathcal{B}(d,m,n,\omega) \subseteq [C_d]^\infty$ which are topologically generated by elements satisfying the same systems of recursion as $\overline{\operatorname{Arb}} f$ up to equality rather than conjugacy. Ultimately we show that these generators have a certain *semirigidity property* which allows us to deduce that $\mathcal{A}$ and $\mathcal{B}$ are conjugate within $[C_d]^\infty$ to $\overline{\operatorname{Arb}} f$ in the periodic and preperiodic cases, respectively.

Definition 4.1. Let $d \geq 2$.

(1) Given an integer $n \geq 1$, let $\mathcal{A} = \mathcal{A}(d,n) \subseteq [C_d]^\infty$ be the closed subgroup defined by
$$\mathcal{A} := \langle\langle a_1, \ldots, a_n \rangle\rangle$$
where
$$a_i := \begin{cases} \sigma(1,\ldots,1,a_n) & \text{if } i = 1, \\ (1\ldots,1,a_{i-1}) & \text{if } i \neq 1. \end{cases}$$
Let $a_\infty := a_1 a_2 \cdots a_n \in \mathcal{A}$.

(2) Given integers $n > m \geq 0$ and $1 \leq \omega < d$, let $\mathcal{B} = \mathcal{B}(d,m,n,\omega) \subseteq [C_d]^\infty$ be the closed subgroup defined by
$$\mathcal{B} := \langle\langle b_1, \ldots, b_n \rangle\rangle$$
where
$$b_i := \begin{cases} \sigma & \text{if } i = 1 \\ (1,\ldots,1,b_n,1,\ldots,1,b_m) & \text{if } i = m+1, \text{where } b_n \text{ is in the } \omega\text{th component} \\ (1,\ldots,1,b_{i-1}) & \text{if } i \neq 1, m+1. \end{cases}$$
Let $b_\infty := b_1 b_2 \cdots b_n \in \mathcal{B}$.

Notation 4.2. Given an integer $1 \leq i \leq n$ we let $\widehat{\mathcal{A}}_i$ denote the normal subgroup of $\mathcal{A}$ topologically generated by all conjugates of a_j with $j \neq i$, and define $\widehat{\mathcal{B}}_i$ analogously.

Similarly, if $1 \leq i,j \leq n$, then $\widehat{\mathcal{B}}_{i,j}$ denotes the normal subgroup of $\mathcal{B}$ topologically generated by all conjugates of b_k with $k \neq i,j$. In $\mathcal{B}$, we write $b_{i,j} := \sigma^j b_i \sigma^{-j}$. Since $\sigma = b_1$, it follows that $b_{i,j} \in \mathcal{B}$.

The recursive descriptions of the model groups imply that $\operatorname{St}_1 \mathcal{A} \subseteq \mathcal{A}^d$ and $\operatorname{St}_1 \mathcal{B} \subseteq \mathcal{B}^d$, which is to say that $\mathcal{A}$ and $\mathcal{B}$ are both self-similar. Furthermore, each coordinate projection is surjective.

Lemma 4.3. *Both $\mathcal{A}$ and $\mathcal{B}$ are self-similar. Moreover, the coordinate projections $\pi_j : \operatorname{St}_1 \mathcal{A} \to \mathcal{A}$ and $\pi_j : \operatorname{St}_1 \mathcal{B} \to \mathcal{B}$ are surjective for all $1 \leq j \leq d$.*

Proof. First consider $\mathcal{A}$. If $2 \leq i \leq d$, then conjugating a_i by powers of a_1 gives us
$$(1,\ldots,1,a_{i-1},1,\ldots,1) \in \operatorname{St}_1 \mathcal{A}$$
where the non-trivial component can be in any coordinate. We also have
$$a_1^d = (a_n,\ldots,a_n) \in \operatorname{St}_1 \mathcal{A}.$$
Therefore $\pi_j(\operatorname{St}_1 \mathcal{A})$ contains the group topologically generated by all of the a_i, which is $\mathcal{A}$. Hence each $\pi_j : \operatorname{St}_1 \mathcal{A} \to \mathcal{A}$ is surjective.

Next consider $\mathcal{B}$. Observe that conjugating b_i with $i \neq 1, m+1$ by powers of $b_1 = \sigma$ gives
$$(1,\ldots,1,b_{i-1},1,\ldots,1) \in \operatorname{St}_1 \mathcal{B}$$

with the non-trivial component in any coordinate. Conjugating b_{m+1} by powers of b_1 gives elements of $\mathrm{St}_1 \mathcal{B}$ with either b_m or b_n in any prescribed coordinate. Thus $\pi_j(\mathrm{St}_1 \mathcal{B})$ contains the group topologically generated by the b_i, namely $\mathcal{B}$. Hence each $\pi_j : \mathrm{St}_1 \mathcal{B} \to \mathcal{B}$ is surjective. □

4.1. **Orders of generators.** The defining systems of recurrence for the model groups along with the recursive formula for the characters χ_ℓ (Definition 2.16) allow us to calculate the values of χ_ℓ at each generator. We make frequent use of these calculations, including in the determination of the orders of the generators.

Lemma 4.4. *Let $1 \leq i \leq n$ and $\ell \geq 1$, then*

(1) $\chi_\ell(a_i) = \begin{cases} 1 & \text{if } \ell \equiv i \bmod n, \\ 0 & \text{otherwise.} \end{cases}$

(2) $\chi_\ell(b_i) = \begin{cases} 1 & \ell = i \leq m \\ 1 & \ell \geq i > m \text{ and } \ell \equiv i \bmod n-m \\ 0 & \text{otherwise.} \end{cases}$

Proof. (1) If $\ell = 1$, then by definition of the a_i we have

$$\chi_1(a_i) = \begin{cases} 1 & \text{if } i = 1, \\ 0 & \text{otherwise.} \end{cases}$$

If $\ell > 1$, then the definition of a_i and the recursive formula for χ_ℓ imply that

$$\chi_\ell(a_i) = \chi_{\ell-1}(a_{i-1})$$

where we interpret the subscript of a_{i-1} modulo n. Hence we may conclude by induction that

$$\chi_\ell(a_i) = \chi_1(a_{i-\ell+1}) = \begin{cases} 1 & \text{if } \ell \equiv i \bmod n, \\ 0 & \text{otherwise.} \end{cases}$$

(2) We proceed similarly for the b_i. Starting with $\ell = 1$, the definition of the b_i imply that

$$\chi_1(b_i) = \begin{cases} 1 & \text{if } i = 1 \\ 0 & \text{otherwise.} \end{cases}$$

If $\ell > 1$, then

$$\chi_\ell(b_i) = \begin{cases} \chi_{\ell-1}(b_{i-1}) & \text{if } i \neq 1, m+1, \\ 0 & \text{if } i = 1, \\ \chi_{\ell-1}(b_m) + \chi_{\ell-1}(b_n) & \text{if } i = m+1. \end{cases}$$

These relations imply that $\chi_\ell(b_i) = 0$ whenever $\ell < i$. Furthermore, if $i \leq m$, then

$$\chi_\ell(b_i) = \begin{cases} 1 & \text{if } \ell = i, \\ 0 & \text{otherwise.} \end{cases} \tag{4}$$

Next we calculate $\chi_\ell(b_n)$ for all $\ell \geq 1$. If $\ell < n - m$, then $n - \ell + 1 > m + 1$ and

$$\chi_\ell(b_n) = \chi_1(b_{n-\ell+1}) = 0.$$

If $\ell \geq n - m$, then

$$\chi_\ell(b_n) = \chi_{\ell-(n-m)}(b_m) + \chi_{\ell-(n-m)}(b_n).$$

Hence if $\ell = r + q(n-m)$ with $0 \leq r < n - m$, then a simple induction implies that

$$\chi_\ell(b_n) = \chi_r(b_n) + \sum_{s=0}^{q-1} \chi_{r+s(n-m)}(b_m) = \sum_{s=0}^{q-1} \chi_{r+s(n-m)}(b_m).$$

Observe that (4) implies that $\chi_{r+s(n-m)}(b_m) = 1$ if and only if $r+s(n-m) = m$ for some $0 \le s < q$, which is equivalent to $\ell > m$ and $\ell \equiv m \bmod n-m$. Therefore

$$\chi_\ell(b_n) = \begin{cases} 1 & \text{if } \ell > m \text{ and } \ell \equiv m \bmod n-m, \\ 0 & \text{otherwise.} \end{cases}$$

Now suppose that $m < i \le n$ and $\ell \ge i$. Then

$$\chi_\ell(b_i) = \chi_{\ell-i+m}(b_m) + \chi_{\ell-i+m}(b_n).$$

Our calculations imply that

$$\chi_{\ell-i+m}(b_m) = \begin{cases} 1 & \text{if } \ell = i, \\ 0 & \text{otherwise,} \end{cases}$$

and

$$\chi_{\ell-i+m}(b_n) = \begin{cases} 1 & \text{if } \ell > i \text{ and } \ell \equiv i \bmod n-m, \\ 0 & \text{otherwise.} \end{cases}$$

Therefore if $m < i \le n$ and $\ell \ge i$, then

$$\chi_\ell(b_i) = \begin{cases} 1 & \text{if } \ell \equiv i \bmod n-m, \\ 0 & \text{otherwise.} \end{cases}$$

Putting this all together we have

$$\chi_\ell(b_i) = \begin{cases} 1 & \ell = i \le m. \\ 1 & \ell \ge i > m \text{ and } \ell \equiv i \bmod n-m, \\ 0 & \text{otherwise.} \end{cases} \qquad \square$$

Recall that $a_\infty = a_1 \cdots a_n$ and $b_\infty = b_1 \cdots b_n$. Thus Lemma 4.4 implies that

$$\chi_\ell(a_\infty) = \chi_\ell(b_\infty) = 1$$

for all $\ell \ge 1$. Hence Lemma 2.18 implies that a_∞ and b_∞ are both strict odometers.

Proposition 4.5. *Let $1 \le i \le n$ and $\ell \ge 0$, then*

(1) $\mathrm{ord}_\ell(a_i) = d^{\left\lfloor \frac{\ell-i}{n} \right\rfloor + 1}$.

(2) $\mathrm{ord}_\ell(b_i) = \begin{cases} d & \textit{if } \ell \ge i, \\ 1 & \textit{otherwise.} \end{cases}$

In particular, the map $\mathbb{Z}_d \to \langle\langle a_i \rangle\rangle$ defined by $m \mapsto a_i^m$ is an isomorphism and each b_i has order d.

Proof. (1) It suffices to prove that for all $1 \le i \le n$ and all $\ell \ge 0$,

$$\log_d(\mathrm{ord}_\ell(a_i)) = \left\lfloor \frac{\ell - i}{n} \right\rfloor + 1. \tag{5}$$

Note that $a_i =_0 1$ implies $\log_d(\mathrm{ord}_0(a_i)) = 0$ for all $1 \le i \le n$. If $1 < i \le n$, then $a_i = (1, \ldots, 1, a_{i-1})$ implies that

$$\log_d(\mathrm{ord}_\ell(a_i)) = \log_d(\mathrm{ord}_{\ell-1}(a_{i-1})).$$

Since $a_1 = \sigma(1, \ldots, 1, a_n)$ and σ has order d, we have

$$\begin{aligned} \log_d(\mathrm{ord}_\ell(a_1)) &= \log_d(\mathrm{ord}_\ell(a_1^d)) + 1 \\ &= \log_d(\mathrm{ord}_\ell((a_n, \ldots, a_n))) + 1 \\ &= \log_d(\mathrm{ord}_{\ell-1}(a_n)) + 1. \end{aligned}$$

Thus for all $\ell > 1$,

$$\log_d(\mathrm{ord}_\ell(a_i)) = \begin{cases} \log_d(\mathrm{ord}_{\ell-1}(a_n)) + 1 & \text{if } i = 1, \\ \log_d(\mathrm{ord}_{\ell-1}(a_{i-1})) & \text{otherwise.} \end{cases}$$

Furthermore, these recursive identities completely determine $\mathrm{ord}_\ell(a_i)$ for all $\ell \geq 0$ and all $1 \leq i \leq n$. On the other hand, let

$$\alpha_{\ell,i} := \left\lfloor \frac{\ell - i}{n} \right\rfloor + 1.$$

We will show that $\alpha_{\ell,i}$ satisfies the same recursive identities and initial values. First note that for $1 \leq i \leq n$,

$$\alpha_{0,1} = \left\lfloor \frac{-i}{n} \right\rfloor + 1 = 0.$$

If $2 \leq i \leq n$, then

$$\alpha_{\ell,i} = \left\lfloor \frac{\ell - i}{n} \right\rfloor + 1 = \left\lfloor \frac{(\ell-1)-(i-1)}{n} \right\rfloor + 1 = \alpha_{\ell-1,i-1}.$$

Finally,

$$\alpha_{\ell,1} = \left\lfloor \frac{\ell - 1}{n} \right\rfloor + 1 = \left(\left\lfloor \frac{(\ell-1)-n}{n} \right\rfloor + 1 \right) + 1 = \alpha_{\ell-1,n} + 1.$$

Thus (5) holds for all $\ell \geq 0$ and $1 \leq i \leq n$. Therefore

$$\log_d(\mathrm{ord}_\ell(a_i)) = \alpha_{\ell,i} = \left\lfloor \frac{\ell - i}{n} \right\rfloor + 1.$$

(2) Let $1 \leq i \leq n$. If $\ell < i$, then the recursive formulas for b_i tell us that b_i acts trivially on T_d^ℓ, hence that $\mathrm{ord}_\ell(b_i) = 1$. Lemma 4.4 implies that $\chi_i(b_i) = 1 \in \mathbb{Z}/d\mathbb{Z}$ for all i. Hence the order of each b_i is a multiple of d. On the other hand, note that the elements b_i^d satisfy the system of recursions

$$\begin{aligned} x_1 &= 1 \\ x_{m+1} &= (1, \ldots, 1, x_n, 1, \ldots, 1, x_m) \\ x_i &= (1, \ldots, 1, x_{i-1}) \text{ if } i \neq 1, m+1, \end{aligned}$$

which are also satisfied by $x_i = 1$. Hence Lemma 2.11 implies that $b_i^d = 1$ for each $1 \leq i \leq n$. Therefore each b_i has order exactly d. Hence $\mathrm{ord}_\ell(b_i) = d$ for all $\ell \geq i$. □

4.2. **Branching.** A closed subgroup $G \subseteq [S_d]^\infty$ is called a *weakly branch group* if G acts transitively on every level of the tree T_d^∞ and G contains an infinite closed normal subgroup K such that $K^d \subseteq G$, in which case we say that G is *weakly branched over* K. If K has finite index in G, then we say that G is a *branch group* and that G *is branched over* K. Whenever a subgroup K of G has the property that K^d is also a subgroup of G, we say that K *branches in* G. Weakly branch and branch groups are an important, natural class of just infinite groups; see Grigorchuk [Gri00], Bartholdi, Grigorchuk, and Šuniḱ [BGŠ03], or Nekrashevych [Nek05] (note that some sources say "regular branch" where we use just "branch").

We show $\mathcal{A}$ is weakly branch in Lemma 4.6, and in in Proposition 4.14 we prove that $\mathcal{B}$ is branch for all but one choice of the defining parameters. This has interesting arithmetic implications; for example, in Proposition 6.3 we show that the constant field extension of $K(f^{-\infty}(t))/K(t)$ is finite whenever $\mathcal{B}$ is branch.

Lemma 4.6. *The group $\mathcal{A}$ is weakly branch over $\widehat{\mathcal{A}}_n$. Furthermore, $\widehat{\mathcal{A}}_i \cap \langle\langle a_i \rangle\rangle = 1$ for $1 \leq i \leq n$ and $\mathrm{St}_1\, \mathcal{A} = \langle\langle a_1^d \rangle\rangle \widehat{\mathcal{A}}_n^d$.*

Proof. The group $\widehat{\mathcal{A}}_1$ is topologically generated by all conjugates of a_i with $2 \le i \le n$. If $2 \le i \le n$, then every $\mathcal{A}$-conjugate of $a_i = (1, \ldots, 1, a_{i-1})$ belongs to $\widehat{\mathcal{A}}_n^d$, hence $\widehat{\mathcal{A}}_1 \subseteq \widehat{\mathcal{A}}_n^d$. On the other hand, $\widehat{\mathcal{A}}_n^d$ is generated by all the $\mathcal{A}$-conjugates of elements of the form $(1, \ldots, 1, a_{i-1}, 1, \ldots, 1)$ with $2 \le i \le n$ where the a_{i-1} can be in any component. These elements are conjugates of a_i by powers of a_1, hence belong to $\widehat{\mathcal{A}}_1$. Therefore $\widehat{\mathcal{A}}_1 = \widehat{\mathcal{A}}_n^d$. Hence $\mathcal{A}$ is weakly branch over $\widehat{\mathcal{A}}_n$.

By Proposition 2.8, it suffices to show that $\widehat{\mathcal{A}}_i \cap \langle\langle a_i \rangle\rangle =_\ell 1$ for each $1 \le i \le n$ and for all $\ell \ge 0$. We proceed by induction on ℓ. The $\ell = 0$ case is trivial, so suppose that $\ell \ge 1$ and that $\widehat{\mathcal{A}}_i \cap \langle\langle a_i \rangle\rangle =_{\ell-1} 1$ for each $1 \le i \le n$. Then $\widehat{\mathcal{A}}_n \cap \langle\langle a_n \rangle\rangle =_{\ell-1} 1$ and

$$\mathrm{St}_1 \mathcal{A} \cap \langle\langle a_1 \rangle\rangle = \langle\langle a_1^d \rangle\rangle = \langle\langle (a_n, \ldots, a_n) \rangle\rangle$$

imply that

$$\widehat{\mathcal{A}}_1 \cap \langle\langle a_1 \rangle\rangle \subseteq \widehat{\mathcal{A}}_n^d \cap \langle\langle (a_n, \ldots, a_n) \rangle\rangle \subseteq (\widehat{\mathcal{A}}_n \cap \langle\langle a_n \rangle\rangle)^d =_\ell 1.$$

Next suppose that $2 \le i \le n$. Observe that $a_i = (1, \ldots, 1, a_{i-1}) \in 1^{d-1} \times \langle\langle a_{i-1} \rangle\rangle$ and

$$\widehat{\mathcal{A}}_i \cap (1^{d-1} \times \mathcal{A}) \subseteq 1^{d-1} \times \widehat{\mathcal{A}}_{i-1}.$$

Thus

$$\begin{aligned}
\widehat{\mathcal{A}}_i \cap \langle\langle a_i \rangle\rangle &\subseteq \widehat{\mathcal{A}}_i \cap (1^{d-1} \times \mathcal{A}) \cap \langle\langle a_i \rangle\rangle \\
&\subseteq (1^{d-1} \times \widehat{\mathcal{A}}_{i-1}) \cap \langle\langle (1, \ldots, 1, a_{i-1}) \rangle\rangle \\
&\subseteq 1^{d-1} \times (\widehat{\mathcal{A}}_{i-1} \cap \langle\langle a_{i-1} \rangle\rangle) \\
&=_\ell 1.
\end{aligned}$$

This completes our induction.

The definitions of $\mathcal{A}$ and $\widehat{\mathcal{A}}_1$ imply that $\mathcal{A} = \langle\langle a_1 \rangle\rangle \widehat{\mathcal{A}}_1$. Then by $\widehat{\mathcal{A}}_1 \subseteq \mathrm{St}_1 \mathcal{A}$, $\mathrm{St}_1 \mathcal{A} \cap \langle\langle a_1 \rangle\rangle = \langle\langle a_1^d \rangle\rangle$, and (1) we have

$$\mathrm{St}_1 \mathcal{A} = \langle\langle a_1^d \rangle\rangle \widehat{\mathcal{A}}_1 = \langle\langle a_1^d \rangle\rangle \widehat{\mathcal{A}}_n^d. \qquad \square$$

Lemma 4.6 allows us to construct a useful family of abelian quotients of $\mathcal{A}$.

Proposition 4.7. *For each $1 \le i \le n$, there is a continuous surjection $\eta_i : \mathcal{A} \to \mathbb{Z}_d$ such that $\ker(\eta_i) = \widehat{\mathcal{A}}_i$ and*

$$\eta_i(a_j) = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. The quotient $\mathcal{A}/\widehat{\mathcal{A}}_i$ is topologically generated by the image of a_i, hence Lemma 4.6(2) and Proposition 4.5 together imply that $\mathcal{A}/\widehat{\mathcal{A}}_i \cong \langle\langle a_i \rangle\rangle \cong \mathbb{Z}_d$. We define $\eta_i : \mathcal{A} \to \mathbb{Z}_d$ as the composition

$$\eta_i : \mathcal{A} \longrightarrow \mathcal{A}/\widehat{\mathcal{A}}_i \cong \mathbb{Z}_d.$$

Therefore $\ker(\eta_i) = \widehat{\mathcal{A}}_i$. Since $a_j \in \widehat{\mathcal{A}}_i$ for $i \ne j$ and the isomorphism $\langle\langle a_i \rangle\rangle \cong \mathbb{Z}_d$ maps $a_i \mapsto 1$ by construction, we conclude that

$$\eta_i(a_j) = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{otherwise.} \end{cases} \qquad \square$$

The maps η_i and χ_i combine to give all the abelian quotients of $\mathcal{A}$ and $\mathcal{B}$, respectively.

Proposition 4.8 (Abelianizations)**.**

(1) *The map $\eta : \mathcal{A} \to \mathbb{Z}_d^n$ defined by*

$$\eta(a) := (\eta_1(a), \ldots, \eta_n(a))$$

is a surjective homomorphism which induces an isomorphism $\mathcal{A}^{\mathrm{ab}} \cong \mathbb{Z}_d^n$.

(2) The map $\chi : \mathcal{B} \to (\mathbb{Z}/d\mathbb{Z})^n$ defined by

$$\chi(b) := (\chi_1(b), \ldots, \chi_n(b))$$

is a surjective homomorphism which induces an isomorphism $\mathcal{B}^{\mathrm{ab}} \cong (\mathbb{Z}/d\mathbb{Z})^n$.

Proof. (1) Proposition 4.7 implies that the generators a_i map under η to the standard basis in $\mathbb{Z}_d^n$, hence η is a surjective homomorphism Since $\mathbb{Z}_d^n$ is abelian, η factors through the abelianization, and since the a_i topologically generate $\mathcal{A}$ and $\langle\langle a_i \rangle\rangle \cong \mathbb{Z}_d$, the abelianization factors through $\mathbb{Z}_d^n$. Hence η induces an isomorphism $\mathcal{A}^{\mathrm{ab}} \cong \mathbb{Z}_d^n$.

(2) If $1 \leq i \leq n$, then Lemma 4.4 implies that the generators b_i map under χ to the standard basis in $(\mathbb{Z}/d\mathbb{Z})^n$. Thus χ is a surjective homomorphism. On the other hand, since $\mathcal{B}$ is generated by the b_i which each have order d by Proposition 4.5(2), it follows that $\mathcal{B}^{\mathrm{ab}}$ has order at most d^n, hence χ induces an isomorphism $\mathcal{B}^{\mathrm{ab}} \cong (\mathbb{Z}/d\mathbb{Z})^n$. □

Next we turn to the group $\mathcal{B}$. Our analysis naturally splits along several cases. Consider the hypotheses,

- (A_1) $\omega \neq d/2$,
- (A_2) $m > 1$ and $(d, n) \neq (2, m+1)$,
- (A_3) $d = 2$, $m > 2$, and $n = m + 1$,
- (B_1) $d > 2$, $m = 1$, and $\omega = d/2$,
- (B_2) $d = 2$, $m = 1$, and $n > 2$,
- (C) $d = 2$, $m = 2$, and $n = 3$.
- (D) $d = 2$, $m = 1$, and $n = 2$.

We let (A) refer to the assumption (A_1),(A_2), or (A_3), and let (B) refer to (B_1) or (B_2). Note that these hypotheses exhaust all possible cases with $d \geq 2$, $1 \leq m < n$, and $1 \leq \omega < d$. When $d = 2$, we have $\omega = d/2 = 1$ by default. Case (D) is exceptional; it corresponds, up to conjugacy, to the Chebyshev polynomial of degree 2.

According to these cases, we define a subgroup $\mathcal{N}$ over which $\mathcal{B}$ is regular branch in all cases except (D). We start by observing that a natural subgroup branches.

Lemma 4.9. *The subgroup $\widehat{\mathcal{B}}_{m,n}$ branches, meaning $(\widehat{\mathcal{B}}_{m,n})^d = \widehat{\mathcal{B}}_{1,m+1} \subseteq \mathcal{B}$.*

Proof. The group $\widehat{\mathcal{B}}_{1,m+1}$ is generated by all conjugates of $b_i = (1, \ldots, 1, b_{i-1})$ with $i \neq 1, m+1$. Since $\sigma = b_1 \in \mathcal{B}$, it follows that $(\widehat{\mathcal{B}}_{m,n})^d = \widehat{\mathcal{B}}_{1,m+1} \subseteq \mathcal{B}$. □

However, the subgroup $\widehat{\mathcal{B}}_{m,n}$ is not always finite index, nor is it the maximal normal subgroup with this property. We define $\mathcal{N}$ as an extension of $\widehat{\mathcal{B}}_{m,n}$ by cases.

Definition 4.10. Let $\mathcal{N} \subseteq \mathcal{B}$ be the closed normal subgroup defined in cases as follows:

1. If (A), then $\mathcal{N}$ is the closed normal subgroup generated by $[b_m, b_n]$ and $\widehat{\mathcal{B}}_{m,n}$.
2. If (B) or (C), then $\mathcal{N}$ is the closed normal subgroup generated by $\widehat{\mathcal{B}}_{m,n}$, $[b_m, [b_m, b_n]]$, and $[b_n, [b_m, b_n]]$.
3. If (D), then $\mathcal{N}$ is the trivial subgroup.

We determine the quotients $\mathcal{B}/\mathcal{N}$ in Lemma 4.13; in particular showing that $\mathcal{N}$ has finite index. In cases (A) and (B), the quotient $\mathcal{B}/\mathcal{N}$ is induced by a quotient of $[C_d]^\infty$. The failure of this in case (C) is what distinguishes it from (B). We show that $\mathcal{N}^d \subseteq \mathcal{B}$ in Proposition 4.14.

Definition 4.11. Let $\psi_A : [C_d]^\infty \to (\mathbb{Z}/d\mathbb{Z})^2$ be the map defined by

$$\psi_A(g) = (\chi_m(g), \chi_n(g)).$$

If $g = \sigma^i(g_1, \ldots, g_d) \in [C_d]^\infty$, then let $\chi' : [C_d]^\infty \to \mathbb{Z}/d\mathbb{Z}$ be the function defined by

$$\chi'(g) := \sum_{i=1}^{d} i\chi_{n-1}(g_i) \in \mathbb{Z}/d\mathbb{Z}.$$

Recall that $\mathcal{H}_d$ is the dth Heisenberg group (see Section 2.5). Let $\psi_B : [C_d]^\infty \to \mathcal{H}_d$ be the map defined by

$$\psi_B(g) := \sigma^{-\chi_1(g)}(\chi'(g), \chi_n(g)).$$

Lemma 4.12. *The maps ψ_A and ψ_B are surjective homomorphisms, and*

$$\mathcal{N} = \begin{cases} \ker\psi_A \cap \mathcal{B} & \text{if } (A), \\ \ker\psi_B \cap \mathcal{B} & \text{if } (B). \end{cases}$$

Proof. First suppose (A). As the direct product of homomorphisms, ψ_A is clearly a homomorphism. The elements b_m and b_n are mapped by ψ_A to generators of $(\mathbb{Z}/d\mathbb{Z})^2$, while the other b_i are in the kernel, hence $\widehat{\mathcal{B}}_{m,n} \subseteq \mathcal{N}$. Since $(\mathbb{Z}/d\mathbb{Z})^2$ is abelian, it follows that $[b_m, b_n] \in \ker\psi_A$, and therefore $\mathcal{N} \subseteq \ker\psi_A$. On the other hand, $\mathcal{N}$ clearly has index at most $d^2 = [\mathcal{B} : \ker\psi_A \cap \mathcal{B}]$, thus $\mathcal{N} = \ker\psi_A \cap \mathcal{B}$.

Next suppose (B). Although χ' is not a homomorphism on $[C_d]^\infty$, it restricts to one on $\mathrm{St}_1[C_d]^\infty$. Therefore ψ_B restricts to a homomorphism on $C_d = \langle\sigma\rangle$ and on $\mathrm{St}_1[C_d]^\infty$. Let $g := (g_1, \ldots, g_d) \in \mathrm{St}_1[C_d]^\infty$. Since $[C_d]^\infty = C_d \ltimes \mathrm{St}_1[C_d]^\infty$, to show that ψ_B is a homomorphism on $[C_d]^\infty$, it suffices to show that

$$\psi_B(\sigma)^{-1}\psi_B(g)\psi_B(\sigma) = \psi_B(\sigma^{-1}g\sigma).$$

The left hand side simplifies to

$$\psi_B(\sigma)^{-1}\psi_B(g)\psi_B(\sigma) = \sigma(\chi'(g), \chi_n(g))\sigma^{-1} = (\chi'(g) - \chi_n(g), \chi_n(g)).$$

While the right hand side is

$$\psi_B(\sigma^{-1}g\sigma) = \psi_B(g_2, \ldots, g_d, g_1) = \Big(\sum_{i=1}^{d} i\chi_{n-1}(g_{i+1}), \chi_n(\sigma^{-1}g\sigma)\Big) = \Big(\sum_{i=1}^{d} i\chi_{n-1}(g_{i+1}), \chi_n(g)\Big),$$

where

$$\begin{aligned}\sum_{i=1}^{d} i\chi_{n-1}(g_{i+1}) &= \sum_{i=1}^{d}(i+1)\chi_{n-1}(g_{i+1}) - \chi_{n-1}(g_{i+1}) \\ &= \Big(\sum_{i=1}^{d}(i+1)\chi_{n-1}(g_{i+1})\Big) - \Big(\sum_{i=1}^{d}\chi_{n-1}(g_{i+1})\Big) \\ &= \chi'(g) - \chi_n(g).\end{aligned}$$

Therefore $\psi_B : [C_d]^\infty \to \mathcal{H}_d$ is a homomorphism.

In case (B), we have $m = 1$, so $b_m = b_1 = \sigma$, and $b_n = (1, \ldots, 1, b_{n-1})$. The definition of ψ_B implies that $\psi_B(b_n) = (0, 1)$, which, in combination with $\psi_B(\sigma^{-1}) = \sigma$, generates $\mathcal{H}_d$, so ψ_B is surjective. The definition of ψ_B implies that $b_i \in \ker\psi_B$ for all $i \neq m, n$, hence $\widehat{\mathcal{B}}_{m,n} \subseteq \ker\psi_B$. Observe that

$$\psi_B([b_m, b_n]) = [\sigma^{-1}, (0, 1)] = (-1, 0),$$

which commutes with $\psi_B(b_m) = \sigma^{-1}$ and $\psi_B(b_n) = (0, 1)$. Therefore $[b_m, [b_m, b_n]]$ and $[b_n, [b_m, b_n]]$ both belong to $\ker\psi_B$ as well. Therefore $\mathcal{N} \subseteq \ker\psi_B \cap \mathcal{B}$, which is to say there is a surjective homomorphism $\mathcal{B}/\mathcal{N} \to \mathcal{B}/(\ker\psi_B \cap \mathcal{B}) \cong \mathcal{H}_d$.

On the other hand, the definition of $\mathcal{N}$ and Lemma 2.23 imply there is a surjective homomorphism $\mathcal{H}_d \to \mathcal{B}/\mathcal{N}$. Hence $\mathcal{B}/\mathcal{N} \cong \mathcal{H}_d$ and $\mathcal{N} = \ker\psi_B \cap \mathcal{B}$. ∎

Remark. We use this description of $\mathcal{N}$ as the intersection of a normal subgroup of $[C_d]^\infty$ with $\mathcal{B}$ provided by Lemma 4.12 when analyzing the normalizer of $\mathcal{B}$ in Section 5.2. However, this does not hold in case (C): if $\mathcal{N}'$ is the normal closure of $\mathcal{N}$ in $[C_2]^\infty$, then a calculation shows that $\mathcal{N}$ has index 2 in $\mathcal{N}' \cap \mathcal{B}$, whereas $\mathcal{N}$ has index 8 in $\mathcal{B}$ (see Appendix A).

With these, we are now able to show that $\mathcal{N}$ has finite index in $\mathcal{B}$,

Lemma 4.13. *The quotients $\mathcal{B}/\mathcal{N}$ are as follows:*
- *(1) If (A), then $\mathcal{B}/\mathcal{N} \cong (\mathbb{Z}/d\mathbb{Z})^2$,*
- *(2) If (B) or (C), then $\mathcal{B}/\mathcal{N} \cong \mathcal{H}_d$,*
- *(3) If (D), then $\mathcal{B} \cong \mathcal{B}/\mathcal{N} \cong \widehat{D}_\infty$, the pro-2 dihedral group.*

Proof. The (A) and (B) cases follow immediately from Lemma 4.12. In case (C), the definition of $\mathcal{N}$ and Lemma 2.23 imply there is a surjective homomorphism $\mathcal{H}_2 \to \mathcal{B}/\mathcal{N}$. On the other hand, $[\mathcal{B} : \mathcal{N}] \geq [\mathcal{B} : \mathcal{N}]_4$ and a computer calculation (see Appendix A) shows that $[\mathcal{B} : \mathcal{N}]_4 = 8 = |\mathcal{H}_2|$. Thus this surjection is an isomorphism.

Finally suppose (D), in which case $n = 2$ so $\mathcal{B} = \langle\langle b_1, b_2\rangle\rangle$ and $\mathcal{N}$ is trivial. Both b_1 and b_2 have order 2 by Proposition 4.5, while the identity $(b_1b_2)^2 = (b_1b_2, b_2b_1)$ implies that b_1b_2 is an odometer, hence $\langle\langle b_1b_2\rangle\rangle \cong \mathbb{Z}_2$. Therefore $\mathcal{B} \cong \mathcal{B}/\mathcal{N}$ is isomorphic to the pro-2 dihedral group $\widehat{D}_\infty$. □

Proposition 4.14. *In cases (A), (B), and (C), $\mathcal{B}$ is branch over $\mathcal{N}$.*

Proof. We proved in Lemma 4.13 that $\mathcal{N}$ has finite index in cases (A), (B), and (C). Hence it suffices to prove that $\mathcal{N}^d \subseteq \mathcal{B}$.

Recall that $b_{i,j} := \sigma^j b_i \sigma^{-j}$. If (A_1), then $\omega \neq d/2$ which implies $g := [b_{m+1,\omega}, b_{m+1}] \in \mathcal{B}$ is supported in only the ωth component. Since $\pi_\omega(g) = [b_m, b_n]$ and $\mathcal{N} = [b_m, b_n]\widehat{\mathcal{B}}_{m,n}$, we conclude that $\mathcal{N}^d \subseteq \mathcal{B}$.

If (A_2), then $m > 1$ and $\sigma = b_1 \in \widehat{\mathcal{B}}_{m,n}$. Our assumption that $(d, n) \neq (2, m + 1)$ implies that there exists some j such that $b_{m,j}$ commutes with b_n. Therefore

$$[b_m, b_n] \equiv [b_{m,j}, b_n] \equiv 1 \bmod \widehat{\mathcal{B}}_{m,n},$$

or equivalently $[b_m, b_n] \in \widehat{\mathcal{B}}_{m,n}$. This implies that $\mathcal{N} = \widehat{\mathcal{B}}_{m,n}$ in case (A_2). Therefore (1) implies that $\mathcal{N}^d \subseteq \mathcal{B}$.

If (A_3), then $b_2 = (1, \sigma) \in \widehat{\mathcal{B}}_{m,n}$ and both b_{m-1} and b_{n-1} are supported in exactly one coordinate. Thus

$$[b_m, b_n] = (1, [b_{m-1}, b_{n-1}]) \equiv (1, [b_{m-1,1}, b_{n-1}]) \equiv 1 \bmod \widehat{\mathcal{B}}_{m,n}.$$

Hence in this case we also have $\mathcal{N} = \widehat{\mathcal{B}}_{m,n}$ and therefore $\mathcal{N}^d \subseteq \mathcal{B}$.

Next suppose (B_1): $d > 2$, $\omega = d/2$, and $m = 1$. Observe that

$$[b_n, [b_m, b_n]] = [b_n, b_{n,-1}^{-1} b_n] = [b_n, b_{n,-1}^{-1}]^{b_n}.$$

Since $\omega \neq d - 1$, we have $[b_n, b_{n,-1}^{-1}] = 1$. Therefore, $[b_n, [b_m, b_n]] = 1$. Consider the element $g = [b_{m+1,\omega}, [b_{m+1,\omega}, b_{m+1}]] \in \mathcal{B}$. Note that g is supported in at most the ωth and dth coordinates. We have $\pi_\omega(g) = [b_m, [b_m, b_n]]$, and $\pi_d(g) = [b_n, [b_m, b_n]] = 1$. Therefore g is an element of $\mathcal{B}$ supported in a single coordinate with support $[b_m, [b_m, b_n]]$. Since $\mathcal{N}$ is the normal subgroup generated by $\widehat{\mathcal{B}}_{m,n}$, $[b_m, [b_m, b_n]]$ and $[b_n, [b_m, b_n]]$, (1) implies that $\mathcal{N}^d \subseteq \mathcal{B}$.

If (B_2), then $d = 2$, $m = 1$, and $n > 2$. Thus $b_n = (1, b_{n-1})$ and

$$[b_m, b_n] = [\sigma, (1, b_{n-1})] = (b_{n-1}, b_{n-1}),$$

which clearly commutes with both σ and b_n. Hence $\mathcal{N}_B = \widehat{\mathcal{B}}_{m,n}$.

Similarly in case (C) where $d = 2$, $m = 2$, and $n = 3$, we have

$$[b_2, b_3] = (1, [\sigma, b_2]) = (1, (\sigma, \sigma)),$$

which commutes with $b_2 = (1, \sigma)$ and $b_3 = (1, (1, \sigma))$. Therefore $\mathcal{N} = \widehat{\mathcal{B}}_{m,n}$ in this case as well. Thus (1) implies that $\mathcal{N}^d \subseteq \mathcal{B}$ in cases (B_2) and (C).

In cases (A), (B), and (C) the subgroup $\mathcal{N}$ has finite index by Lemma 4.13. Hence $\mathcal{B}$ is a regular branch group in each of those cases. □

4.3. **Characterizing the level one stabilizer.** We previously observed that $\mathrm{St}_1 \mathcal{A} \subseteq \mathcal{A}^d$ and $\mathrm{St}_1 \mathcal{B} \subseteq \mathcal{B}^d$ (see the remark before Lemma 4.3). In this section we characterize $\mathrm{St}_1 \mathcal{A}$ and $\mathrm{St}_1 \mathcal{B}$ as subgroups of $\mathcal{A}^d$ and $\mathcal{B}^d$, respectively. In the pre-periodic case, the branching property is essential.

Lemma 4.15. *There exists an involution $\tau : \mathcal{B}/\mathcal{N} \to \mathcal{B}/\mathcal{N}$ uniquely determined by*

$$\tau(b_m) \equiv b_n \bmod \mathcal{N}.$$

Proof. The quotient $\mathcal{B}/\mathcal{N}$ is generated by b_m and b_n, so any involution exchanging them is unique.

In case (A), observe that $\mathcal{B}/\mathcal{N} \cong (\mathbb{Z}/d\mathbb{Z})^2$ where the generators b_m and b_n are identified with $(1, 0)$ and $(0, 1)$, respectively, and exchanging them is an involution.

If (B) or (C), then $\mathcal{B}/\mathcal{N} \cong \mathcal{H}_d$ by Lemma 4.13, under which b_m and b_n are sent to the g_1 and g_2 from the presentation of Lemma 2.23. In Lemma 2.24, we observed that $\mathcal{H}_d$ has an involution exchanging g_1 and g_2, and therefore $\mathcal{B}/\mathcal{N}$ has an involution exchanging b_m and b_n.

If (D), then $\mathcal{N} = 1$ and Lemma 4.13 implies that $\mathcal{B}$ is isomorphic to the pro-2 dihedral group. There is a unique involution τ of $\mathcal{B}$ satisfying $\tau(b_1) = b_2$. □

Lemma 4.16. *If $\ell \le n$, then $\mathcal{B} =_\ell [C_d]^\ell$.*

Proof. We proceed by induction on ℓ. The base case $\ell = 0$ is trivial, so suppose that $0 < \ell \le n$ and the claim is true for $\ell - 1$. Since $b_i =_\ell 1$ if and only if $\ell < i$, it follows that each b_i is supported in at most one component in $\rho_\ell(\mathcal{B})$ and that $\mathrm{St}_1 \mathcal{B} =_\ell \mathcal{B}^d$. Our inductive hypothesis implies that $\mathcal{B} =_{\ell-1} [C_d]^{\ell-1}$, hence

$$\mathrm{St}_1 \mathcal{B} =_\ell ([C_d]^{\ell-1})^d = \mathrm{St}_1 [C_d]^\ell.$$

Since $b_1 = \sigma$, we conclude that $\mathcal{B} =_\ell [C_d]^\ell$. □

Proposition 4.17. *The level one stabilizer in the periodic case is*

$$\mathrm{St}_1 \mathcal{A} = \{(g_1, \dots, g_d) \in \mathcal{A}^d : \eta_n(g_i) = \eta_n(g_j) \text{ for all } 1 \le i, j \le n\}.$$

The level one stabilizer in the pre-periodic cases is

$$\mathrm{St}_1 \mathcal{B} = \begin{cases} \{(g_1, \dots, g_d) \in \mathcal{B}^d : \chi_m(g_i) = \chi_n(g_{i+\omega}) \text{ for all } 1 \le i \le n\} & \text{if } (A), \\ \{(g_1, \dots, g_d) \in \mathcal{B}^d : \tau(g_i) \equiv g_{i+\omega} \bmod \mathcal{N} \text{ for all } 1 \le i \le n\} & \text{if } (B), (C), \text{ or } (D). \end{cases}$$

Proof. First consider the periodic case. Let $\mathcal{S} \subseteq \mathcal{A}^d$ be the subgroup defined by

$$\mathcal{S} := \{(g_1, \dots, g_d) \in \mathcal{A}^d : \eta_n(g_i) = \eta_n(g_j) \text{ for all } 1 \le i, j \le n\}.$$

Lemma 4.6(3) implies that $\mathrm{St}_1 \mathcal{A} = \langle\langle a_1^d \rangle\rangle \widehat{\mathcal{A}}_n^d$. Since $a_1^d = (a_n, \dots, a_n)$ and $\widehat{\mathcal{A}}_n = \ker(\eta_n)$, we have $\mathrm{St}_1 \mathcal{A} \subseteq \mathcal{S}$. On the other hand, if $g := (g_1, \dots, g_d) \in \mathcal{S}$, then Lemma 4.6(2) implies that for each i we can write $g_i = a_n^{k_i} h_i$ for some $k_i \in \mathbb{Z}_d$ and some $h_i \in \widehat{\mathcal{A}}_n$. The definition of $\mathcal{S}$ implies that $k_i = \eta_n(g_i) = \eta_n(g_j) = k_j$ for all i, j. Hence $g = a_1^{dk_1}(h_1, \dots, h_d) \in \mathrm{St}_1 \mathcal{A}$. Therefore $\mathrm{St}_1 \mathcal{A} = \mathcal{S}$.

Next consider the preperiodic case. Suppose (A). Let $\mathcal{S} \subseteq \mathcal{B}^d$ be the subgroup defined by

$$\mathcal{S} := \{(g_1, \dots, g_d) \in \mathcal{B}^d : \chi_m(g_i) = \chi_n(g_{i+\omega}) \text{ for all } i\}.$$

If $i \ne 1, m+1$, then

$$b_i \in \widehat{\mathcal{B}}_{1,m+1} = \widehat{\mathcal{B}}_{m,n}^d \subseteq \mathcal{S},$$

where the equality follows from Lemma 4.9. Recall that $b_{m+1} = (1, \dots, 1, b_n, 1, \dots, 1, b_m)$ where b_n is in the ωth component. Hence $b_{m+1} \in \mathcal{S}$. The group $\mathcal{S}$ is closed under conjugation by elements of $\mathcal{B}$, thus $\mathrm{St}_1 \mathcal{B} = \widehat{\mathcal{B}}_1 \subseteq \mathcal{S}$.

Now suppose that $g = (g_1, \ldots, g_d) \in \mathcal{S}$. Consider the element

$$h := (h_1, \ldots, h_d) = b_{m+1,1}^{\chi_m(g_1)} \cdots b_{m+1,d}^{\chi_m(g_d)} \in \mathrm{St}_1 \mathcal{B}.$$

Proposition 4.14 implies that $\mathcal{B}/\mathcal{N}$ is abelian, hence

$$h_{i+\omega} \equiv b_m^{\chi_m(g_{i+\omega})} b_n^{\chi_m(g_i)} \equiv b_m^{\chi_m(g_{i+\omega})} b_n^{\chi_n(g_{i+\omega})} \equiv g_{i+\omega} \bmod \mathcal{N},$$

where the second congruence follows from $g \in \mathcal{S}$. Thus $g \equiv h \bmod \mathcal{N}^d$ and Proposition 4.14 implies $g \in \mathrm{St}_1 \mathcal{B}$. Therefore $\mathrm{St}_1 \mathcal{B} = \mathcal{S}$ in case (A).

Next suppose either (B) or (C). Let $\mathcal{S} \subseteq \mathcal{B}^d$ be the subgroup defined by

$$\mathcal{S} := \{(g_1, \ldots, g_d) \in \mathcal{B}^d : \tau(g_i) \equiv g_{i+\omega} \bmod \mathcal{N} \text{ for all } i\}.$$

Since $\widehat{\mathcal{B}}_{m,n} \subseteq \mathcal{N}$, for any $i \neq 1, m+1$,

$$b_i \in \widehat{\mathcal{B}}_{1,m+1} = \widehat{\mathcal{B}}_{m,n}^d \subseteq \mathcal{S}.$$

The definition of τ implies that

$$b_{m+1} \equiv (1, \ldots, 1, \tau(b_m), 1, \ldots, 1, b_m) \bmod \mathcal{N}^d,$$

hence that $b_{m+1} \in \mathcal{S}$. The group $\mathcal{S}$ is closed under conjugation by $\mathcal{B}$ and $\mathrm{St}_1 \mathcal{B} = \widehat{\mathcal{B}}_1$, hence $\mathrm{St}_1 \mathcal{B} \subseteq \mathcal{S}$.

For the reverse inclusion, suppose that $g = (g_1, \ldots, g_d) \in \mathcal{S}$. Lemma 4.13 and the presentation Lemma 2.23 implies that for each g_i we have

$$g_i \equiv b_m^{r_i} b_n^{s_i} [b_m, b_n]^{t_i} \bmod \mathcal{N},$$

for some unique $r_i, s_i, t_i \in \mathbb{Z}/d\mathbb{Z}$. Since $g \in \mathcal{S}$, it follows that

$$g_{i+\omega} \equiv \tau(g_i) \equiv b_n^{r_i} b_m^{s_i} [b_n, b_m]^{t_i} \bmod \mathcal{N}.$$

Recall that $\omega = d/2$ in cases (B) and (C), hence $\pi_i(b_{m+1,i}) = b_m$ and $\pi_i(b_{m+1,i+\omega}) = b_n$. For $1 \leq i \leq \omega$, let

$$h_i' := b_{m+1,i}^{r_i} b_{m+1,i+\omega}^{s_i} [b_{m+1,i}, b_{m+1,i+\omega}]^{t_i} \in \mathcal{B}.$$

Then h_i' is supported in the ith and $(i+\omega)$th coordinates and

$$\pi_i(h_i') = b_m^{r_i} b_n^{s_i} [b_m, b_n]^{t_i} \equiv g_i \bmod \mathcal{N}$$

$$\pi_{i+\omega}(h_i') = b_n^{r_i} b_m^{s_i} [b_n, b_m]^{t_i} \equiv g_{i+\omega} \bmod \mathcal{N}.$$

Let $h = (h_1, \ldots, h_d) := h_1' \cdots h_\omega' \in \mathcal{B}$. The above congruences imply that $g \equiv h \bmod \mathcal{N}^d$, hence Proposition 4.14 implies that $g \in \mathrm{St}_1 \mathcal{B}$. Therefore $\mathrm{St}_1 \mathcal{B} = \mathcal{S}$.

Finally, suppose (D). Lemma 4.13 implies that $\mathcal{N} = 1$ and $\mathcal{B}$ is pro-2 dihedral. Recall that $b_\infty := b_2 b_1$. Hence every element of $\mathcal{B}$ may be uniquely expressed as $b_\infty^\varepsilon \sigma^i$ for some $\varepsilon \in \mathbb{Z}_2$ and $i \in \mathbb{Z}/2\mathbb{Z}$. Let

$$\mathcal{S} := \{(g, \tau(g)) \in \mathcal{B}^2\}.$$

The stabilizer $\mathrm{St}_1 \mathcal{B}$ is topologically generated by $b_\infty^2 = (b_\infty, b_\infty^{-1})$ and $b_\infty \sigma = b_2 = (b_2, b_1)$, both of which belong to $\mathcal{S}$. Hence $\mathrm{St}_1 \mathcal{B} \subseteq \mathcal{S}$. On the other hand,

$$(b_\infty^\varepsilon, \tau(b_\infty^\varepsilon)) = (b_\infty^\varepsilon, b_\infty^{-\varepsilon}) = b_\infty^{2\varepsilon} \in \mathcal{B},$$

and

$$(b_\infty^\varepsilon \sigma, \tau(b_\infty^\varepsilon \sigma)) = (b_\infty^\varepsilon \sigma, b_\infty^{-\varepsilon} b_2) = b_\infty^{2\varepsilon - 1} \sigma \in \mathcal{B}.$$

Thus $\mathrm{St}_1 \mathcal{B} = \mathcal{S}$. □

In Proposition 4.14 we showed that $\mathcal{N}^d \subseteq \mathcal{B}$. Since $\mathcal{N}$ is a normal subgroup of $\mathcal{B}$, it follows that $\mathcal{N}^d$ is also normal in $\mathcal{B}$. We identify the quotients $\mathcal{B}/\mathcal{N}^d$ in Lemma 4.18; the index of $\mathcal{N}^d$ is essential for our calculation of the orders of finite level truncations of $\mathcal{B}$.

Lemma 4.18. *The quotients $\mathcal{B}/\mathcal{N}^d$ are as follows*

(1) *If (A), then $\mathcal{B}/\mathcal{N}^d \cong [C_d]^2$.*

(2) *If (B) or (C), then $\mathcal{B}/\mathcal{N}^d \cong \langle\sigma\rangle \ltimes (\mathcal{B}/\mathcal{N})^\omega$ where σ acts on $(\mathcal{B}/\mathcal{N})^\omega$ by*

$$\sigma^{-1}(g_1,\ldots,g_\omega)\sigma := (g_2,\ldots,g_\omega,\tau(g_1)).$$

Hence

$$[\mathcal{B}:\mathcal{N}^d] = \begin{cases} d^{d+1} & \text{if } (A), \\ d^{3d/2+1} & \text{if } (B) \text{ or } (C). \end{cases}$$

Proof. Lemma 4.9 says $\widehat{\mathcal{B}}_{1,m+1} = \widehat{\mathcal{B}}_{m,n}$ while Proposition 4.14 implies that $(\widehat{\mathcal{B}}_{m,n})^d \subseteq \mathcal{N}^d$. Hence $\mathcal{B}/\mathcal{N}^d$ factors through $\mathcal{B}/\widehat{\mathcal{B}}_{1,m+1}$ and hence is generated by $b_1 = \sigma$ and b_{m+1}. Furthermore,

$$\mathrm{St}_1\,\mathcal{B}/\mathcal{N}^d \subseteq \mathcal{B}^d/\mathcal{N}^d \cong (\mathcal{B}/\mathcal{N})^d.$$

(1) Suppose (A). Since $\mathcal{B}/\mathcal{N}$ is abelian, it follows that $b_{m+1,i}$ and $b_{m+1,j}$ commute modulo $\mathcal{B}/\mathcal{N}^d$ for all i and j. Thus $\mathcal{B}/\mathcal{N}^d$ is a quotient of $[C_d]^2$. On the other hand, suppose that

$$\sigma^j(g_1,\ldots,g_d) := \sigma^j b_{m+1,1}^{e_1}\cdots b_{m+1,d}^{e_d}.$$

Then $\chi_m(g_i) = e_i$, which by Lemma 4.13(1) implies that these d^{d+1} elements are all distinct in $\mathcal{B}/\mathcal{N}^d$. Therefore $\mathcal{B}/\mathcal{N}^d \cong [C_d]^2$ and $[\mathcal{B}:\mathcal{N}^d] = d^{d+1}$ in case (A).

(2) Suppose (B) or (C). Proposition 4.17(2) implies that $\sigma^j(g_1,\ldots,g_d) \in \mathcal{B}$ if and only if $\tau(g_i) \equiv g_{i+\omega} \bmod \mathcal{N}$ for all i. Hence the map $\sigma^j(g_1,\ldots,g_d) \mapsto \sigma^j(g_1,\ldots,g_\omega)$ defines a surjective homomorphism from $\mathcal{B}$ onto $\langle\sigma\rangle \ltimes (\mathcal{B}/\mathcal{N})^\omega$ with kernel $\mathcal{N}^d$. Since $|\mathcal{B}/\mathcal{N}| = d^3$ by Lemma 4.13(2) and $\omega = d/2$ in cases (B) and (C), we conclude that $[\mathcal{B}:\mathcal{N}^d] = d^{3d/2+1}$. □

4.4. **Semirigidity.** The following lemma shows that the generators of $\mathcal{A}$ and $\mathcal{B}$ satisfy a weak form of rigidity in the sense of the inverse Galois problem: if our distinguished generators are replaced by arbitrary conjugates, they still generate the group.

Lemma 4.19. *If $u_i \in \mathcal{A}$ and $v_i \in \mathcal{B}$ for $1 \le i \le n$ are arbitrary elements, then*

$$\mathcal{A} = \langle\langle u_1a_1u_1^{-1},\ldots,u_na_nu_n^{-1}\rangle\rangle,$$
$$\mathcal{B} = \langle\langle v_1b_1v_1^{-1},\ldots,v_nb_nv_n^{-1}\rangle\rangle.$$

Proof. First consider $\mathcal{A}$. Let $a_i' := u_ia_iu_i^{-1}$ and define $\mathcal{A}' := \langle\langle a_1',\ldots,a_n'\rangle\rangle \subseteq \mathcal{A}$. We claim that $\mathcal{A}' = \mathcal{A}$. The groups $\mathcal{A}$ and $\mathcal{A}'$ are both closed, hence it suffices by Proposition 2.8 to prove that $\mathcal{A}' =_\ell \mathcal{A}$ for all $\ell \ge 0$; we proceed by induction on ℓ.

The base case $\ell = 0$ is immediate since $\mathcal{A}$ and $\mathcal{A}'$ are both trivial at level $\ell = 0$. Now suppose that $\ell \ge 1$ and $\mathcal{A}' =_{\ell-1} \mathcal{A}$. Since $u\mathcal{A}u^{-1} = \mathcal{A}$ for all $u \in \mathcal{A}$, we may conjugate everything by u_1^{-1} to assume without loss of generality that $a_1 = a_1' \in \mathcal{A}'$ and $u_1 = 1$. Thus it suffices to show that $a_i \in_\ell \mathcal{A}'$ for all $2 \le i \le n$.

If $1 \le i < n$, then after potentially replacing a_{i+1}' with a conjugate by a power of a_1', there is some element $w_i \in \mathcal{A}$ such that

$$a_{i+1}' = u_{i+1}a_{i+1}u_{i+1}^{-1} = (1,\ldots,1,w_ia_iw_i^{-1}).$$

We also have

$$a_1'^d = a_1^d = (a_n,\ldots,a_n).$$

Let $\pi_d : \mathrm{St}_1\,\mathcal{A}' \to \mathcal{A}$ denote projection onto the dth coordinate. The above observations imply that $\pi_d(\mathrm{St}_1\,\mathcal{A}')$ is a subgroup of $\mathcal{A}$ containing $\mathcal{A}$-conjugates of each of the a_i, hence $\pi_d(\mathrm{St}_1\,\mathcal{A}') =_{\ell-1} \mathcal{A}$ by our inductive hypothesis. Thus for $2 \le i \le n$ there exists elements $w_i' \in \mathrm{St}_1\,\mathcal{A}'$ such that $\pi_d(w_i') =_{\ell-1} w_{i-1}^{-1}$, hence

$$a_i = (1,\ldots,1,a_{i-1}) =_\ell w_i'a_i'w_i'^{-1} \in \mathcal{A}'.$$

That completes the induction, hence proves that $\mathcal{A} = \langle\langle u_1 a_1 u_1^{-1}, \ldots, u_n a_n u_n^{-1} \rangle\rangle$.

Next consider $\mathcal{B}$. Let $b_i' := v_i b_i v_i^{-1}$ and define $\mathcal{B}' := \langle\langle b_1', \ldots, b_n' \rangle\rangle \subseteq \mathcal{B}$. We claim that $\mathcal{B}' = \mathcal{B}$. The groups $\mathcal{B}$ and $\mathcal{B}'$ are both closed, hence it suffices by Proposition 2.8 to prove that $\mathcal{B}' =_\ell \mathcal{B}$ for all $\ell \geq 0$; we proceed by induction on ℓ.

The base case $\ell = 0$ is again immediate; suppose that $\ell \geq 1$ and $\mathcal{B}' =_{\ell-1} \mathcal{B}$. Since $v\mathcal{B}v^{-1} = \mathcal{B}$ for all $v \in \mathcal{V}$, we may conjugate everything by v_1^{-1} to assume without loss of generality that $\sigma = b_1 = b_1' \in \mathcal{B}'$ and $v_1 = 1$. Thus it suffices to show that $b_i \in_\ell \mathcal{B}'$ for all $2 \leq i \leq n$.

If $1 \leq i < n$ and $i \neq m$, then after potentially replacing b_{i+1}' with a conjugate by a power of σ, there is some element $w_i \in \mathcal{B}$ such that

$$b_{i+1}' = v_{i+1} b_{i+1} v_{i+1}^{-1} = (1, \ldots, 1, w_i b_i w_i^{-1}).$$

Similarly after potentially replacing b_{m+1}' with a conjugate by a power of σ, there are elements $w_m, w_n \in \mathcal{B}$ such that

$$b_{m+1}' = v_{m+1} b_{m+1} v_{m+1}^{-1} = (1, \ldots, 1, w_n b_n w_n^{-1}, 1, \ldots, 1, w_m b_m w_m^{-1}).$$

Conjugating by powers of σ gives us all cyclic shifts of these elements in $\mathcal{B}'$. It follows that $\pi_d(\mathrm{St}_1 \mathcal{B}')$ contains $\mathcal{B}$-conjugates of b_i for each $1 \leq i \leq n$. Thus $\pi_d(\mathrm{St}_1 \mathcal{B}') =_{\ell-1} \mathcal{B}$ by our inductive hypothesis.

If $i \neq 1, m+1$ and $v \in \mathcal{B}$, then $\pi_d(\mathrm{St}_1 \mathcal{B}') =_{\ell-1} \mathcal{B}$ implies that there exists a $w_i' \in \mathrm{St}_1 \mathcal{B}'$ such that $\pi_d(w_i') =_{\ell-1} v w_{i-1}^{-1}$. Hence

$$w_i' b_i' w_i'^{-1} =_\ell (1, \ldots, 1, v b_{i-1} v^{-1}) \in_\ell \mathcal{B}'$$

and we conclude that $\widehat{\mathcal{B}}_{1,m+1} \subseteq_\ell \mathcal{B}'$. Observe that

$$\begin{aligned} b_{m+1}' &= v_{m+1} b_{m+1} v_{m+1}^{-1} \\ &= (1, \ldots, 1, w_n b_n w_n^{-1}, 1, \ldots, 1, w_m b_m w_m^{-1}) \\ &= (1 \ldots, 1, [w_n, b_n] b_n, 1, \ldots, 1, [w_m, b_m] b_m) \\ &= (1, \ldots, 1, [w_n, b_n], 1, \ldots, 1, [w_m, b_m]) b_{m+1} \end{aligned}$$

Without loss of generality we may assume that $v_{m+1} \in \widehat{\mathcal{B}}_{m+1}$. Since $\widehat{\mathcal{B}}_{m+1} = (\widehat{\mathcal{B}}_{m,n})^d$, it follows that $w_m, w_n \in \widehat{\mathcal{B}}_{m,n}$. Note that $[w_m, b_m]$ and $[w_n, b_n]$ belong to $\widehat{\mathcal{B}}_{m,n}$ since $\widehat{\mathcal{B}}_{m,n}$ is a normal subgroup of $\mathcal{B}$. Therefore $b_{m+1} \in_\ell \mathcal{B}'$. This completes the induction and proves that $\mathcal{B} = \langle\langle v_1 b_1 v_1^{-1}, \ldots, v_n b_n v_n^{-1} \rangle\rangle$. □

Remark. When $d = p$ is prime, the groups $\mathcal{A}, \mathcal{A}', \mathcal{B}, \mathcal{B}'$ from the proof of Lemma 4.19 are pro-p groups. In that case, $\mathcal{A}' = \mathcal{A}$ and $\mathcal{B}' = \mathcal{B}$ follow from observing that $\mathcal{A}'$ and $\mathcal{B}'$ surject onto $\mathcal{A}^{ab}$ and $\mathcal{B}^{ab}$, combined with the observation that the Frattini subgroup of a p-group contains the commutator; semirigidity holds for all pro-p groups. This is the approach taken by Pink when analyzing the $d = 2$ case (see [Pin13c, Lem. 1.3.2]).

We now prove the first main result of this section. This result generalizes the main assertion of [Pin13c, Thm. 2.4.1] in the case of a periodic critical point.

Theorem 4.20 (Semirigidity of $\mathcal{A}$)**.** *If $a_i' \in [C_d]^\infty$ are elements such that $a_i' \sim a_i'$ in $[C_d]^\infty$ for $1 \leq i \leq n$, then there exists an element $w \in [C_d]^\infty$ and elements $u_i \in \mathcal{A}$ such that for each i,*

$$w a_i' w^{-1} = u_i a_i u_i^{-1}.$$

In particular, $w\langle\langle a_1', \ldots, a_n' \rangle\rangle w^{-1} = \mathcal{A}$.

Proof. For $\ell \geq 0$ let $(*_\ell)$ be the claim,

> $(*_\ell)$: If $a_i' \in [C_d]^\infty$ are elements such that $a_i' \sim_\ell a_i$ in $[C_d]^\infty$ for $1 \leq i \leq n$, then there exists an element $w \in [C_d]^\infty$ and elements $u_i \in \mathcal{A}$ such that $w a_i' w^{-1} =_\ell u_i a_i u_i^{-1}$ for each i.

If $(*_\ell)$, then Proposition 2.8 implies that proving $(*_\ell)$ for all $\ell \geq 0$ furnishes u_i and w as in the statement of the theorem, while Lemma 4.19 implies that $\mathcal{A} =_\ell \langle\langle u_1 a_1 u_1^{-1}, \ldots, u_n a_n u_n^{-1} \rangle\rangle$. So it suffices to verify $(*_\ell)$, which we do by induction on ℓ. The base case is immediate.

Suppose that $\ell \geq 1$ and that $(*_{\ell-1})$ holds.

Simultaneously conjugating all of the a_i' we may assume without loss of generality that $a_1 = a_1'$. If $2 \leq i \leq n$, then any conjugate of $a_i = (1, \ldots, 1, a_{i-1})$ has a single non-trivial component. Hence there is an integer j_i such that

$$a_1^{j_i} a_i' a_1^{-j_i} =_\ell (1, \ldots, 1, a_{i-1}'')$$

for some element $a_{i-1}'' \in [C_d]^\infty$. Proposition 2.2 implies that $a_{i-1} \sim_{\ell-1} a_{i-1}''$ in $[C_d]^\infty$. Let $a_n'' := a_n$, so that $a_i \sim_{\ell-1} a_i''$ for all $1 \leq i \leq n$. Then $(*_{\ell-1})$ implies there is an element $w' \in [C_d]^\infty$ and elements $u_i' \in \mathcal{A}$ such that for each i,

$$w' a_i'' w'^{-1} =_{\ell-1} u_i' a_i u_i'^{-1}$$

Define $w := (u_n'^{-1} w', \ldots, u_n'^{-1} w') \in [C_d]^\infty$; note that w commutes with σ. Let $u_1 = 1$, then

$$\begin{aligned} w a_1' w^{-1} &= w a_1 w^{-1} \\ &= \sigma w (1, \ldots, 1, a_n) w^{-1} \\ &= \sigma (1, \ldots, 1, u_n'^{-1} (w' a_n'' w'^{-1}) u_n') \\ &=_\ell \sigma (1, \ldots, 1, a_n) \\ &= u_1 a_1 u_1^{-1}. \end{aligned}$$

In particular, w commutes with a_1.

Next suppose that $2 \leq i \leq n$. Since $u_n'^{-1} u_{i-1}' \in \mathcal{A}$, Lemma 4.3 implies there is a $v_i' \in \mathrm{St}_1 \mathcal{A}$ such that $\pi_d(v_i) = u_n'^{-1} u_{i-1}'$. Therefore,

$$\begin{aligned} w(a_1^{j_i} a_i' a_1^{-j_i}) w^{-1} &=_\ell w(1, \ldots, 1, a_{i-1}'') w^{-1} \\ &= (1, \ldots, 1, u_n'^{-1} w' a_{i-1}'' w'^{-1} u_n') \\ &=_\ell (1, \ldots, 1, u_n'^{-1} u_{i-1}' a_{i-1} u_{i-1}'^{-1} u_n') \\ &= v_i (1, \ldots, 1, a_{i-1}) v_i^{-1} \\ &= v_i a_i v_i^{-1}. \end{aligned}$$

Let $u_i := a_1^{-j_i} v_i$. Since a_1 commutes with w, the above identity is equivalent to

$$w a_i' w^{-1} =_\ell u_i a_i u_i^{-1}.$$

Then $u_i \in \mathcal{A}$ for all i and $w a_i' w^{-1} =_\ell u_i a_i u_i^{-1}$ which completes our induction. □

Next we prove the analog of Theorem 4.20 for the group $\mathcal{B}$. First, a technical lemma.

Lemma 4.21. *If $u_1, u_2 \in \mathcal{B}$, then there exists an element $v \in \mathcal{B}$ and $i, j \in \mathbb{Z}/d\mathbb{Z}$ such that $(1, \ldots, 1, v u_1 b_n^i, 1, \ldots, 1, v u_2 b_m^j) \in \mathcal{B}$, where the $v u_1 b_n^i$ term is in the ωth component.*

Proof. We proceed by cases. First suppose (A). Then $(1, \ldots, 1, v u_1 b_n^i, 1, \ldots, 1, v u_2 b_m^j) \in \mathcal{B}$ is equivalent, by Proposition 4.17, to

$$\chi_n(v) + \chi_n(u_1) + i = \chi_m(v) + \chi_m(u_2) + j,$$

and either

$$\chi_m(v) + \chi_m(u_1) = \begin{cases} \chi_n(v) + \chi_n(u_2) & \text{if } \omega = d/2, \\ 0 & \text{if } \omega \neq d/2. \end{cases}$$

Either system of equations has a solution for $i, j \in \mathbb{Z}/d\mathbb{Z}$ and $v \in \mathcal{B}$.

Next suppose (B), (C), or (D). Then $(1,\ldots,1,vu_1b_n^i,1,\ldots,1,vu_2b_m^j) \in \mathcal{B}$ is equivalent, by Proposition 4.17, to

$$\tau(vu_1b_n^i) \equiv vu_2b_m^j \bmod \mathcal{N}. \tag{6}$$

Let $w_1 := vu_1$ and $w_2 := u_1^{-1}u_2$, so that (6) becomes

$$\tau(w_1)b_m^i \equiv \tau(w_1b_n^i) \equiv w_1w_2b_m^j \bmod \mathcal{N},$$

or equivalently,

$$w_1^{-1}\tau(w_1) \equiv w_2b_m^{j-i} \bmod \mathcal{N}.$$

Writing $w_1 = b_m^ab_n^b[b_m,b_n]^c$ and $w_2 = b_m^rb_n^s[b_m,b_n]^t$ this expands to

$$b_n^{a-b}b_m^{b-a}[b_m,b_n]^{-a^2-2c} \equiv b_n^sb_m^{r+j-i}[b_m,b_n]^{t+s(j-i)} \bmod \mathcal{N}.$$

Comparing exponents, we then see that (6) is equivalent to the following system of congruences having a solution for a,b,c,i,j in terms of r,s,t

$$\begin{aligned} a-b &\equiv s \bmod d \\ b-a &\equiv r+j-i \bmod d \\ -a^2-2c &\equiv t+s(j-i) \bmod d. \end{aligned}$$

The first two congruences determine b and are equivalent to $i-j \equiv r+s \bmod d$. Hence the system reduces to solving

$$a^2+2c \equiv (r+s)s-t \bmod d.$$

This congruence may be solved by setting $a \equiv 0,1 \bmod d$ depending on the parity of the right hand side. In particular, a solution always exists. □

Theorem 4.22 (Semirigidity of $\mathcal{B}$)**.** *If $b_i' \in [C_d]^\infty$ are elements such that $b_i' \sim b_i$ in $[C_d]^\infty$ for $1 \leq i \leq n$, then there exists an element $w \in [C_d]^\infty$ and elements $u_i \in \mathcal{B}$ such that for each i,*

$$wb_i'w^{-1} = u_ib_iu_i^{-1}.$$

In particular, $\langle\langle b_1',\ldots,b_n'\rangle\rangle = w\mathcal{B}w^{-1}$.

Proof. As in the proof of Theorem 4.20, by combining Proposition 2.8 and Lemma 4.19, it suffices to prove the following proposition for all $\ell \geq 0$,

> $(*_\ell)$: If $b_i' \in [C_d]^\infty$ are elements such that $b_i' \sim_\ell b_i$ in $[C_d]^\infty$ for $1 \leq i \leq n$, then there exists an element $w \in [C_d]^\infty$ and elements $u_i \in \mathcal{B}$ such that $wb_i'w^{-1} =_\ell u_ib_iu_i^{-1}$.

We proceed by induction on ℓ. The base case is trivial; suppose $\ell \geq 1$ and that $(*_{\ell-1})$ is true. Replacing all the b_i by a simultaneous conjugation we may assume that $b_1' = b_1 = \sigma$. If $i \neq 1, m+1$, then $b_i' \sim_\ell b_i$ implies there is some integer j_i such that

$$b_i' =_\ell \sigma^{j_i}(1,\ldots,1,b_{i-1}'')\sigma^{-j_i},$$

where $b_{i-1}'' \in [C_d]^\infty$. Similarly, $b_{m+1}' \sim_\ell b_{m+1}$ implies there is some j_{m+1} such that

$$b_{m+1}' =_\ell \sigma^{j_{m+1}}(1,\ldots,1,b_n'',1,\ldots,1,b_m'')\sigma^{-j_{m+1}},$$

where $b_m'', b_n'' \in [C_d]^\infty$ and the b_n'' is in the ωth coordinate. Proposition 2.2 implies that $b_i \sim_{\ell-1} b_i''$ for each $1 \leq i \leq n$. Thus by $(*_{\ell-1})$ we have an element $w' \in [C_d]^\infty$ and elements $u_i' \in \mathcal{B}$ such that for each $1 \leq i \leq n$,

$$w'b_i''w'^{-1} =_{\ell-1} u_i'b_iu_i'^{-1}.$$

Let $v \in \mathcal{B}$ and $i,j \in \mathbb{Z}/d\mathbb{Z}$ be the elements provided by Lemma 4.21 associated to $u_n', u_m' \in \mathcal{B}$ such that

$$v_{n+1} := (1,\ldots,1,vu_n'b_n^i,1\ldots,1,vu_m'b_m^j) \in \mathcal{B}.$$

Define $w := (vw', \dots, vw') \in [C_d]^\infty$; note that w commutes with σ. Lemma 4.3 implies that for $i \neq 1, m+1$ there exists an element $v_i \in \operatorname{St}_1 \mathcal{B}$ such that $\pi_d(v_i) = vu'_{i-1}$. Let $u_1 = 1$ and for $i \neq 1$ let $u_i = \sigma^{j_i} v_i$. Hence each $u_i \in \mathcal{B}$

Since w commutes with σ and $u_1 = 1$, we have

$$wb'_1 w^{-1} = \sigma = u_1 b_1 u_1^{-1}.$$

If $i = m+1$, then

$$\begin{aligned} wb'_{m+1} w^{-1} &=_\ell w\sigma^{j_{m+1}}(1, \dots, 1, b''_n, 1 \dots, 1, b''_m)\sigma^{-j_{m+1}} w^{-1} \\ &= \sigma^{j_{m+1}}(1, \dots, 1, vw'b''_n w'^{-1} v^{-1}, 1, \dots, 1, vw'b''_m w'^{-1} v^{-1})\sigma^{-j_{m+1}} \\ &=_\ell \sigma^{j_{m+1}}(1, \dots, 1, vu'_n b_n u_n'^{-1} v^{-1}, 1, \dots, 1, vu'_m b_m u_m'^{-1} v^{-1})\sigma^{-j_{m+1}} \\ &= \sigma^{j_{m+1}} v_{m+1} b_{m+1} v_{m+1}^{-1} \sigma^{-j_{m+1}} \\ &= u_{m+1} b_{m+1} u_{m+1}^{-1}. \end{aligned}$$

Finally, if $i \neq 1, m+1$, then

$$\begin{aligned} wb'_i w^{-1} &=_\ell w\sigma^{j_i}(1, \dots, 1, b''_{i-1})\sigma^{-j_i} w^{-1} \\ &= \sigma^{j_i}(1, \dots, 1, vw'b''_{i-1} w'^{-1} v^{-1})\sigma^{-j_i} \\ &=_\ell \sigma^{j_i}(1, \dots, 1, vu'_{i-1} b_{i-1} u_{i-1}'^{-1} v^{-1})\sigma^{-j_i} \\ &= \sigma^{j_i} v_i b_i v_i^{-1} \sigma^{-j_i} \\ &= u_i b_i u_i^{-1}. \end{aligned}$$

That completes our induction, hence our proof. ☐

Semirigidity allows us to identify $\overline{\operatorname{Arb}} f$ with $\mathcal{A}$ or $\mathcal{B}$ in the periodic and preperiodic cases, respectively

Corollary 4.23. *Let $f \in K[x]$ be a unicritical PCF polynomial with degree d coprime to* $\operatorname{char} K$*. Suppose that f has n distinct finite post-critical points. Let $\overline{\operatorname{Arb}} f = \langle\langle c_1, c_2, \dots, c_n \rangle\rangle$ be as defined in Section 3.6.*

(1) In the periodic case, there exists a $w \in [C_d]^\infty$ and $u_i \in \mathcal{A} = \mathcal{A}(d, n)$ such that $wc_i w^{-1} = u_i a_i u_i^{-1}$ for each $1 \leq i \leq n$. In particular, $w \overline{\operatorname{Arb}} f w^{-1} = \mathcal{A}$.

(2) In the preperiodic case, there exists a $w \in [C_d]^\infty$ and $u_i \in \mathcal{B} = \mathcal{B}(d, m, n, \omega)$ such that $wc_i w^{-1} = u_i b_i u_i^{-1}$ for each $1 \leq i \leq n$. In particular, $w \overline{\operatorname{Arb}} f w^{-1} = \mathcal{B}$.

Proof. First consider the periodic case. Proposition 3.15 implies that the generators c_i satisfy the same cyclic conjugate recurrences as the a_i. Hence Proposition 2.13 implies $c_i \sim a_i$ in $[C_d]^\infty$ for each i. Therefore Theorem 4.20 yields the conclusion. The preperiodic case is the same, *mutatis mutandis.* ☐

Remark. Recall that $c_i \in \overline{\operatorname{Arb}} f$ is the image of an inertia generator over the point $p_i = f^i(0)$. If $v \in \overline{\operatorname{Arb}} f$ is any element, then $vc_i v^{-1}$ is also an inertia generator over p_i. Thus after replacing c_i by the appropriate conjugates, Corollary 4.23 implies that $\overline{\operatorname{Arb}} f = \langle\langle c_1, c_2, \dots, c_n \rangle\rangle$ where the c_i are inertia generators and satisfy the recursive identities defining the model groups. Note, however, that with this choice of generators we typically will not have $c_1 c_2 \cdots c_n$ equal to the standard odometer.

5. Finite level truncations, Hausdorff dimension, and normalizers

The semirigidity result of the preceding section establishes that $\overline{\operatorname{Arb}} f$ is isomorphic to a model group $\mathcal{A}(d, n)$ or $\mathcal{B}(d, m, n, \omega)$ according to the combinatorics of its critical orbit. In this section we establish additional properties of the model groups—hence $\overline{\operatorname{Arb}} f$—including determining the order of their finite level truncations, calculating their Hausdorff dimension, and analyzing the structure of their normalizers in $[C_d]^\infty$. The latter is essential to our determination of the constant field extensions in Section 6.

5.1. **Finite level truncations.** If $K \subseteq H \subseteq [C_d]^\infty$ are subgroups and $\ell \geq 0$, then we define

$$[H : K]_\ell := [\rho_\ell(H) : \rho_\ell(K)] = [H \operatorname{St}_\ell[C_d]^\infty : K \operatorname{St}_\ell[C_d]^\infty].$$

Note that $[H : K]_\ell$ is a weakly increasing function of ℓ and that $[H : K] \geq [H : K]_\ell$ for all $\ell \geq 0$. For K open, $[H : K] = \lim_{\ell\to\infty}[H : K]_\ell$, so the sequence necessarily stabilizes. We say the index of K *stabilizes at* k if $[H : K]_\ell = [H : K]$ for all $\ell \geq k$. In Lemma 5.1 we determine when the index stabilizes for the subgroups $\mathcal{N}$ and $\mathcal{N}^d$ in $\mathcal{B}$.

Lemma 5.1. *The table below shows the index of stabilization for $\mathcal{N}$ and $\mathcal{N}^d$ in cases $(A), (B), (C)$.*

	$\mathcal{N}$	$\mathcal{N}^d$
(A)	n	$m+1$
(B)	n	$n+1$
(C)	$n+1$	$n+2$

Furthermore, $[\mathcal{B} : \mathcal{N}]_3 = 4$ and $[\mathcal{B} : \mathcal{N}^d]_4 = 8$.

Proof. Suppose (A). Lemma 4.13 implies that

$$\mathcal{N} = \mathcal{B} \cap \ker(\chi_m) \cap \ker(\chi_n).$$

From this and $\operatorname{St}_\ell[C_d]^\infty \subseteq \ker(\chi_\ell)$, it follows that $\operatorname{St}_\ell \mathcal{B} \subseteq \mathcal{N}$ for all $\ell \geq n$. Therefore the index of $\mathcal{N}$ stabilizes at n. The proof of Lemma 4.18 shows that $\mathcal{B}/\mathcal{N}^d$ is generated by σ and b_{m+1} in case (A). These elements generate a subgroup of order $d^{d+1} = [\mathcal{B} : \mathcal{N}^d]$ at each level $\ell \geq m+1$. Therefore $\mathcal{N}^d$ stabilizes at $m+1$.

Suppose (B). Lemma 4.13 implies that $[\mathcal{B} : \mathcal{N}] = d^3$ and $\mathcal{N} = \ker \psi_B \cap \mathcal{B}$. The formulas defining ψ_B only depend on the level n truncation of $\mathcal{B}$, hence $\operatorname{St}_n \mathcal{B} \subseteq \mathcal{N}$. Therefore $\mathcal{N}$ stabilizes at level n and $\mathcal{N}^d$ stabilizes at level $n+1$.

Finally, suppose (C). Lemma 4.13 implies that $[\mathcal{B} : \mathcal{N}] = 8$. In this case we do not have a description of $\mathcal{N}$ as the kernel of a homomorphism depending on the χ_i characters. We check via computer calculation (see Appendix A) that $[\mathcal{B} : \mathcal{N}]_3 = 4$ and $[\mathcal{B} : \mathcal{N}^d]_4 = 8$, while $[\mathcal{B} : \mathcal{N}]_4 = 8 = [\mathcal{B} : \mathcal{N}]$ and $[\mathcal{B} : \mathcal{N}^d]_5 = 16 = [\mathcal{B} : \mathcal{N}^d]$. Therefore the index of $\mathcal{N}$ stabilizes at level 4 and the index of $\mathcal{N}^d$ stabilizes at level 5. □

Recall that for a subgroup $H \subseteq [C_d]^\infty$ we write $\operatorname{ord}_\ell(H)$ to denote the order of $\rho_\ell(H) \subseteq [C_d]^\ell$. Note that $\log_d |[C_d]^\ell| = [\ell]_d$ where

$$[\ell]_d := \frac{d^\ell - 1}{d-1} = 1 + d + \ldots + d^{\ell-1}.$$

We will make several uses of the identity $[\ell+k]_d = d^k[\ell]_d + [k]_d$ in the proof of the next proposition.

Proposition 5.2. *Let $\ell \geq 0$,*

(1) In the periodic case, let q_ℓ and r_ℓ be the unique integers such that $\ell = q_\ell n + r_\ell$ and $0 \leq r_\ell < n$. Then,

$$\log_d \operatorname{ord}_\ell(\mathcal{A}) = [\ell]_d - d^{r_\ell}[q_\ell]_{d^n} + q_\ell.$$

(2) In the preperiodic case except (D), let $\delta := \log_d[\mathcal{B} : \mathcal{N}]$ and $\varepsilon := \log_d[\mathcal{B} : \mathcal{N}^d]$.

(a) If (A) or (B) and $\ell \geq n$*, then,*

$$\log_d \operatorname{ord}_\ell(\mathcal{B}) = \begin{cases} [\ell]_d & \text{if } \ell \leq n \\ [\ell]_d + (\varepsilon - 1)[\ell - n]_d - \delta[\ell - n + 1]_d + \delta & \text{if } \ell > n. \end{cases}$$

(b) If (C), then

$$\log_2 \operatorname{ord}_\ell(\mathcal{B}) = \begin{cases} 2^\ell - 1 & \text{if } \ell \leq 3, \\ 13 & \text{if } \ell = 4, \\ 11 \cdot 2^{\ell-4} + 2 & \text{if } \ell > 4. \end{cases}$$

(c) If (D), then

$$\log_d \operatorname{ord}_\ell(\mathcal{B}) = \ell + 1.$$

Proof. (1) First consider the periodic case. Let $\alpha_\ell := \log_d \operatorname{ord}_\ell(\mathcal{A})$ and let $\gamma_\ell := \log_d \operatorname{ord}_\ell(\widehat{\mathcal{A}}_n)$. Lemma 4.6 implies that $\mathcal{A} = \langle\langle a_1 \rangle\rangle \widehat{\mathcal{A}}_n^d$ and $\mathcal{A} = \langle\langle a_n \rangle\rangle \widehat{\mathcal{A}}_n$. Thus by Proposition 4.5 we have

$$\begin{aligned} \alpha_\ell &= \gamma_\ell + \lfloor \tfrac{\ell - n}{n} \rfloor + 1 = \gamma_\ell + \lfloor \tfrac{\ell}{n} \rfloor, \\ \alpha_\ell &= d\gamma_{\ell-1} + \lfloor \tfrac{\ell-1}{n} \rfloor + 1. \end{aligned}$$

Eliminating α_ℓ gives a recurrence for γ_ℓ with initial value $\gamma_0 = 0$ and

$$\gamma_\ell = d\gamma_{\ell-1} + \lfloor \tfrac{\ell-1}{n} \rfloor - \lfloor \tfrac{\ell-n}{n} \rfloor = \begin{cases} d\gamma_{\ell-1} + 1 & \text{if } n \nmid \ell, \\ d\gamma_{\ell-1} & \text{if } n \mid \ell. \end{cases}$$

Define $\gamma'_\ell := [\ell]_d - d^{r_\ell}[q_\ell]_{d^n}$. We show that γ'_ℓ satisfies the same recurrence as γ_ℓ. Note that $\gamma'_0 = 0$, so their initial conditions agree. Now suppose $\ell > 0$. If $n \nmid \ell$, then $\ell - 1 = q_\ell n + r_\ell - 1$ where $0 \leq r_\ell - 1 < n$. Hence $q_{\ell-1} = q_\ell$ and $r_{\ell-1} = r_\ell - 1$. Thus

$$\begin{aligned} \gamma'_\ell &= [\ell]_d - d^{r_\ell}[q_\ell]_{d^n} \\ &= d[\ell - 1]_d - d^{r_\ell}[q_\ell]_{d^n} + 1 \\ &= d([\ell - 1]_d - d^{r_\ell - 1}[q_\ell]_{d^n}) + 1 \\ &= d([\ell - 1]_d - d^{r_{\ell-1}}[q_{\ell-1}]_{d^n}) + 1 \\ &= d\gamma'_{\ell-1} + 1. \end{aligned}$$

Otherwise, $n \mid \ell$ so $r_\ell = 0$ and $\ell - 1 = (q_\ell - 1)n + (n - 1)$. Hence $q_{\ell-1} = q_\ell - 1$ and $r_{\ell-1} = n - 1$. Therefore,

$$\begin{aligned} \gamma'_\ell &= [\ell]_d - [q_\ell]_{d^n} \\ &= (d[\ell - 1]_d + 1) - (d^n[q_\ell - 1]_{d^n} + 1) \\ &= d([\ell - 1]_d - d^{n-1}[q_\ell - 1]_{d^n}) \\ &= d([\ell - 1]_d - d^{r_{\ell-1}}[q_{\ell-1}]_{d^n}) \\ &= d\gamma'_{\ell-1}. \end{aligned}$$

Therefore, $\gamma_\ell = \gamma'_\ell$ for all $\ell \geq 0$. Note that $q_\ell = \lfloor \frac{\ell}{n} \rfloor$. Hence for all $\ell \geq 0$,

$$\alpha_\ell = [\ell]_d - d^{r_\ell}[q_\ell]_{d^n} + q_\ell.$$

(2) Next consider the preperiodic case. If $\ell \leq n$, then Lemma 4.16 implies that $\mathcal{B} =_\ell [C_d]^\ell$. Hence

$$\log_d \operatorname{ord}_\ell(\mathcal{B}) = \log_d \operatorname{ord}_\ell([C_d]^\ell) = [\ell]_d.$$

Now suppose that $\ell > n$. First suppose (A), (B), or (C). Let $\beta_\ell := \log_d \mathrm{ord}_\ell(\mathcal{B})$. Proposition 4.14 implies that $\mathcal{N}^d \subseteq \mathcal{B}$ and $[\mathcal{B} : \mathcal{N}] < \infty$. Define

$$\begin{aligned}\gamma_\ell &:= \log_d \mathrm{ord}_\ell(\mathcal{N}),\\ \delta_\ell &:= \log_d [\mathcal{B} : \mathcal{N}]_\ell,\\ \varepsilon_\ell &:= \log_d [\mathcal{B} : \mathcal{N}^d]_\ell.\end{aligned}$$

Then for all $\ell \geq 1$,

$$\begin{aligned}\beta_\ell &= \gamma_\ell + \delta_\ell\\ \beta_\ell &= d\gamma_{\ell-1} + \varepsilon_\ell.\end{aligned}$$

Eliminating β_ℓ gives the recursion

$$\gamma_\ell = d\gamma_{\ell-1} + \varepsilon_\ell - \delta_\ell.$$

Suppose either (A) or (B). Lemma 5.1 implies that $\delta_\ell = \delta := [\mathcal{B} : \mathcal{N}]$ for all $\ell \geq n$ and $\varepsilon_\ell = \varepsilon := [\mathcal{B} : \mathcal{N}^d]$ for all $\ell \geq n+1$. Hence the recursion simplifies, for all $\ell \geq n+1$, to

$$\gamma_\ell = d\gamma_{\ell-1} + \varepsilon - \delta.$$

From this, a straightforward induction implies for all $k \geq 1$

$$\gamma_{n+k} = d^k \gamma_n + (\varepsilon - \delta)[k]_d \tag{7}$$

Since $\gamma_n = [n]_d - \delta$ we can further simplify this to

$$\begin{aligned}\gamma_{n+k} &= d^k \gamma_n + (\varepsilon - \delta)[k]_d\\ &= d^k [n]_d - d^k \delta + (\varepsilon - \delta)[k]_d\\ &= [n+k]_d + (\varepsilon - 1)[k]_d - \delta[k+1]_d.\end{aligned}$$

Setting $\ell = n + k$ and regrouping terms we have for $\ell \geq n+1$,

$$\gamma_\ell = [\ell]_d + (\varepsilon - 1)[\ell - n]_d - \delta[\ell - n + 1]_d.$$

From $\beta_\ell = \gamma_\ell + \delta_\ell$ and $\delta_\ell = \delta$ when $\ell \geq n$, we conclude that

$$\beta_\ell = \gamma_\ell + \delta = [\ell]_d + (\varepsilon - 1)[\ell - n]_d - \delta[\ell - n + 1]_d + \delta.$$

Now suppose (C); hence $d = 2$ and $n = 3$. Then Lemma 5.1 implies that $\delta_\ell = \delta = 3$ for $\ell \geq 4$; $\varepsilon_\ell = \varepsilon = 4$ for $\ell \geq 5$; $\delta_3 = 2$; and $\varepsilon_4 = 3$. Note that

$$\gamma_4 = 2\gamma_3 + \varepsilon_4 - \delta_4 = 2(\beta_3 - \delta_3) + \varepsilon_4 - \delta_4 = 2([3]_2 - 2) + 3 - 3 = 10.$$

Hence

$$\beta_4 = \gamma_4 + \delta_4 = 10 + 3 = 13.$$

If $\ell > 4$, then

$$\gamma_\ell = d\gamma_{\ell-1} + \varepsilon - \delta,$$

and a simple induction implies that for all $k \geq 1$,

$$\gamma_{k+4} = 2^k \gamma_4 + [k]_2 = 11 \cdot 2^k - 1.$$

Thus for $\ell > 4$,

$$\beta_\ell = \gamma_\ell + \delta = 11 \cdot 2^{\ell-4} + 2.$$

Suppose (D). The identity $(b_1 b_2)^2 = (b_1 b_2, b_2 b_1)$ implies that

$$\mathrm{ord}_\ell(b_1 b_2) = 2\mathrm{ord}_{\ell-1}(b_1 b_2).$$

Since $\mathrm{ord}_0(b_1 b_2) = 1$, it follows by induction that $\mathrm{ord}_\ell(b_1 b_2) = 2^\ell$. Hence $\rho_\ell(\mathcal{B})$ is isomorphic to the dihedral group of order $2^{\ell+1} = d^{\ell+1}$. Thus $\log_d \mathrm{ord}_\ell(\mathcal{B}) = \ell + 1$ in case (D). □

Given a subgroup $H \subseteq [C_d]^\infty$, we define the *Hausdorff dimension* of H to be

$$\mu_{\text{haus}}(H) := \lim_{\ell\to\infty} \frac{\log_d \operatorname{ord}_\ell(H)}{\log_d \operatorname{ord}_\ell([C_d]^\infty)} = \lim_{\ell\to\infty} \frac{\log_d \operatorname{ord}_\ell(H)}{[\ell]_d},$$

provided the limit exists.

Corollary 5.3.

(1) In the periodic case,

$$\mu_{\text{haus}}(\mathcal{A}) = 1 - \frac{d-1}{d^n-1}.$$

(2) In the preperiodic case,

(a) If (A) or (B) then, using the notation of Proposition 5.2 we have,

$$\mu_{\text{haus}}(\mathcal{B}) = 1 + \frac{\varepsilon-1}{d^n} - \frac{\delta}{d^{n-1}}.$$

(b) If (C), then

$$\mu_{\text{haus}}(\mathcal{B}) = \frac{11}{16}.$$

(c) If (D), then $\mu_{\text{haus}}(\mathcal{B}) = 0$.

Proof. (1) Consider the periodic case. Using the notation from Proposition 5.2(1), observe that

$$d^{r_\ell}[q_\ell]_{d^n} = d^{\ell-n} + d^{\ell-2n} + \ldots + d^{r_\ell}.$$

Hence

$$\lim_{\ell\to\infty} \frac{d^{r_\ell}[q_\ell]_{d^n}}{[\ell]_d} = \frac{\sum_{i=1}^\infty d^{-ni}}{\sum_{i=1}^\infty d^{-i}} = \frac{d^{-n}(1-d^{-n})^{-1}}{d^{-1}(1-d^{-1})^{-1}} = \frac{d-1}{d^n-1}.$$

Therefore Proposition 5.2(1) implies

$$\mu_{\text{haus}}(\mathcal{A}) = \lim_{\ell\to\infty} \frac{[\ell]_d - d^{r_\ell}[q_\ell]_{d^n} + q_\ell}{[\ell]_d} = 1 - \frac{d-1}{d^n-1}.$$

(2) Now consider the preperiodic case.

(2a) Suppose (A) or (B). Note that for any $k \geq 0$,

$$\lim_{\ell\to\infty} \frac{[\ell-k]_d}{[\ell]_d} = \frac{\sum_{i=1}^\infty d^{-k-i}}{\sum_{i=1}^\infty d^{-i}} = \frac{1}{d^k}.$$

Thus Proposition 5.2(2a) implies

$$\mu_{\text{haus}}(\mathcal{B}) = \lim_{\ell\to\infty} \frac{[\ell]_d + (\varepsilon-1)[\ell-n]_d - \delta[\ell-n+1]_d}{[\ell]_d} = 1 + \frac{\varepsilon-1}{d^n} - \frac{\delta}{d^{n-1}}.$$

(2b) Suppose (C). Then

$$\mu_{\text{haus}}(\mathcal{B}) = \lim_{\ell\to\infty} \frac{11\cdot 2^{\ell-4}+2}{[\ell]_2} = \lim_{\ell\to\infty} \frac{11\cdot 2^{\ell-4}+2}{2^\ell-1} = \frac{11}{16}.$$

(2c) Suppose (D). Proposition 5.2(2b) implies that

$$\mu_{\text{haus}}(\mathcal{B}) = \lim_{\ell\to\infty} \frac{\ell+1}{[\ell]_d} = 0. \qquad \square$$

The following table combines the results of Proposition 5.2, Corollary 5.3, and the index calculations of Lemma 4.13 and Lemma 4.18.

	$\log_d \operatorname{ord}_\ell(\mathcal{B})$	$\mu_{\text{haus}}(\mathcal{B})$
(A)	$[\ell]_d + d[\ell - n]_d - 2[\ell - n + 1]_d + 2$	$1 - \frac{1}{d^{n-1}}$
(B)	$[\ell]_d + \frac{3d}{2}[\ell - n]_d - 3[\ell - n + 1]_d + 3$	$1 - \frac{3}{2d^{n-1}}$
(C)	$11 \cdot 2^{\ell-4} + 2$	$\frac{11}{16}$
(D)	$\ell + 1$	0

5.2. **Normalizers.** Let $N(\mathcal{A})$ and $N(\mathcal{B})$ denote the normalizers of $\mathcal{A}$ and $\mathcal{B}$, respectively, in $[C_d]^\infty$. Some understanding of these normalizers is required for our analysis of the constant field extensions $\widehat{K}_f/K$. For example, in Proposition 5.8 we show that $N(\mathcal{B})/\mathcal{B}$ has finite exponent except in case (D), which translates into $\widehat{K}_f/K$ being finite in those cases.

Lemma 5.4.

(1) $\operatorname{St}_1 N(\mathcal{A}) \subseteq N(\mathcal{A})^d$ *and* $\operatorname{St}_1 N(\mathcal{B}) \subseteq N(\mathcal{B})^d$*,*
(2) If $v \in N(\mathcal{B})$ *and* $b \in \mathcal{B}$*, then* $\chi_\ell(v^{-1}bv) = \chi_\ell(b)$ *for each* $\ell \geq 1$*,*
(3) In cases (B)*,* (C)*, or* (D)*, for every* $v \in N(\mathcal{B})$*, then there exists an element* $u \in \mathcal{B}$ *such that* $v^{-1}bv \equiv u^{-1}bu \bmod \mathcal{N}$ *for all* $b \in \mathcal{B}$*.*

Proof. (1) Since $N(\mathcal{A})$ is contained in $[C_d]^\infty$, it preserves the level structure on $\mathcal{A}$. Therefore, $N(\mathcal{A})$ and its subgroup $\operatorname{St}_1 N(\mathcal{A})$ normalize $\operatorname{St}_1 \mathcal{A}$. Lemma 4.3 implies that $\pi_i(\operatorname{St}_1 \mathcal{A}) = \mathcal{A}$ for each $1 \leq i \leq d$, which in turn implies that $\operatorname{St}_1 N(\mathcal{B}) \subseteq N(\mathcal{B})^d$. The same argument applies to $\mathcal{B}$.

(2) The functions χ_ℓ are defined on all of $[C_d]^\infty$ and $N(\mathcal{B}) \subseteq [C_d]^\infty$ by definition. Hence $\chi_\ell(v^{-1}bv) = \chi_\ell(b)$ for all $v \in N(\mathcal{B})$ and all $b \in \mathcal{B}$.

(3) Suppose either (B), (C), or (D). Let $v \in N(\mathcal{B})$. Part (2) implies that

$$v^{-1}b_m v \equiv b_m[b_m, b_n]^i \bmod \mathcal{N}$$
$$v^{-1}b_n v \equiv b_n[b_m, b_n]^j \bmod \mathcal{N}$$

for some $i, j \in \mathbb{Z}/d\mathbb{Z}$. Let $u = b_m^{-j} b_n^i \in \mathcal{B}$. A straightforward calculation using the identity

$$b_m b_n \equiv b_n b_m [b_m, b_n] \bmod \mathcal{N}$$

implies that $v^{-1}b_m v \equiv u^{-1}b_m u \bmod \mathcal{N}$ and $v^{-1}b_n v \equiv u^{-1}b_n u \bmod \mathcal{N}$, hence that

$$v^{-1}bv \equiv u^{-1}bu \bmod \mathcal{N}$$

for all $b \in \mathcal{B}$. □

Lemma 5.5 characterizes the action of the normalizer $N(\mathcal{A})$ on the generators a_i.

Lemma 5.5. *If* $w \in N(\mathcal{A})$*, then for each* $1 \leq i \leq n$ *there exists an* $\varepsilon_i \in 1 + d\mathbb{Z}_d$ *such that* $w^{-1}a_i w \sim a_i^{\varepsilon_i}$ *in* $\mathcal{A}$*.*

Proof. By Proposition 2.8, it suffices to prove the following proposition for all $1 \leq i \leq n$ and $\ell \geq 0$,

$(*_{i,\ell})$: If $w \in N(\mathcal{A})$, then there exists an $\varepsilon_i \in 1 + d\mathbb{Z}_d$ and a $v_i \in \mathcal{A}$ such that $w^{-1}a_i w =_\ell v_i^{-1} a_i^{\varepsilon_i} v_i$.

Note that if $w \in N(\mathcal{A})$, we may replace w by wa_1^k in order to assume without loss of generality that $w := (w_1, \ldots, w_d) \in \operatorname{St}_1 \mathcal{A} \subseteq \mathcal{A}^d$, where the last containment follows from Lemma 5.4.

We proceed by induction to show that $(*_{i-1,\ell-1})$ implies $(*_{i,\ell})$ for $1 \leq i \leq n$ and $\ell \geq 1$, where the i subscripts are interpreted modulo n. First observe that $(*_{i,0})$ holds trivially for each $1 \leq i \leq n$ since $a_i =_0 1$. Now suppose that $2 \leq i \leq n$, $\ell \geq 1$, and that $(*_{i-1,\ell-1})$ is true. Since $a_i = (1, \ldots, 1, a_{i-1})$, we have

$$w^{-1}a_i w = (1, \ldots, 1, w_d^{-1} a_{i-1} w_d),$$

where $w_d^{-1}a_{i-1}w_d \in \mathcal{A}$. The inductive hypothesis $(*_{i-1,\ell-1})$ implies there is some $\varepsilon_i \in 1+d\mathbb{Z}_d$ and some $v_d \in \mathcal{A}$ such that

$$w_d^{-1}a_{i-1}w_d =_{\ell-1} v_d^{-1}a_{i-1}^{\varepsilon_i}v_d.$$

Lemma 4.3 provides an element $v_i \in \mathrm{St}_1(\mathcal{A})$ such that $\pi_d(v_i) = v_d$. Thus

$$w^{-1}a_iw = (1,\dots,1,w_d^{-1}a_{i-1}w_d) =_\ell (1,\dots,1,v_d^{-1}a_{i-1}^{\varepsilon_i}v_d) = v_i^{-1}a_i^{\varepsilon_i}v_i,$$

hence $(*_{i,\ell})$ is true.

Next suppose that $(*_{n,\ell-1})$ is true for some $\ell \geq 1$. Then

$$w^{-1}a_1^dw = (w_1^{-1}a_nw_1,\dots,w_d^{-1}a_nw_d) \in \mathrm{St}_1(\mathcal{A}) \subseteq \mathcal{A}^d.$$

In particular, $w_1^{-1}a_nw_1 \in \mathcal{A}$ and $(*_{n,\ell-1})$ implies that there exists an $\varepsilon_1 \in 1+d\mathbb{Z}_d$ and a $v_1 \in \mathcal{A}$ such that

$$w_1^{-1}a_nw_1 =_{\ell-1} v_1^{-1}a_n^{\varepsilon_1}v_1. \tag{8}$$

On the other hand, if $w^{-1}a_1w := \sigma(u_1,\dots,u_d)$, then

$$w^{-1}a_1^dw = (u_du_{d-1}\cdots u_2u_1, u_1u_du_{d-1}\cdots u_2,\dots,u_{d-1}\cdots u_2u_1u_d),$$

which implies that

$$w_1^{-1}a_nw_1 = u_du_{d-1}\cdots u_2u_1. \tag{9}$$

Recall that $\mathrm{St}_1(\mathcal{A}) = \langle\langle a_1^d\rangle\rangle\widehat{\mathcal{A}}_n^d$ by Lemma 4.6(3). Hence there are $c_i \in \widehat{\mathcal{A}}_n$ for $1 \leq i \leq d$ and some $k \in \mathbb{Z}_d$ such that

$$a_1^{-1}w^{-1}a_1w = (u_1,u_2,\dots,u_{d-1},a_n^{-1}u_d) = (c_1a_n^m,\dots,c_da_n^m).$$

Therefore

$$\eta_n(u_j) = \begin{cases} m & \text{if } 1 \leq j < d, \\ m+1 & \text{if } j = d., \end{cases} \tag{10}$$

which implies that $u_du_{d-1}\cdots u_2u_1 \equiv a_n^{1+dm} \bmod \widehat{\mathcal{A}}_n$. Using (8) we see that

$$v_1^{-1}a_n^{\varepsilon_1}v_1 =_{\ell-1} w_1^{-1}a_nw_1 = u_du_{d-1}\cdots u_2u_1 \equiv a_n^{1+dm} \bmod \widehat{\mathcal{A}}_n.$$

Thus $a_n^{\varepsilon_1} =_{\ell-1} a_n^{1+dm}$.

Now we let $v := y_1y_2y_3$ where

$$\begin{aligned} y_1 &:= (1,a_n^m,a_n^{2m},\dots,a_n^{(d-1)m}) \\ y_2 &:= (v_1,\dots,v_1) \\ y_3 &:= (1,u_1^{-1},(u_2u_1)^{-1},\dots,(u_{d-1}\cdots u_2u_1)^{-1}) \end{aligned}$$

Proposition 4.17 implies that $v \in \mathrm{St}_1\mathcal{A}$.

We finish the proof by showing $w^{-1}a_1w =_\ell v^{-1}a_1^{1+dm}v$ in several steps, starting by conjugating

$$a_1^{1+dm} = \sigma(a_n^m,\dots,a_n^m,a_n^{1+m}).$$

by y_1 to get

$$\begin{aligned} y_1^{-1}a_1^{1+dm}y_1 &= (1,a_n^{-m},a_n^{-2m},\dots,a_n^{-(d-1)m})\sigma(a_n^m,\dots,a_n^m,a_n^{1+m})(1,a_n^m,a_n^{2m},\dots,a_n^{(d-1)m}) \\ &= \sigma(a_n^{-m},a_n^{-2m},\dots,a_n^{-(d-1)m},1)(a_n^m,\dots,a_n^m,a_n^{1+m})(1,a_n^m,a_n^{2m},\dots,a_n^{(d-1)m}) \\ &= \sigma(1,\dots,1,a_n^{1+dm}). \end{aligned}$$

Next, conjugating by y_2 gives us

$$(y_1y_2)^{-1}a_1^{1+dm_2}(y_1y_2) = \sigma(1,\dots,1,v_1^{-1}a_n^{1+dm}v_1) =_\ell \sigma(1,\dots,1,w_1^{-1}a_nw_1).$$

Finally conjugating by y_3 and using (9) we have

$$\begin{aligned}v^{-1}a_1^{1+dm}v &= (y_1y_2y_3)^{-1}a_1^{1+dm}(y_1y_2y_3)\\ &=_\ell (1, u_1, \ldots, u_{d-1}\cdots u_2u_1)\sigma(1, \ldots, 1, w_1^{-1}a_nw_1)(1, u_1^{-1}, \ldots, (u_{d-1}\cdots u_2u_1)^{-1})\\ &= \sigma(u_1, \ldots, u_{d-1}\cdots u_2u_1, 1)(1, \ldots, 1, w_1^{-1}a_nw_1)(1, u_1^{-1}, \ldots, (u_{d-1}\cdots u_2u_1)^{-1})\\ &= \sigma(u_1, u_2, \ldots, u_{d-1}, (w_1^{-1}a_nw_1)(u_{d-1}\cdots u_2u_1)^{-1})\\ &= \sigma(u_1, u_2, \ldots, u_{d-1}, u_d)\\ &= w^{-1}a_1w.\end{aligned}$$

This completes the proof of $(*_{1,\ell})$ and hence our induction. □

Lemma 5.5 allows us to characterize $N(\mathcal{A})$ as a subset of $N(\mathcal{A})^d$.

Lemma 5.6. *Let $v \in N(\mathcal{A})^d$. Then $v \in N(\mathcal{A})$ if and only if $v^{-1}a_\infty v \in \mathcal{A}$.*

Proof. Let $v := (v_1, \ldots, v_d) \in N(\mathcal{A})^d$. If $2 \leq i \leq n$, then Lemma 5.5 implies there is some $u_d \in \mathcal{A}$ and some $\varepsilon_i \in 1 + d\mathbb{Z}_d$ such that

$$v^{-1}a_iv = (1, \ldots, 1, v_d^{-1}a_{i-1}v_d) = (1, \ldots, 1, u_d^{-1}a_{i-1}^{\varepsilon_i}u_d).$$

Lemma 4.3 implies there is an element $u \in \mathrm{St}_1\,\mathcal{A}$ such that $\pi_d(u) = u_d$, hence $v^{-1}a_iv = u^{-1}a_i^{\varepsilon_i}u \in \mathcal{A}$. Thus the identity $a_\infty = a_1a_2\cdots a_n$ implies that $v \in N(\mathcal{A})$ if and only if $v^{-1}a_\infty v \in \mathcal{A}$. □

Next we prove an analog of Lemma 5.6 for $\mathcal{B}$.

Lemma 5.7. *Let $v \in N(\mathcal{B})^d$. Then $v^{-1}b_iv \in \mathcal{B}$ for $i \neq 1, m+1$. Furthermore, $v \in N(\mathcal{B})$ if and only if $v^{-1}b_iv \in \mathcal{B}$ for $i = 1, m+1$.*

Proof. In cases (A) or (B), Lemma 4.12 implies that $\mathcal{N} = \ker\psi_A \cap \mathcal{B}$ or $\ker\psi_B \cap \mathcal{B}$, respectively. Hence $N(\mathcal{B}) \subseteq [C_d]^\infty$ normalizes $\mathcal{N}$. In case (C), Lemma 5.1 implies that $\mathrm{St}_4(\mathcal{B}) \subseteq \mathcal{N}$. The group $N(\mathcal{B})$ normalizes $\mathrm{St}_4(\mathcal{B}) = \mathrm{St}_4[C_d]^\infty \cap \mathcal{B}$. We then check via a computer calculation (see Appendix A) that $N(\mathcal{B})$ normalizes $\mathcal{N}/\mathrm{St}_4\,\mathcal{B}$. Hence $N(\mathcal{B})_4$ normalizes $\mathcal{N}$ in case (C) as well. In case (D) we have $\mathcal{N} = 1$, hence $N(\mathcal{B})$ trivially normalizes $\mathcal{N}$. Therefore in all cases $N(\mathcal{B})^d$ normalizes $\mathcal{N}^d$. Since $\widehat{\mathcal{B}}_{1,m+1} \subseteq \mathcal{N}^d$, we have $v^{-1}b_iv \in \mathcal{N}^d \subseteq \mathcal{B}$ for all $i \neq 1, m+1$. Thus $v \in N(\mathcal{B})$ if and only if $v^{-1}b_iv \in \mathcal{B}$ for $i = 1, m+1$. □

We give a detailed analysis of the constant field extensions for unicritical PCF polynomials in Section 6. In the preperiodic cases, except for case (D), the constant field extensions are always finite. Mere finiteness can be deduced from Proposition 5.8, though we obtain far more refined control over these extensions in Section 5.4.

Proposition 5.8. *If (A), (B), or (C), then $N(\mathcal{B})/\mathcal{B}$ has a finite exponent.*

Proof. Lemma 5.1 implies that $\mathrm{St}_{n'}\,\mathcal{B} \subseteq \mathcal{N}$. Let ε be the exponent of $[C_d]^{n'} \cong [C_d]^\infty/\mathrm{St}_{n'}[C_d]^\infty$. We will show that $v^\varepsilon \in \mathcal{B}$ for all $v \in N(\mathcal{B})$. By Proposition 2.8, it suffices to show that $v^\varepsilon \in_\ell \mathcal{B}$ for all $\ell \geq 0$ and all $v \in N(\mathcal{B})$. We proceed by induction. The case $\ell = 0$ is immediate. Suppose that $\ell \geq 1$ and that $v^\varepsilon \in_{\ell-1} \mathcal{B}$ for all $v \in N(\mathcal{B})$. Let $v \in N(\mathcal{B})$. Note that $[\sigma, v] \in \mathcal{B}$, hence for each i

$$(\sigma^iv)^\varepsilon \equiv \sigma^{i\varepsilon}v^\varepsilon \equiv v^\varepsilon \bmod \mathcal{B}.$$

Thus we may suppose without loss of generality that $v = (v_1, \ldots, v_d) \in \mathrm{St}_1\,N(\mathcal{B}) \subseteq N(\mathcal{B})^d$, where the last containment is Lemma 5.4(1). Our inductive hypothesis implies that

$$v^\varepsilon = (v_1^\varepsilon, \ldots, v_d^\varepsilon) \in_\ell \mathcal{B}^d.$$

Thus for each i we have $v_i^\varepsilon \in_{\ell-1} \mathcal{B}$ and by the definition of ε we have $v_i^\varepsilon \in \mathrm{St}_{n'}[C_d]^\infty$. Hence $v_i^\varepsilon \in_{\ell-1} \mathrm{St}_{n'}\,\mathcal{B} \subseteq \mathcal{N}$. Therefore $v^\varepsilon \in_\ell \mathcal{N}^d \subseteq \mathcal{B}$, where the last containment is by Proposition 4.14. This completes our induction. □

5.3. **Odometers.** Recall the elements $a_\infty \in \mathcal{A}$ and $b_\infty \in \mathcal{B}$ defined by

$$a_\infty = a_1 a_2 \cdots a_n = \sigma(1, \ldots, 1, a_n a_1 \cdots a_{n-1}) = \sigma(1, \ldots, 1, a_n a_\infty a_n^{-1}),$$
$$b_\infty = b_1 b_2 \cdots b_n = \sigma(1, \ldots, 1, b_n, 1, \ldots, 1, b_1 \cdots b_{n-1}) = \sigma(1, \ldots, 1, b_n, 1, \ldots, 1, b_\infty b_n^{-1}).$$

In Section 6 we reduce the analysis of constant field extensions $\widehat{K}_{f,\ell}$ in $\overline{\mathrm{Arb}}\, f$ to the problem of deciding for which $\varepsilon \in \mathbb{Z}_d^\times$ we have $a_\infty \sim_\ell a_\infty^\varepsilon$ in $\mathcal{A}$ and $b_\infty \sim_\ell b_\infty^\varepsilon$ in $\mathcal{B}$. The following proposition is a crucial step in this reduction.

Proposition 5.9. *Let $v \in [C_d]^\infty$. Then*
(1) $v^{-1} a_\infty v \in \mathcal{A}$ if and only if $v \in N(\mathcal{A})$,
(2) $v^{-1} b_\infty v \in \mathcal{B}$ if and only if $v \in N(\mathcal{B})$.

Proof. (1) If $v \in N(\mathcal{A})$, then clearly $v^{-1} a_\infty v \in \mathcal{A}$. For the other direction we prove by induction on $\ell \geq 0$ that if $v \in [C_d]^\infty$ is an element such that $v^{-1} a_\infty v \in_\ell \mathcal{A}$, then $v \in_\ell N(\mathcal{A})$. The base case is trivial, so suppose $\ell \geq 1$ and that the assertion holds for $\ell - 1$.

Let $v \in [C_d]^\infty$ be an element such that $v^{-1} a_\infty v \in_\ell \mathcal{A}$. Replacing v by $v a_\infty^k$ we may assume that $v =_1 1$, hence $v = (v_1, \ldots, v_d)$. Then $v^{-1} a_\infty^d v \in \mathcal{A}$ and we calculate

$$v^{-1} a_\infty^d v = (v_1^{-1} a_n a_\infty a_n^{-1} v_1, \ldots, v_d^{-1} a_n a_\infty a_n^{-1} v_d) \in_\ell \mathrm{St}_1 \mathcal{A} \subseteq \mathcal{A}^d.$$

Therefore $(v_i^{-1} a_n) a_\infty (v_i^{-1} a_n)^{-1} \in_{\ell-1} \mathcal{A}$ for each i, which by our inductive hypothesis implies that $v_i \in_{\ell-1} N(\mathcal{A})$. Hence $v \in_\ell N(\mathcal{A})^d$. Then Lemma 5.6 implies that $v \in_\ell N(\mathcal{A})$, which completes our induction.

(2) The forward direction is immediate. For the reverse direction it suffices by Proposition 2.8 to prove by induction on $\ell \geq 0$ that if $v \in [C_d]^\infty$ is an element such that $v^{-1} b_\infty v \in_\ell \mathcal{B}$, then $v \in_\ell N(\mathcal{B})$. If $\ell \leq n$, then Lemma 4.16 implies that $\mathcal{B} =_\ell [C_d]^\infty$ which renders both assertions immediate. We now assume that $\ell > n$ and that the claim holds for $\ell - 1$.

Now suppose that $v \in [C_d]^\infty$ is an element such that $v^{-1} b_\infty v \in_\ell \mathcal{B}$. Replacing v by $b_\infty^i v$ for some i, we may assume without loss of generality that $v = (v_1, \ldots, v_d)$ for some $v_i \in [C_d]^\infty$.

Observe that $b_\infty^d = (b_\infty, \ldots, b_\infty, b_n b_\infty b_n^{-1}, \ldots, b_n b_\infty b_n^{-1})$. Since $v^{-1} b_\infty^d v \in_\ell \mathrm{St}_1 \mathcal{B} \subseteq \mathcal{B}^d$, we have $v_i^{-1} b_\infty v_i \in_{\ell-1} \mathcal{B}$ for $1 \leq i \leq \omega$ and $v_i^{-1} b_n^{-1} b_\infty b_n v_i \in_{\ell-1} \mathcal{B}$ for $\omega < i \leq d$. Hence the inductive hypothesis implies that $v_i \in_{\ell-1} N(\mathcal{B})$ for each i. Thus $v \in_\ell N(\mathcal{B})^d$.

Lemma 5.7 implies that $v^{-1} b_i v \in_\ell \mathcal{B}$ for all $i \neq 1, m+1$ and that $v \in_\ell N(\mathcal{B})$ if and only if $v^{-1} b_i v \in_\ell \mathcal{B}$ for $i = 1, m+1$. Since $b_\infty = b_1 \cdots b_n$ and $v^{-1} b_\infty v \in_\ell \mathcal{B}$, to prove that $v \in_\ell N(\mathcal{B})$, it suffices to show that $v^{-1} b_1 v = v^{-1} \sigma v \in_\ell \mathcal{B}$, or equivalently that

$$u = (u_1, \ldots, u_d) := [\sigma, v] \in_\ell \mathcal{B}.$$

We proceed by comparing u to $u' = (u'_1, \ldots, u'_d) := \sigma^{-1} v^{-1} b_\infty v$, which is in $\mathrm{St}_1 \mathcal{B} \subseteq \mathcal{B}^d$. Note that

$$u = \sigma^{-1} v^{-1} \sigma v = (\sigma^{-1} v^{-1} \sigma b_2 \cdots b_n v)(v^{-1} (b_2 \cdots b_n)^{-1} v) = u' v^{-1} b_\infty^{-1} \sigma v. \tag{11}$$

Since $b_\infty^{-1} \sigma \in \mathrm{St}_1 \mathcal{B} \subseteq \mathcal{B}^d$ and $v \in_\ell N(\mathcal{B})^d$, we conclude that $u \in_\ell \mathcal{B}^d$.

Observe that $b_\infty^{-1} \sigma = (1, \ldots, 1, b_n^{-1}, 1, \ldots, 1, b_n b_\infty^{-1})$, hence comparing coordinates in (11) we have

$$u_i = \begin{cases} u'_i & i \neq \omega, d, \\ u'_\omega v_\omega^{-1} b_n^{-1} v_\omega & i = \omega, \\ u'_d v_d^{-1} b_n b_\infty^{-1} v_d & i = d. \end{cases}$$

First consider case (A). Since $u' = \sigma^{-1}(v^{-1} b_\infty v) \in \mathcal{B}$, Proposition 4.17(1) implies that $\chi_m(u'_i) = \chi_n(u'_{i+\omega})$ for each i. Thus $\chi_m(u_i) = \chi_n(u_{i+\omega})$ for $i \neq d$. Furthermore,

$$\chi_m(u_d) = \chi_m(u'_d) + \chi_m(v_d^{-1} b_n b_\infty^{-1} v_d) = \chi_n(u'_\omega) - 1 = \chi_n(u'_\omega) + \chi_n(v_\omega^{-1} b_n^{-1} v_\omega) = \chi_n(u_\omega).$$

Therefore Proposition 4.17(1) implies that $u \in_\ell \mathcal{B}$ This completes our induction in case (A).

Next suppose either (B), (C), or (D). It is still the case that $u' \in \mathcal{B}$, so we apply Proposition 4.17(2) which says that $\tau(u'_i) \equiv u'_{i+\omega} \bmod \mathcal{N}$ for each i. Thus $\tau(u_i) \equiv u_{i+\omega} \bmod \mathcal{N}$ for $i \neq \omega, d$. We calculate that

$$u'_{\omega-1} \cdots u'_1 u'_d = v_\omega^{-1} b_\infty b_n^{-1} v_d$$
$$u'_{d-1} \cdots u'_\omega = v_d^{-1} b_n v_\omega.$$

Then $\tau(u'_{\omega-1} \cdots u'_1 u'_d) \equiv u'_{d-1} \cdots u'_\omega \bmod \mathcal{N}$ implies that $\tau(v_\omega^{-1} b_\infty b_n^{-1} v_d) \equiv v_d^{-1} b_n v_\omega \bmod \mathcal{N}$. In particular,

$$\chi_m(v_\omega^{-1} v_d) = \chi_m(v_\omega^{-1} b_\infty b_n^{-1} v_d) - 1 = \chi_n(v_d^{-1} b_n v_\omega) - 1 = \chi_n(v_d^{-1} v_\omega). \tag{12}$$

To show that $[\sigma, v] \in_\ell \mathcal{B}$, it only remains to show that $\tau(u_\omega) \equiv u_d \bmod \mathcal{N}$. Observe that

$$u_{\omega-1} \cdots u_1 u_d = u'_{\omega-1} \cdots u'_1 u'_d v_d^{-1} b_n b_\infty^{-1} v_d = v_\omega^{-1} v_d$$
$$u_{d-1} \cdots u_\omega = u'_{d-1} \cdots u'_\omega v_\omega^{-1} b_n^{-1} v_\omega = v_d^{-1} v_\omega.$$

Hence it suffices to show that

$$\tau(v_\omega^{-1} v_d) \equiv v_d^{-1} v_\omega \equiv (v_\omega^{-1} v_d)^{-1} \bmod \mathcal{N},$$

which, because $v_\omega^{-1} v_d \in_\ell \mathcal{B}$, is equivalent to (12). Therefore $[\sigma, v] \in_\ell \mathcal{B}$. Hence in any case we have $v \in_\ell N(\mathcal{B})$. □

5.4. **Power conjugators.** In Lemma 2.15 we showed that $c_\infty \sim c_\infty^\varepsilon$ for any $\varepsilon \in 1 + d\mathbb{Z}_d$. Here we introduce a subgroup of elements $\hat{c}_\varepsilon \in [C_d]^\infty$ which realize these conjugations. We use this subgroup and its conjugates to determine the structure of the constant field extension $\widehat{K}_{f,\ell}$.

Let $v := (1, \ldots, 1, b_n, \ldots, b_n)$ where the first b_n occurs in the $(\omega+1)$th component. Given $\varepsilon = 1 + de \in 1 + d\mathbb{Z}_d$ where $e \in \mathbb{Z}_d$, let $\hat{a}_\varepsilon, \hat{b}_\varepsilon \in [C_d]^\infty$ be the elements defined recursively by

$$\hat{a}_\varepsilon := a_1^d (1, a_\infty^e, a_\infty^{2e}, \ldots, a_\infty^{(d-1)e})(\hat{a}_\varepsilon, \ldots, \hat{a}_\varepsilon) a_1^{-d},$$
$$\hat{b}_\varepsilon := v(1, b_\infty^e, b_\infty^{2e}, \ldots, b_\infty^{(d-1)e})(\hat{b}_\varepsilon, \ldots, \hat{b}_\varepsilon) v^{-1}.$$

Recall that

$$a_\infty = a_1 a_2 \cdots a_n = \sigma(1, \ldots, 1, a_n a_1 \cdots a_{n-1}) = \sigma(1, \ldots, 1, a_n a_\infty a_n^{-1}),$$
$$b_\infty = b_1 b_2 \cdots b_n = \sigma(1, \ldots, 1, b_n, 1, \ldots, 1, b_1 \cdots b_{n-1}) = \sigma(1, \ldots, 1, b_n, 1, \ldots, 1, b_\infty b_n^{-1}),$$
$$c_\infty = \sigma(1, \ldots, 1, c_\infty).$$

Lemma 5.10. *Let $\varepsilon, \varepsilon_1, \varepsilon_2 \in 1 + d\mathbb{Z}_d$, then*

(1) $\hat{a}_\varepsilon a_\infty \hat{a}_\varepsilon^{-1} = a_\infty^\varepsilon$ and $\hat{b}_\varepsilon b_\infty c_\varepsilon^{-1} = b_\infty^\varepsilon$,
(2) $\hat{a}_{\varepsilon_1} \hat{a}_{\varepsilon_2} = \hat{a}_{\varepsilon_1 \varepsilon_2}$ and $\hat{b}_{\varepsilon_1} \hat{b}_{\varepsilon_2} = \hat{b}_{\varepsilon_1 \varepsilon_2}$,
(3) If $0 \leq \ell \leq \infty$, then $\hat{a}_\varepsilon \in_\ell \mathcal{A}$ if and only if $a_\infty \sim_\ell a_\infty^\varepsilon$ in $\mathcal{A}$, and $\hat{b}_\varepsilon \in_\ell \mathcal{B}$ if and only if $b_\infty \sim_\ell b_\infty^\varepsilon$ in $\mathcal{B}$,
(4) $\chi_1(\hat{b}_\varepsilon) = 0$ and $\chi_\ell(\hat{b}_\varepsilon) = \binom{d}{2} e$ for $\ell > 1$.

Proof. (1) Given $\varepsilon = 1 + de \in 1 + d\mathbb{Z}_d$, let $\hat{c}_\varepsilon \in [C_d]^\infty$ be the element defined recursively by

$$\hat{c}_\varepsilon := (1, c_\infty^e, c_\infty^{2e}, \ldots, c_\infty^{(d-1)e})(\hat{c}_\varepsilon, \ldots, \hat{c}_\varepsilon).$$

We calculate

$$\begin{aligned}\hat{c}_\varepsilon c_\infty \hat{c}_\varepsilon^{-1} &= (1, c_\infty^e, c_\infty^{2e}, \ldots, c_\infty^{(d-1)e}) \sigma(1, \ldots, 1, \hat{c}_\varepsilon c_\infty \hat{c}_\varepsilon^{-1})(1, c_\infty^{-e}, c_\infty^{-2e}, \ldots, c_\infty^{-(d-1)e}) \\ &= \sigma(c_\infty^e, \ldots, c_\infty^e, (\hat{c}_\varepsilon c_\infty \hat{c}_\varepsilon^{-1}) c_\infty^{-(d-1)e}).\end{aligned}$$

On the other hand,

$$c_\infty^\varepsilon = c_\infty c_\infty^{de} = \sigma(c_\infty^e, \ldots, c_\infty^e, c_\infty^{1+e}) = \sigma(c_\infty^e, \ldots, c_\infty^e, c_\infty^\varepsilon c_\infty^{-(d-1)e}).$$

Hence Lemma 2.11 implies $\hat{c}_\varepsilon c_\infty \hat{c}_\varepsilon^{-1} = c_\infty^\varepsilon$.

First, the periodic case. Let $w \in [C_d]^\infty$ be the element defined recursively by

$$w = a_1^d(w, \dots, w) = (a_n w, \dots, a_n w).$$

Note that w commutes with σ. Then

$$wc_\infty w^{-1} = \sigma(1, \dots, 1, a_n w c_\infty w^{-1} a_n^{-1}).$$

Since $wc_\infty w^{-1}$ satisfies the same recurrence as a_∞, we conclude that $wc_\infty w^{-1} = a_\infty$ by Lemma 2.11. Observe that

$$\begin{aligned} w\hat{c}_\varepsilon w^{-1} &= a_1^d(1, wc_\infty^e w^{-1}, \dots, wc_\infty^{(d-1)e} w^{-1})(w\hat{c}_\varepsilon w^{-1}, \dots, w\hat{c}_\varepsilon w^{-1})a_1^{-d} \\ &= a_1^d(1, a_\infty^e, \dots, a_\infty^{(d-1)e})(w\hat{c}_\varepsilon w^{-1}, \dots, w\hat{c}_\varepsilon w^{-1})a_1^{-d}, \end{aligned}$$

hence $w\hat{c}_\varepsilon w^{-1}$ satisfies the same recurrence as $\hat{a}_\varepsilon$, implying that $w\hat{c}_\varepsilon w^{-1} = \hat{a}_\varepsilon$ by Lemma 2.11. Therefore

$$\hat{a}_\varepsilon a_\infty \hat{a}_\varepsilon^{-1} = (w\hat{c}_\varepsilon w^{-1})(wc_\infty w^{-1})(w\hat{c}_\varepsilon^{-1} w^{-1}) = w\hat{c}_\varepsilon c_\infty \hat{c}_\varepsilon^{-1} w^{-1} = wc_\infty^\varepsilon w^{-1} = a_\infty^\varepsilon.$$

For the preperiodic case, let $w' \in [C_d]^\infty$ be the element defined recursively by

$$w' = v(w', \dots, w'),$$

where $v = (1, \dots, 1, b_n, \dots, b_n)$ and the first b_n occurs in the $(\omega+1)$th component. Observe that

$$\begin{aligned} w' c_\infty w'^{-1} &= (1, \dots, 1, b_n, \dots, b_n)\sigma(1, \dots, 1, w' c_\infty w'^{-1})(1, \dots, 1, b_n^{-1}, \dots, b_n^{-1}) \\ &= \sigma(1, \dots, 1, b_n, 1, \dots, 1, w' c_\infty w'^{-1} b_n^{-1}). \end{aligned}$$

Hence $w' c_\infty w'^{-1}$ satisfies the same recurrence as b_∞, implying $w' c_\infty w'^{-1} = b_\infty$.

Observe that

$$\begin{aligned} w'\hat{c}_\varepsilon w'^{-1} &= v(1, w' c_\infty^e w'^{-1}, \dots, w' c_\infty^{(d-1)e} w'^{-1})(w'\hat{c}_\varepsilon w'^{-1}, \dots, w'\hat{c}_\varepsilon w'^{-1})v^{-1} \\ &= v(1, b_\infty^e, \dots, b_\infty^{(d-1)e})(w'\hat{c}_\varepsilon w'^{-1}, \dots, w'\hat{c}_\varepsilon w'^{-1})v^{-1}. \end{aligned}$$

Therefore $w'\hat{c}_\varepsilon w'^{-1} = \hat{b}_\varepsilon$, again by Lemma 2.11. We then calculate

$$\hat{b}_\varepsilon b_\infty \hat{b}_\varepsilon^{-1} = (w'\hat{c}_\varepsilon w'^{-1})(w' c_\infty w'^{-1})(w'\hat{c}_\varepsilon^{-1} w'^{-1}) = w'\hat{c}_\varepsilon c_\infty \hat{c}_\varepsilon^{-1} w'^{-1} = w' c_\infty^\varepsilon w'^{-1} = b_\infty^\varepsilon.$$

(2) Let $\varepsilon_1 = 1 + de_1$ and $\varepsilon_2 = 1 + de_2$. We calculate

$$\begin{aligned} \hat{c}_{\varepsilon_1}\hat{c}_{\varepsilon_2} &= (1, c_\infty^{e_1}, \dots, c_\infty^{(d-1)e_1})(\hat{c}_{\varepsilon_1}, \dots, \hat{c}_{\varepsilon_1})(1, c_\infty^{e_2}, \dots, c_\infty^{(d-1)e_2})(\hat{c}_{\varepsilon_2}, \dots, \hat{c}_{\varepsilon_2}) \\ &= (1, c_\infty^{e_1+\varepsilon_1 e_2}, \dots, c_\infty^{(d-1)(e_1+\varepsilon_1 e_2)})(\hat{c}_{\varepsilon_1}\hat{c}_{\varepsilon_2}, \dots, \hat{c}_{\varepsilon_1}\hat{c}_{\varepsilon_2}). \end{aligned}$$

Note that

$$\varepsilon_1\varepsilon_2 = 1 + de_1 + de_2 + d^2 e_1 e_2 = 1 + d(e_1 + \varepsilon_1 e_2).$$

Hence $\hat{c}_{\varepsilon_1}\hat{c}_{\varepsilon_2}$ satisfies the same recurrence as $\hat{c}_{\varepsilon_1\varepsilon_2}$ and we conclude that $\hat{c}_{\varepsilon_1}\hat{c}_{\varepsilon_2} = \hat{c}_{\varepsilon_1\varepsilon_2}$. Conjugating by w and w' gives us $\hat{a}_{\varepsilon_1}\hat{a}_{\varepsilon_2} = \hat{a}_{\varepsilon_1\varepsilon_2}$ and $\hat{b}_{\varepsilon_1}\hat{b}_{\varepsilon_2} = \hat{b}_{\varepsilon_1\varepsilon_2}$.

(3) The forward direction is an immediate consequence of (2). For the converse direction, suppose that there exists some $g \in \mathcal{A}$ such that $ga_\infty g^{-1} =_\ell a_\infty^\varepsilon = \hat{a}_\varepsilon a_\infty \hat{a}_\varepsilon^{-1}$. Then $[g^{-1}\hat{a}_\varepsilon, a_\infty] =_\ell 1$ and Proposition 2.21 implies that $\hat{a}_\varepsilon \in_\ell g\langle\langle a_\infty \rangle\rangle \subseteq \mathcal{A}$. The same argument, *mutatis mutandis*, implies that $\hat{b}_\varepsilon \in_\ell \mathcal{B}$ if and only if $b_\infty \sim_\ell b_\infty^\varepsilon$ in $\mathcal{B}$.

(4) By construction we have $\chi_1(\hat{b}_\varepsilon) = 0$. If $\ell > 1$, we calculate

$$\chi_\ell(\hat{b}_\varepsilon) = \sum_{i=0}^{d-1} \chi_{\ell-1}(b_\infty^{ie}\hat{b}_\varepsilon) = \sum_{i=0}^{d-1} \big(ie + \chi_{\ell-1}(\hat{b}_\varepsilon)\big) = \binom{d}{2}e + d\chi_{\ell-1}(\hat{b}_\varepsilon) \equiv \binom{d}{2}e \bmod d. \qquad \square$$

In Theorem 6.5 we prove that for polynomials f in the preperiodic case, there is a direct correspondence between elements in the Galois group of the constant field extension $\widehat{K}_f/K$ and $\varepsilon \in \mathbb{Z}_d^\times$ such that $\hat{b}_\varepsilon \in \mathcal{B}$. Under this correspondence, the following proposition allows us to precisely determine $\mathrm{Gal}(\widehat{K}_f/K)$. This is a substantial refinement of Proposition 5.8.

Proposition 5.11. *Let $\varepsilon \in 1 + d\mathbb{Z}_d$, let $0 \leq \ell \leq \infty$, and let*

$$\kappa := \begin{cases} d^2/\gcd(d,\omega) & \text{if } (A) \text{ and either } m > 1 \text{ or } d \text{ odd,} \\ d^2/\gcd(d,\omega + d/2) & \text{if } (A) \text{ and } m = 1 \text{ and } d \text{ even,} \\ 4d & \text{if } (B) \text{ or } (C). \end{cases}$$

Then

(1) $a_\infty \sim_\ell a_\infty^\varepsilon$ *in* $\mathcal{A}$ *if and only if* $\varepsilon \equiv 1 \bmod d^{\lfloor (\ell-1)/n \rfloor + 1}$,
(2) $b_\infty \sim_\ell b_\infty^\varepsilon$ *in* $\mathcal{B}$ *if and only if* $\ell \leq n$ *or* $\ell > n$ *and* $\varepsilon \equiv 1 \bmod \kappa$.

Proof. (1) Lemma 5.10 implies that $a_\infty \sim_\ell a_\infty^\varepsilon$ if and only if $\hat{a}_\varepsilon \in_\ell \mathcal{A}$. A simple induction implies that $\hat{a}_\varepsilon \in \mathcal{A}^d$; let $\hat{a}_\varepsilon := (h_1, \ldots, h_d)$. Recall that Proposition 4.17 implies that

$$\mathrm{St}_1 \mathcal{A} = \{(g_1, \ldots, g_d) \in \mathcal{A}^d : \eta_n(g_i) = \eta_n(g_j) \text{ for all } i, j\}.$$

Proposition 4.5 implies that $\mathrm{ord}_\ell(a_n) = d^{\lfloor \ell/n \rfloor}$, hence $\hat{a}_\varepsilon \in_\ell \mathcal{A}$ if and only if $\eta_n(h_i) \equiv \eta_n(h_j) \bmod d^{\lfloor (\ell-1)/n \rfloor}$ for each i and j. The definition of $\hat{a}_\varepsilon$ implies that $\eta_n(h_i) = (i-1)e$. Hence in particular we must have

$$e \equiv \eta_n(h_2) \equiv \eta_n(h_1) \equiv 0 \bmod d^{\lfloor (\ell-1)/n \rfloor}.$$

This is also clearly sufficient. Therefore $a_\infty \sim_\ell a_\infty^\varepsilon$ in $\mathcal{A}$ if and only if $\varepsilon \equiv 1 \bmod d^{\lfloor (\ell-1)/n \rfloor + 1}$.

(2) If $\ell \leq n$, then Lemma 4.16 implies that $\mathcal{B} =_\ell [C_d]^\infty$ and Lemma 2.15 implies that $b_\infty \sim_\ell b_\infty^\varepsilon$ for all $\varepsilon \in 1 + d\mathbb{Z}_d$.

Suppose that $\ell > n$. Lemma 5.10 implies that $b_\infty \sim_\ell b_\infty^\varepsilon$ in $\mathcal{B}$ if and only if $\hat{b}_\varepsilon \in_\ell \mathcal{B}$. Our argument above implies that $\hat{b}_\varepsilon \in_{\ell-1} \mathcal{B}$, hence that $\hat{b}_\varepsilon \in_\ell \mathcal{B}^d$. Thus we may use the criteria provided by Proposition 4.17 to characterize when $\hat{b}_\varepsilon \in_\ell \mathrm{St}_1 \mathcal{B}$. Note that these criteria are vacuous for $\ell \leq n$ but impose constraints for all $\ell > n$. Let us write $\hat{b}_\varepsilon := (g_1, \ldots, g_d)$. The definition of $\hat{b}_\varepsilon$ implies that for all $k \geq 1$,

$$\chi_k(g_i) = \chi_k(b_\infty^{ie} \hat{b}_\varepsilon) = ie + \chi_k(\hat{b}_\varepsilon) = \begin{cases} ie & k = 1, \\ ie + \binom{d}{2}e & k > 1. \end{cases} \tag{13}$$

Suppose (A). Proposition 4.17(1) implies that $\hat{b}_\varepsilon \in_\ell \mathcal{B}$ if and only if $\chi_m(g_i) = \chi_n(g_{i+\omega})$ for each i. By (13), this is equivalent to

$$ie + \chi_m(\hat{b}_\varepsilon) \equiv (i + \omega)e + \chi_n(\hat{b}_\varepsilon) \bmod d,$$

which reduces to

$$\omega e \equiv \chi_m(\hat{b}_\varepsilon) - \chi_n(\hat{b}_\varepsilon) \bmod d,$$

or equivalently

$$d \text{ divides } \begin{cases} \omega e & m > 1, \\ \omega e + \binom{d}{2}e & m = 1. \end{cases}$$

If $m > 1$, this is equivalent to $d/\gcd(d,\omega)$ dividing e, hence $\varepsilon \equiv 1 \bmod d^2/\gcd(d,\omega)$. If $m = 1$, then $\hat{b}_\varepsilon \in_\ell \mathcal{B}$ is equivalent to $d/\gcd\left(d, \omega + \binom{d}{2}\right)$ dividing e. Note that

$$\binom{d}{2} \equiv \begin{cases} 0 \bmod d & \text{if } d \text{ odd,} \\ \frac{d}{2} \bmod d & \text{if } d \text{ even.} \end{cases}$$

Hence

$$\gcd\left(d, \omega + \binom{d}{2}\right) = \begin{cases} \gcd(d, \omega) & \text{if } d \text{ odd,} \\ \gcd(d, \omega + d/2) & \text{if } d \text{ even.} \end{cases}$$

Therefore if $m = 1$ and d odd, then $\hat{b}_\varepsilon \in_\ell \mathcal{B}$ if and only if $\varepsilon \equiv 1 \bmod d^2/\gcd(d,\omega)$. If $m = 1$ and d is even, then $\hat{b}_\varepsilon \in_\ell \mathcal{B}$ if and only if $\varepsilon \equiv 1 \bmod d^2/\gcd(d, \omega + d/2)$.

Suppose (B) or (C). Proposition 4.17(2) implies that $\hat{b}_\varepsilon \in_\ell \mathcal{B}$ if and only if

$$\tau(g_i) \equiv g_{i+\omega} \bmod \mathcal{N}$$

for each $0 \le i < \omega$, or equivalently

$$\tau(b_\infty^{ie}\hat{b}_\varepsilon) \equiv b_n b_\infty^{(i+\omega)e}\hat{b}_\varepsilon b_n^{-1} \bmod \mathcal{N}. \tag{14}$$

Since $b_\infty \equiv b_m b_n \bmod \mathcal{N}$, it follows by a simple induction that for all $k \in \mathbb{Z}$,

$$b_\infty^k \equiv b_m^k b_n^k [b_n, b_m]^{\binom{k}{2}} \bmod \mathcal{N}.$$

Suppose that $a, b, c \in \mathbb{Z}/d\mathbb{Z}$ satisfy

$$\hat{b}_\varepsilon \equiv b_m^a b_n^b [b_n, b_m]^c \bmod \mathcal{N}.$$

We calculate

$$\begin{aligned} \tau(b_\infty^{ie}\hat{b}_\varepsilon) &\equiv (b_n^{ie} b_m^{ie} [b_n, b_m]^{-\binom{ie}{2}})(b_n^a b_m^b [b_n, b_m]^{-c}) \bmod \mathcal{N} \\ &\equiv b_m^{ie+b} b_n^{ie+a} [b_n, b_m]^{-\binom{ie}{2} - c + (ie)^2 + bie + ab} \bmod \mathcal{N}, \\ b_n b_\infty^{(i+\omega)e}\hat{b}_\varepsilon b_n^{-1} &\equiv b_n (b_m^{(i+\omega)e} b_n^{(i+\omega)e} [b_n, b_m]^{\binom{(i+\omega)e}{2}})(b_m^a b_n^b [b_n, b_m]^c) b_n^{-1} \bmod \mathcal{N} \\ &\equiv b_m^{(i+\omega)e+a} b_n^{(i+\omega)e+b} [b_n, b_m]^{\binom{(i+\omega)e}{2} + c + a(i+\omega)e + (i+\omega)e + a} \bmod \mathcal{N}. \end{aligned}$$

Comparing exponents of b_n and b_m we conclude that $\omega e \equiv 0 \bmod d$. Since $\omega = d/2$ in cases (B) and (C), this is equivalent to e being even. Comparing exponents of $[b_n, b_m]$ we have

$$-\binom{ie}{2} - c + (ie)^2 + bie + ab \equiv \binom{(i+\omega)e}{2} + c + a(i+\omega)e + (i+\omega)e + a \bmod d,$$

which simplifies to

$$\frac{de(de-2)}{8} + 2c - ab + a + (a-b)ie \equiv 0 \bmod d. \tag{15}$$

In case (C) we have $d = 2$, $m = 2$, and $i = 0$. Thus Lemma 5.10(4) implies $a = \chi_m(\hat{b}_\varepsilon) = e = \chi_n(\hat{b}_\varepsilon) = b$. Hence (15) reduces to

$$\frac{e(e-1)}{2} \equiv 0 \bmod 2.$$

Since e is even, this is equivalent to 4 dividing e.

Now suppose we are in case (B). Since $m = 1$, Lemma 5.10(4) implies $a = \chi_m(\hat{b}_\varepsilon) = 0$ and $b = \chi_n(\hat{b}_\varepsilon) = \binom{d}{2}e$. Furthermore, Lemma 4.12 implies that

$$2c = 2\chi'(\hat{b}_\varepsilon) = 2\sum_{i=1}^{d} i\chi_{n-1}(g_i) = 2\sum_{i=1}^{d} i^2 e + i\chi_{n-1}(\hat{b}_\varepsilon) \equiv \frac{d(d+1)(2d+1)e}{3} \equiv \frac{(2d^2+1)de}{3} \bmod d.$$

Since (15) holds for all i, we have

$$\begin{aligned} (a-b)e &\equiv 0 \bmod d, \\ \frac{de(de-2)}{8} + 2c - ab + a &\equiv 0 \bmod d. \end{aligned}$$

Substituting for a, b, c gives us

$$\binom{d}{2}e^2 \equiv 0 \bmod d$$

$$\frac{de(de-2)}{8} + \frac{(2d^2+1)de}{3} \equiv 0 \bmod d. \tag{16}$$

Since e is even, $\binom{d}{2}e \equiv 0 \bmod d$. If 3 does not divide d, then $\frac{(2d^2+1)de}{3} \equiv 0 \bmod d$. Let $e' = e/2$. Then

$$0 \equiv \frac{de(de-2)}{8} \equiv \frac{de'(de'-1)}{2} \bmod d.$$

Since d is even, this reduces to $\frac{de'}{2} \equiv 0 \bmod d$, which is equivalent to 4 dividing e. If 3 divides d, then (16) is equivalent to

$$6k = 3e'(de'-1) + 2e$$

for some integer k. This implies 3 divides e and 4 divides e. Hence in either case, (16) is equivalent to 4 dividing e in case (B). Therefore $\hat{b}_\varepsilon \in_\ell \mathcal{B}$ if and only if 4 divides e, or equivalently, if and only if $\varepsilon \equiv 1 \bmod 4d$. □

6. Constant field extensions and the outer action

In this section we study the constant field extensions $\widehat{K}_{f,\ell}/K$ in iterated pre-image extensions $K(f^{-\infty}(t))/K(t)$. We show that for any polynomial f with degree d coprime to $\operatorname{char} K$, the constant field extension is contained in the pro-d cyclotomic extension $K(\zeta_{d^\infty})/K$ and is completely encoded within the structure of $\overline{\operatorname{Arb}} f$. Then we turn to the case of unicritical polynomials where we can leverage our analysis of $\overline{\operatorname{Arb}} f$ to precisely determine $\widehat{K}_{f,\ell}/K$ for all $\ell \geq 1$. In particular, show that $\operatorname{Gal}(\widehat{K}_f/K)$ has a faithful outer action on $\overline{\operatorname{Arb}} f$ which factors through the cyclotomic character. Applying the results of Section 5 tells us how much farther the action factors.

6.1. Preliminary bounds.

First we establish a general upper bound on the constant field extensions of a polynomial.

Proposition 6.1. *Let $f(x) \in K[x]$ be a polynomial of degree $d \geq 2$, and assume that d is coprime to* $\operatorname{char} K$. *Then the constant field extension $\widehat{K}_{f,\ell}$ is contained in $K(\zeta_{d^\ell})$, and hence $\widehat{K}_f \subseteq K(\zeta_{d^\infty})$.*

Proof. Our assumption that f is a polynomial and d is coprime to $\operatorname{char} K$ implies that f has a totally tamely ramified fixed point at ∞. Therefore

$$K(\!(1/t)\!)(f^{-\ell}(t)) = K(\zeta_{d^\ell})(\!(1/t^{1/d^\ell})\!).$$

Hence

$$\widehat{K}_{f,\ell} = K^{\mathrm{sep}} \cap K(f^{-\ell}(t)) \subseteq K^{\mathrm{sep}} \cap K(\!(1/t)\!)(f^{-\ell}(t)) = K(\zeta_{d^\ell}).$$ □

This upper bound on the constant field extension need not be sharp but a result of Hamblen and Jones [HJ24] shows that it is when $f(x) = ax^d + b$ has a periodic critical point.

Corollary 6.2. *Let $f(x) = ax^d + b \in K[x]$ where d is coprime to* $\operatorname{char} K$ *and* 0 *is periodic under f, then $\widehat{K}_f = K(\zeta_{d^\infty})$.*

Proof. Hamblen and Jones [HJ24, Thm. 2.1] prove, in this situation, that $K(\zeta_{d^\infty}) \subseteq \widehat{K}_f$. Therefore Proposition 6.1 implies that $K(\zeta_{d^\infty}) = \widehat{K}_f$. □

The situation when 0 is strictly preperiodic under $f(x) = ax^d + b$ is quite different. Note that $K(\zeta_d) \subseteq \widehat{K}_f$ for any unicritical polynomial $f(x) = ax^d + b$ since

$$K(f^{-1}(t)) = K(\zeta_d, \left(\tfrac{t-b}{a}\right)^{1/d}). \tag{17}$$

In the preperiodic case we can prove the following coarse finiteness result.

Proposition 6.3. *If $f(x) = ax^d + b \in K[x]$ where d is coprime to the characteristic of K and 0 is strictly preperiodic under f, then $\widehat{K}_f$ is a finite extension of $K(\zeta_d)$, hence of K, unless $d = 2$ and f is conjugate over K to $x^2 - 2$, the degree 2 Chebyshev polynomial.*

Proof. Recall that $\mathrm{Gal}(\widehat{K}_f/K) \cong \mathrm{Arb}\, f/\overline{\mathrm{Arb}}\, f$. Corollary 4.23 implies that there is a $w \in [C_d]^\infty$ such that $w\,\overline{\mathrm{Arb}}\, f w^{-1} = \mathcal{B}$ for the appropriate values of d, m, n, ω. Since $\overline{\mathrm{Arb}}\, f$ is a normal subgroup of $\mathrm{Arb}\, f$, it follows that $w\,\mathrm{Arb}\, f w^{-1} \subseteq N(\mathcal{B})$ where $N(\mathcal{B})$ is the normalizer of $\mathcal{B}$ in $[C_d]^\infty$. Thus we have an injective homomorphism $\mathrm{Gal}(\widehat{K}_f/K) \hookrightarrow N(\mathcal{B})/\mathcal{B}$.

On the other hand, Proposition 6.1 implies that $\mathrm{Gal}(\widehat{K}_f/K)$ is a quotient of $\mathrm{Gal}(K(\zeta_{d^\infty})/K)$, which is isomorphic to a finitely generated subgroup of $\mathbb{Z}_d^\times$. Proposition 5.8 implies that $N(\mathcal{B})/\mathcal{B}$ has a finite exponent unless $(d, m, n, \omega) = (2, 1, 2, 1)$. A finitely generated abelian group with a finite exponent is finite, therefore $\mathrm{Gal}(\widehat{K}_f/K)$ is finite in all but one case.

If $(d, m, n, \omega) = (2, 1, 2, 1)$, then f is conjugate over K to a polynomial of the form $f(x) = x^2 + c$ such that $f^2(c) = f(c)$ and $c \neq 0$. The only such polynomial is $x^2 - 2$. □

One may extract an explicit degree bound on $\widehat{K}_f/K$ from the argument in Proposition 6.3, however in Theorem 1.6 we refine this result to an exact determination of $\widehat{K}_{f,\ell}/K$ for all $\ell \geq 1$.

The finiteness of the constant field extensions shown in Proposition 6.3 stems from the fact that $\overline{\mathrm{Arb}}\, f$ is a branch group in each preperiodic case except case (D) (see Proposition 4.14). In case (D) and the periodic case, the group $\overline{\mathrm{Arb}}\, f$ is not a branch group and the constant field extension is all or nearly all of the cyclotomic extension $K(\zeta_{d^\infty})/K$. This suggests the following question:

Question. Does $\overline{\mathrm{Arb}}\, f$ being a branch group imply that $\widehat{K}_f/K$ is a finite extension for any polynomial $f(x)$? For any rational function?

As the proof of Proposition 6.3 shows, for an affirmative answer it would suffice to prove that $\overline{\mathrm{Arb}}\, f$ regular branch implies $N(\overline{\mathrm{Arb}}\, f)/\overline{\mathrm{Arb}}\, f$ has finite exponent.

6.2. **Outer actions and sharper bounds.** In this section, we show that the structure of the constant field extension can be interpreted group-theoretically, in terms of an outer Galois action. This allows us to refine the results of the preceding subsection and explicitly calculate the constant field extensions.

In the preperiodic cases except for (D), Proposition 6.3 implies that $K(\zeta_d) \subseteq \widehat{K}_{f,\ell} \subseteq K(\zeta_{d^\ell})$. Combining Proposition 5.11 with the branch cycle lemma (Lemma 6.4) we give a precise determination of $\widehat{K}_{f,\ell}$.

Let $P \subseteq \mathbb{P}^1_{K^{\mathrm{sep}}}$ be a Galois-stable set of points and let $K(t)_P/K(t)$ denote the maximal tamely ramified extension of $K^{\mathrm{sep}}(t)$ unramified outside of P. As discussed in the proof of Lemma 3.11, the Galois group of $K^{\mathrm{sep}}(t)_P/K^{\mathrm{sep}}(t)$ is topologically generated by inertia generators over the points in P. The branch cycle lemma (first appearing in Fried [Fri73]) constrains how $\mathrm{Gal}(\widehat{K}_{f,\ell}/K)$ acts on these inertia generators. Our proof of the branch cycle lemma generalizes one appearing in Malle and Matzat [MM99, Thm. 2.6] for $\mathbb{Q}$. In characteristic zero, their proof requires no changes; tameness allows their argument to be extended into positive characteristic.

Lemma 6.4 (Branch Cycle Lemma)**.** *Let K be a field and let $P \subseteq \mathbb{P}^1_{K^{\mathrm{sep}}}$ be a Galois-stable set. Let $\chi_{\mathrm{cyc}} : \mathrm{Gal}(K^{\mathrm{sep}}/K) \to \widehat{\mathbb{Z}}^\times$ be the cyclotomic character defined by*

$$\gamma(\zeta) = \zeta^{\chi_{\mathrm{cyc}}(\gamma)}$$

for all $\gamma \in \mathrm{Gal}(K^{\mathrm{sep}}/K)$ and all roots of unity $\zeta \in K^{\mathrm{sep}}$. Let $b \in P$ and let γ_b be a topological generator for an inertia group over b in $\mathrm{Gal}(K^{\mathrm{sep}}(t)_P/K^{\mathrm{sep}}(t))$. If $\tau \in \mathrm{Gal}(K^{\mathrm{sep}}/K)$ and $\tilde{\tau} \in \mathrm{Gal}(K(t)_P/K(t))$ is any lift, then

$$\tilde{\tau}\gamma_b\tilde{\tau}^{-1} \sim \gamma_{\tau(b)}^{\chi_{\mathrm{cyc}}(\tau)},$$

where the conjugacy takes place in $\mathrm{Gal}(K^{\mathrm{sep}}(t)_P/K^{\mathrm{sep}}(t))$.

Proof. Let $\tau \in \mathrm{Gal}(K^{\mathrm{sep}}/K)$ and let $b \in P$. After a change of coordinates over K, we may assume for simplicity that b and $\tau(b)$ are both finite. Embedding $K^{\mathrm{sep}}(t)_P$ into a separable closure of $K^{\mathrm{sep}}(\!(t-b)\!)$ we have (by our tameness hypothesis) that $K^{\mathrm{sep}}(t)_P K^{\mathrm{sep}}(\!(t-b)\!)/K^{\mathrm{sep}}(\!(t-b)\!)$ is generated by elements $(t-b)^{1/n}$ for all n coprime to $\operatorname{char} K$ and that the Galois group of this extension is topologically generated by a lift of γ_b. We may assume that these elements are compatible in the sense that $((t-b)^{1/mn})^m = (t-b)^{1/n}$. Let (ζ_n) be primitive nth roots of unity in K^{sep} such that

$$\gamma_b(t-b)^{1/n} = \zeta_n(t-b)^{1/n}.$$

Note that γ_b is determined by how it acts on the elements $(t-b)^{1/n}$.

The element $\tilde{\tau}\gamma_b\tilde{\tau}^{-1} \in \mathrm{Gal}(K^{\mathrm{sep}}(t)_P/K^{\mathrm{sep}}(t))$ generates an inertia group over $\tau(b)$. The element $\tilde{\tau}$ may be extended to a K^{sep}-isomorphism of completions such that

$$(t-\tau(b))^{1/n} := \tilde{\tau}((t-b)^{1/n}).$$

Replacing $\gamma_{\tau(b)}$ by a conjugate in $\mathrm{Gal}(K^{\mathrm{sep}}(t)_P/K^{\mathrm{sep}}(t))$, we may assume that

$$\gamma_{\tau(b)}(t-\tau(b))^{1/n} = \zeta_n(t-\tau(b))$$

for all n. We calculate

$$\begin{aligned}\tilde{\tau}\gamma_b\tilde{\tau}^{-1}(t-\tau(b))^{1/n} &= \tilde{\tau}(t-b)^{1/n}\\ &= \tilde{\tau}\zeta_n(t-b)^{1/n}\\ &= \zeta_n^{\chi_{\mathrm{cyc}}(\tau)}(t-\tau(b))^{1/n}\\ &= \gamma_{\tau(b)}^{\chi_{\mathrm{cyc}}(\tau)}(t-\tau(b))^{1/n}.\end{aligned}$$

Since $\tilde{\tau}\gamma_b\tilde{\tau}^{-1}$ and $\gamma_{\tau(b)}^{\chi_{\mathrm{cyc}}(\tau)}$ are determined by how they acts on $(t-\tau(b))^{1/n}$ for all n coprime to $\operatorname{char} K$, we conclude that $\tilde{\tau}\gamma_b\tilde{\tau}^{-1} \sim \gamma_{\tau(b)}^{\chi_{\mathrm{cyc}}(\tau)}$ in $\mathrm{Gal}(K^{\mathrm{sep}}(t)_P/K^{\mathrm{sep}}(t))$. □

Theorem 6.5. *Let K be a field, let $0 \le \ell \le \infty$, and let $f(x) \in K[x]$ be a polynomial with degree d coprime to* $\operatorname{char} K$. *If $\tau \in \mathrm{Gal}(K^{\mathrm{sep}}/K)$, then τ fixes $\widehat{K}_{f,\ell}$ if and only if $\gamma_\infty \sim_\ell \gamma_\infty^{\chi_{\mathrm{cyc}}(\tau)}$ in* $\overline{\mathrm{pIMG}}\, f$.

Proof. Let $P_f \subseteq \mathbb{P}^1_{K^{\mathrm{sep}}}$ be the post-critical set of f. Then by assumption P_f is a Galois-stable set of points. Hence we have the following commutative diagram with exact rows:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathrm{Gal}(K^{\mathrm{sep}}(t)_{P_f}/K^{\mathrm{sep}}(t)) & \longrightarrow & \mathrm{Gal}(K(t)_{P_f}/K(t)) & \longrightarrow & \mathrm{Gal}(K^{\mathrm{sep}}/K) & \longrightarrow & 0\\ & & \downarrow \bar{\rho} & & \downarrow \rho & & \downarrow \widehat{\rho} & & \\ 0 & \longrightarrow & \rho_\ell(\overline{\mathrm{pIMG}}\, f) & \longrightarrow & \rho_\ell(\mathrm{pIMG}\, f) & \longrightarrow & \mathrm{Gal}(\widehat{K}_{f,\ell}/K) & \longrightarrow & 0\end{array} \tag{18}$$

Let $\tau \in \mathrm{Gal}(K^{\mathrm{sep}}/K)$. First suppose that τ fixes $\widehat{K}_{f,\ell}$, hence belongs to the kernel of $\widehat{\rho}$. Thus any lift $\tilde{\tau}$ of τ acts trivially on $\rho_\ell(\overline{\mathrm{pIMG}}\, f)$, which means that $\tilde{\tau}\gamma_\infty\tilde{\tau}^{-1} \sim_\ell \gamma_\infty$ in $\overline{\mathrm{pIMG}}\, f$. On the other hand, the branch cycle lemma implies that $\tilde{\tau}\gamma_\infty\tilde{\tau}^{-1} \sim \gamma_\infty^{\chi_{\mathrm{cyc}}(\tau)}$ in $\overline{\mathrm{pIMG}}\, f$. Therefore $\gamma_\infty \sim_\ell \gamma_\infty^{\chi_{\mathrm{cyc}}(\tau)}$ in $\overline{\mathrm{pIMG}}\, f$.

Next suppose that $\gamma_\infty \sim_\ell \gamma_\infty^{\chi_{\mathrm{cyc}}(\tau)}$ in $\overline{\mathrm{pIMG}}\, f$. If $\tilde{\tau}$ is any lift of τ, then the branch cycle lemma implies that $\tilde{\tau}\gamma_\infty\tilde{\tau}^{-1} \sim \gamma_\infty^{\chi_{\mathrm{cyc}}(\tau)} \sim_\ell \gamma_\infty$ in $\overline{\mathrm{pIMG}}\, f$. Let $\delta \in \overline{\mathrm{pIMG}}\, f$ be an element such that $\tilde{\tau}\gamma_\infty\tilde{\tau}^{-1} =_\ell \delta\gamma_\infty\delta^{-1}$. Then $[\delta^{-1}\tilde{\tau}, \gamma_\infty] =_\ell 1$. Since f is a polynomial, Proposition 3.13 and Proposition 2.21 together imply that $\tilde{\tau} \in_\ell \delta\langle\!\langle\gamma_\infty\rangle\!\rangle \subseteq \overline{\mathrm{pIMG}}\, f$. The exactness of the bottom row of (18) implies that $\widehat{\rho}(\tau) = 1$. Hence τ fixes $\widehat{K}_{f,\ell}$. □

Theorem 6.5 shows that for polynomials, the constant field extension is entirely encoded in the structure of the geometric profinite iterated monodromy group. As a first illustration of this result we determine the constant field extension for post-critically infinite polynomials.

Proposition 6.6 (PCI Constant Field)**.** *Let $f(x) = ax^d + b$ be post-critically infinite. Then $\widehat{K}_{f,\ell} = K(\zeta_d)$ for $1 \le \ell \le \infty$.*

Proof. In Proposition 3.14 we showed that $\overline{\mathrm{Arb}}\, f = [C_d]^\infty$ when f is post-critically infinite. Proposition 3.13 implies that γ_∞ is a strict odometer and Lemma 2.15 implies that $\gamma_\infty \sim_\ell \gamma_\infty^\varepsilon$ in $[C_d]^\infty$ if and only if $\varepsilon \equiv 1 \bmod d$. Therefore Theorem 6.5 implies that $\widehat{K}_{f,\ell}$ is the fixed field of $K(\zeta_{d^\infty})$ corresponding to the subgroup of $\mathbb{Z}_d^\times$ generated by all $\varepsilon \equiv 1 \bmod d$, which is precisely $K(\zeta_d)$. □

Next we use Theorem 6.5 to determine $\widehat{K}_{f,\ell}$ for all $1 \le \ell \le \infty$ in the periodic case; this also provides an alternative proof of Corollary 6.2. Let $\chi_{\mathrm{cyc},d} : \mathrm{Gal}(K(\zeta_{d^\infty})/K) \to \mathbb{Z}_d^\times$ denote the pro-d cyclotomic character of K. Observe that

$$\mathrm{Gal}(K(\zeta_{d^\infty})/K(\zeta_\kappa)) = \{\tau \in \mathrm{Gal}(K(\zeta_{d^\infty})/K) : \chi_{\mathrm{cyc},d}(\tau) \equiv 1 \bmod \kappa\}.$$

Proposition 6.7 (Periodic Constant Field)**.** *Let $f(x) = ax^d + b$ and suppose that 0 is periodic under f with period n. Then $\widehat{K}_{f,\ell} = K(\zeta_{d^{\lfloor \ell/n \rfloor+1}})$ for $\ell \ge 1$. In particular, $\widehat{K}_f = K(\zeta_{d^\infty})$.*

Proof. Recall that $\overline{\mathrm{Arb}}\, f = \langle\langle c_1, \ldots, c_n \rangle\rangle$ and that $c_\infty = c_1 c_2 \cdots c_n$ is the image of γ_∞ in $\overline{\mathrm{Arb}}\, f$. Let $\mathcal{A} = \mathcal{A}(d,n)$. Corollary 4.23 implies that there exists a $w \in [C_d]^\infty$ and elements $u_i \in \mathcal{A}$ such that

$$w c_i w^{-1} = u_i a_i u_i^{-1}$$

for each $1 \le i \le n$. Therefore

$$w c_\infty w^{-1} = (u_1 a_1 u_1^{-1}) \cdots (u_n a_n u_n^{-1}).$$

Note that $wc_\infty w^{-1}$ is a strict odometer in $\mathcal{A}$, hence there exists a $v \in [C_d]^\infty$ such that $v^{-1} a_\infty v = w c_\infty w^{-1} \in \mathcal{A}$. Thus Lemma 5.9 implies that $v \in N(\mathcal{A})$. Therefore $c_\infty \sim_\ell c_\infty^\varepsilon$ in $\overline{\mathrm{Arb}}\, f$ if and only if $a_\infty \sim_\ell a_\infty^\varepsilon$ in $\mathcal{A}$. Proposition 5.11 implies that $a_\infty \sim_\ell a_\infty^\varepsilon$ in $\mathcal{A}$ if and only if $\varepsilon \equiv 1 \bmod d^{\lfloor (\ell-1)/n \rfloor + 1}$. Hence Theorem 6.5 implies that $\widehat{K}_{f,\ell} = K(\zeta_{d^{\lfloor (\ell-1)/n \rfloor+1}})$. □

Finally we determine $\widehat{K}_{f,\ell}$ in the preperiodic case.

Theorem 6.8 (Preperiodic Constant Field)**.** *Let $f(x) = ax^d + b \in K[x]$ where d is coprime to the characteristic of K and 0 is strictly preperiodic. Let $m < n$ be the smallest integers such that $f^m(b) = f^n(b)$ and let $1 < \omega < d$ be such that $f^n(0) = \zeta_d^\omega f^m(0)$. If $(d,m,n,\omega) \neq (2,1,2,1)$ (case (D)), then*

$$K(\zeta_d) \subseteq \widehat{K}_f \subseteq K(\zeta_{2d^2}).$$

More precisely, if $1 \le \ell \le \infty$, then for $\ell \le n$ we have $\widehat{K}_{f,\ell} = K(\zeta_d)$ and for $\ell > n$ we have

$$\widehat{K}_{f,\ell} = \begin{cases} K(\zeta_{d^2/\gcd(d,\omega)}) & \text{if } (A) \text{ and either } m > 1 \text{ or } d \text{ odd}, \\ K(\zeta_{d^2/\gcd(d,\omega+d/2)}) & \text{if } (A) \text{ and } m = 1 \text{ and } d \text{ even}, \\ K(\zeta_{4d}) & \text{if } (B) \text{ or } (C), \\ K(\zeta_{2^\ell} + \zeta_{2^\ell}^{-1}) & \text{if } (D). \end{cases}$$

Proof. Suppose we are not in case (D). The lower bound on $\widehat{K}_f$ comes from (17). Recall that $\overline{\mathrm{Arb}}\, f = \langle\langle c_1, \ldots, c_n \rangle\rangle$ and that $c_\infty = c_1 c_2 \cdots c_n$ is the image of γ_∞ in $\overline{\mathrm{Arb}}\, f$. Let $\mathcal{B} = \mathcal{B}(d,m,n,\omega)$. Corollary 4.23 implies that there exists a $w \in [C_d]^\infty$ and $u_i \in \mathcal{B}$ such that $w c_i w^{-1} = u_i b_i u_i^{-1}$ for each $1 \le i \le n$. Therefore

$$w c_\infty w^{-1} = (u_1 b_1 u_1^{-1}) \cdots (u_n b_n u_n^{-1}).$$

Note that $wc_\infty w^{-1}$ is a strict odometer in $\mathcal{B}$, hence is conjugate to $b_\infty := b_1 b_2 \cdots b_n$ in $[C_d]^\infty$. Proposition 5.9 implies that $wc_\infty w^{-1} \sim b_\infty$ in $N(\mathcal{B})$. Altogether this implies that $c_\infty \sim_\ell c_\infty^\varepsilon$ in $\overline{\mathrm{Arb}}\, f$ if and only if $b_\infty \sim_\ell b_\infty^\varepsilon$ in $\mathcal{B}$.

Proposition 5.11 implies that $b_\infty \sim_\ell b_\infty^\varepsilon$ in $\mathcal{B}$ for all $\varepsilon \in 1 + d\mathbb{Z}_d$ when $\ell \leq n$, and $b_\infty \sim_\ell b_\infty^\varepsilon$ in $\mathcal{B}$ for $\ell > n$ if and only if $\varepsilon \equiv 1 \bmod \kappa$ where

$$\kappa := \begin{cases} d^2/\gcd(d,\omega) & \text{if } (A) \text{ and either } m > 1 \text{ or } d \text{ odd}, \\ d^2/\gcd(d,\omega + d/2) & \text{if } (A) \text{ and } m = 1 \text{ and } d \text{ even}, \\ 4d & \text{if } (B) \text{ or } (C). \end{cases}$$

This translates directly via Theorem 6.5 to the calculations for $\widehat{K}_f$ in cases (A), (B), and (C).

In case (D), $\mathcal{B}$ is isomorphic to the pro-2 dihedral group (Lemma 4.13). We have $b_\infty \sim_\ell b_\infty^\varepsilon$ in $\mathcal{B}$ if and only if $\varepsilon \equiv -1 \bmod 2^\ell$. Hence in this case $\widehat{K}_{f,\ell} = K(\zeta_{2^\ell} + \zeta_{2^\ell}^{-1})$. □

Appendix A. Computer calculations

We use Laurent Bartholdi's FR GAP package [Bar24] to verify calculations in case $(d, m, n, \omega) = (2, 2, 3, 1)$ (case (C)). The code below calculates the indices $[\mathcal{B} : \mathcal{N}]_3$, $[\mathcal{B} : \mathcal{N}]_4$, and the index $[\mathcal{B} : \mathcal{N}' \cap \mathcal{B}]_4$ where $\mathcal{N}'$ is the normal closure of $\mathcal{N}$ in $[C_2]^\infty$.

```
LoadPackage("fr");

# Construct the (discrete) IMG
B := FRGroup("b1=(1,2)", "b2=<1,b1>", "b3=<b3,b2>");
AssignGeneratorVariables(B);

# Branching subgroup
N := NormalClosure(B, [b1, Comm(b2,Comm(b2,b3)), Comm(b3,Comm(b2,b3))]);


# Level 3 and 4 truncations
B3 := PermGroup(B,3);
N3 := PermGroup(N,3);
B4 := PermGroup(B,4);
N4 := PermGroup(N,4);

# Truncated $[C_2]^4$ via FRGroups to ensure compatibility
C4 := PermGroup(FRGroup("c1=(1,2)", "c2=<1,c1>", "c3=<1,c2>", "c4=<1,c3>"),4);

# Normalizer of $B_4$ in $[C_2]^4$
NB4 := Normalizer(C4, B4);

# Normal closure of the branching subgroup in $[C_2]^4$
Np := NormalClosure(C4,N4);

# Results
Print("[B:N]_4 = ", Index(B4,N4),"\n",
      "[B:N]_3 = ", Index(B3,N3), "\n",
      "[B:N'\\cap B]_4 = ", Index(B4,Intersection(Np,B4)),"\n",
      "N(B)_4 normalizes N_4? ",IsNormal(NB4,N4));
```

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