10
The Bockstein Spectral Sequence

"Unlike the previous proofs which made strong use of the infinitesimal structure of Lie groups, the proof given here depends only on the homological structure and can be applied to H-spaces . . . "

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In the early days of combinatorial topology, a topological space of finite type (a polyhedron) had its integral homology determined by sequences of integers—the Betti numbers and torsion coefficients. That this numerical data ought to be interpreted algebraically is attributed to Emmy Noether (see [Alexandroff-Hopf35]).

The torsion coefficients are determined by the the Universal Coefficient theorem; there is a short exact sequence

$$0 \to H_n(X) \otimes \mathbb{Z}/r\mathbb{Z} \xrightarrow{\rho} H_n(X; \mathbb{Z}/r\mathbb{Z}) \to \mathrm{Tor}_{\mathbb{Z}}(H_{n-1}(X), \mathbb{Z}/r\mathbb{Z}) \to 0.$$

To unravel the integral homology from the mod r homology there is also the **Bockstein homomorphism**: Consider the short exact sequence of coefficient rings where red_r is reduction mod r:

$$0 \to \mathbb{Z} \xrightarrow{-\times r} \mathbb{Z} \xrightarrow{\mathrm{red}_r} \mathbb{Z}/r\mathbb{Z} \to 0.$$

The singular chain complex of a space X is a complex, $C_*(X)$, of free abelian groups. Hence we obtain another short exact sequence of chain complexes

$$0 \to C_*(X) \xrightarrow{-\times r} C_*(X) \xrightarrow{\mathrm{red}_r} C_*(X) \otimes \mathbb{Z}/r\mathbb{Z} \to 0,$$

and this gives a long exact sequence of homology groups,

$$\cdots \to H_n(X) \xrightarrow{-\times r} H_n(X) \xrightarrow{\mathrm{red}_{r*}} H_n(X; \mathbb{Z}/r\mathbb{Z}) \xrightarrow{\partial} H_{n-1}(X) \to \cdots .$$

When an element $u \in H_{n-1}(X)$ satisfies $ru = 0$, then, by exactness, there is an element $\bar{u} \in H_{n+1}(X; \mathbb{Z}/r\mathbb{Z})$ with $\partial(\bar{u}) = u$. To unpack what is happening here, we write $\bar{u} = \{c \otimes 1\} \in H_n(X; \mathbb{Z}/r\mathbb{Z})$. Since $\partial(c \otimes 1) = 0$ and $\partial(c) \neq 0$, we see that $\partial(c) = rv$ and the boundary homomorphism takes $\bar{u}$ to $\{v\} \in H_{n-1}(X)$. The Bockstein homomorphism is defined by

$$\beta\colon H_n(X; \mathbb{Z}/r\mathbb{Z}) \to H_{n-1}(X; \mathbb{Z}/r\mathbb{Z}), \quad \bar{u} = \{c \otimes 1\} \mapsto \{v \otimes 1\} = \{\frac{1}{r}\partial c \otimes 1\}.$$

This mapping was introduced by [Bockstein43]. The Bockstein spectral sequence is derived from the long exact sequence when we treat it as an exact couple (§10.1).

One of the motivating problems for the development of the Bockstein spectral sequence comes from the study of Lie groups. Recall that a space X is **torsion-free** when all its torsion coefficients vanish, that is, when $H_i(X)$ is a free abelian group for each i. A remarkable result due to [Bott54, 56] identifies a particular class of torsion-free spaces.

Theorem 10.1. *If (G, e, μ) denotes a connected, simply-connected, compact Lie group, then ΩG is torsion-free.*

Bott's proof of this theorem is a tour-de-force in the use of the analytic structure of a Lie group. The transition to topological consequences is via Morse theory. The essential ingredient is the study of the **diagram** D **associated to** G—a system of subspaces of the tangent space to a maximal torus $T \subset G$ which may be described in terms of "root forms" on G. The fundamental chambers in D carry indices that determine the Poincaré series of the based loop space ΩG. In fact, the Poincaré series has nonzero entries only in even degrees. From this condition for all coefficient fields, it follows that ΩG is torsion-free.

By way of contrast, we recall a celebrated result of [Hopf41]. H-spaces and Hopf algebras made their first appearance in this landmark paper where results about the algebraic topology of Lie groups were shown to depend only on the more fundamental notion of an H-space structure.

Suppose (X, x_0, μ) is an H-space. The commutativity of the diagram

$$\begin{array}{ccccc} X \times X & \xrightarrow{\Delta \times \Delta} & X \times X \times X \times X & \xrightarrow{1 \times T \times 1} & X \times X \times X \times X \\ \downarrow{\scriptstyle \mu} & & & & \downarrow{\scriptstyle \mu \times \mu} \\ X & & \xrightarrow[\Delta]{\hspace{6em}} & & X \times X. \end{array}$$

implies that the coproduct on homology,

$$\Delta_*\colon H_*(X; k) \to H_*(X; k) \otimes H_*(X; k),$$

is an algebra map with respect to the product μ_*. Thus $(H_*(X; k), \mu_*, \Delta_*)$ satisfies the defining property of a Hopf algebra. This algebraic observation implies the following structure result.

Theorem 10.2 ***(***[Hopf41]***).*** *If (X, x_0, μ) is an H-space of the homotopy type of a finite CW-complex and k is a field of characteristic zero, then $H^*(X;k)$ is an exterior algebra on generators of odd degree.*

PROOF: Consider the graded vector space of indecomposable elements in $H^*(X;k)$:

$$Q(H^*(X;k)) = H^+(X;k)/H^+(X;k) \smile H^+(X;k).$$

Let $Q(H^*(X;k)) = k\{x_1, x_2, \dots, x_q\}$ with the generators ordered by degree, $\deg x_1 \leq \deg x_2 \leq \cdots \leq \deg x_q$. Let $x = x_j$ denote first even-dimensional generator, of degree $2m$, and A_x denote the sub-Hopf algebra generated by the odd-dimensional classes x_1 through x_{j-1}.

Recall that if $C \subset B$ is a normal sub-Hopf algebra of C, that is, $I(C) \cdot B = B \cdot I(C)$, then $C//B = B/I(C) \cdot B$ is the quotient Hopf algebra and $I(C)$ and $I(B)$ denote the kernels of the augmentation.

Consider the short exact sequence of Hopf algebras:

$$0 \to A_x \longrightarrow H^*(X;k) \longrightarrow H^*(X;k)//A_x \to 0.$$

Since $H^*(X;k)$ is commutative, A_x is normal in $H^*(X;k)$. The class x goes to a primitive class $\bar{x}$ in $H^*(X;k)//A_x$, that is, $\mu^*(\bar{x}) = \bar{x} \otimes 1 + 1 \otimes \bar{x}$. Since $H^*(X;k)//A_x$ is also a Hopf algebra, we have that μ^* is a homomorphism of algebras and so $\mu^*((\bar{x})^n) = (\mu^*(\bar{x}))^n = (1 \otimes \bar{x} + \bar{x} \otimes 1)^n$. It follows, as in the proof of the binomial theorem, that, for all $n > 0$,

$$\mu^*((\bar{x})^n) = \sum\nolimits_{i=0}^{n} \binom{n}{i} (\bar{x})^i \otimes (\bar{x})^{n-i} \qquad \text{where } (\bar{x})^0 = 1.$$

Since X has the homotopy type of a finite CW-complex, for some N, $H^s(X;k) = \{0\}$ for $s \geq N$. It follows that $(\bar{x})^i = 0$ whenever $2mi \geq N$. However, for the first such t,

$$\mu^*((\bar{x})^t) = \sum\nolimits_{i=0}^{t} \binom{t}{i} (\bar{x})^i \otimes (\bar{x})^{t-i} \neq 0$$

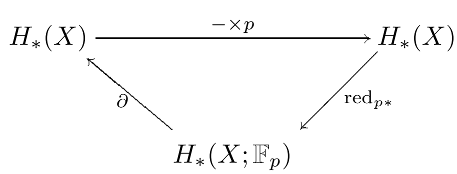
because $\binom{t}{i} \neq 0$ in k and $(\bar{x})^i \otimes (\bar{x})^{t-i} \neq 0$ when $i \geq 1$. Thus, the appearance of $\bar{x} \neq 0$, a primitive of even degree in $H^*(X;k)//A_x$, implies that $(\bar{x})^t \neq 0$ for all $t \geq 1$, and $H^*(X;k)//A_x$ is of infinite dimension over k. Since $H^*(X;k)//A_x$ is a quotient of $H^*(X;k)$, this contradicts the finiteness assumption on X. It follows that $H^*(X;k)$ has only odd degree algebra generators. The theorem follows from Theorem 6.36—a graded commutative Hopf algebra on odd generators is an exterior algebra on those generators. □

The interplay between the homotopy-theoretic properties of H-spaces and the analytic properties of Lie groups has deepened our understanding of such spaces considerably. At first it was believed that H-spaces with nice enough properties need be Lie groups ([Curtis, M71] reviewed this program), but the powerful methods of localization at a prime soon revealed a much richer field of examples including the so-called "Hilton-Roitberg criminal" ([Hilton-Roitberg69]), a manifold and H-space of non-Lie type. The generalization of properties of Lie groups to H-spaces of the homotopy type of a finite complex fueled considerable efforts that include the development of the Bockstein spectral sequence ([Browder61]), the introduction of A_n-structures ([Stasheff63]), new applications of localization ([Zabrodsky70], [Hilton-Mislin-Roitberg75]), and the solution of the torsion conjecture ([Lin82], [Kane86]), which states that ΩX is torsion-free for X a finite, simply-connected H-space. [Dwyer-Wilkerson94] have applied the methods of homotopy fixed point sets developed by [Miller84] and [Lannes92] to recover the algebraic topology of Lie groups from a completely homotopy-theoretical viewpoint ([Dwyer98]).

In this chapter we develop the properties of the Bockstein spectral sequence, especially for applications to H-spaces. We introduce the remarkable notion of ∞-implications due to [Browder61] and apply it to derive certain finiteness results. We then consider some unexpected applications of the Bockstein spectral sequence to differential geometry and to the Adams spectral sequence. The short exact sequence of coefficients that characterizes the Bockstein spectral sequence can also be generalized to other homology theories and to homotopy groups with coefficients (introduced by [Peterson56]). This leads to other Bockstein spectral sequences—for mod r homotopy groups, and for Morava K-theory—whose properties have played a key role in some of the major developments in homotopy theory. These ideas are discussed in §10.2.

10.1 The Bockstein spectral sequence

Although it has a modest form, the Bockstein spectral sequence has led to some remarkable insights, particularly in the study of H-spaces. We recall the construction of the Bockstein spectral sequence here (§2.2). Fix a prime p and carry out the construction of the long exact sequence associated to the exact sequence of coefficients, $0 \to \mathbb{Z} \to \mathbb{Z} \to \mathbb{F}_p \to 0$. Following a suggestion of John Moore, [Browder61] interpreted the long exact sequence as an exact couple:



We denote the E^1-term by $B_n^1 \cong H_n(X;\mathbb{F}_p)$. The first differential is given by $d^1 = \partial \circ \mathrm{red}_{p*} = \beta$, the Bockstein homomorphism. The spectral sequence is singly-graded and the results of Chapter 2 apply to give the following theorem.

Theorem 10.3. *Let X be a connected space of finite type. Then there is a singly-graded spectral sequence $\{B^r_*, d^r\}$, natural with respect to spaces and continuous mappings, with $B^1_n \cong H_n(X;\mathbb{F}_p)$, $d^1 = \beta$, the Bockstein homomorphism, and converging strongly to $(H_*(X)/\text{torsion}) \otimes \mathbb{F}_p$.*

PROOF: Suppose G is a finitely generated abelian group. Then we can write

$$G \cong \bigoplus_i \mathbb{Z} \oplus \bigoplus_j \mathbb{Z}/p^{e_j}\mathbb{Z} \oplus \bigoplus_t \mathbb{Z}/q_t^{r_t}\mathbb{Z},$$

where the q_t are primes not equal to p. The *times* p homomorphism is an isomorphism on $\bigoplus_t \mathbb{Z}/q_t^{r_t}\mathbb{Z}$ and a monomorphism on $\bigoplus_i \mathbb{Z}$. Recall the *p-component of* G is the quotient group

$${}_{(p)}G = G/\{\text{elements of torsion order prime to } p\} \cong \bigoplus_i \mathbb{Z} \oplus \bigoplus_j \mathbb{Z}/p^{e_j}\mathbb{Z}.$$

An nonzero element u in G is **p-divisible** if $u = pv$ for some v in G. The elements in $\bigoplus_t \mathbb{Z}/q_t^{r_t}\mathbb{Z}$ are *infinitely* p-divisible since $- \times p$ is an isomorphism on this summand. No elements in the rest of G can be infinitely p-divisible without violating the condition that G is finitely generated. With these observations we prove the convergence assertion of the theorem.

By Corollary 2.10 we have the short exact sequence

$$0 \to {}^{H_n(X)}/_{(pH_n(X) + \ker p^r)} \to B_n^{r+1} \to p^r H_{n-1}(X) \cap \ker p \to 0.$$

Notice that $B_n^{r+1} = \{0\}$ implies $H_n(X) = pH_n(X) + \ker p^r$. If $u \in H_n(X)$ generates a copy of $\mathbb{Z}$, the $u \notin \ker p^r$. But if $u \in pH_n(X)$, then u is p-divisible. Writing $u = pv_1$, it follows that v_1 is also p-divisible. Continuing in this manner, we conclude that u is infinitely p-divisible, a contradiction to finite generation. It follows that ${}_{(p)}H_n(X) = \ker p^r$ and so ${}_{(p)}H_n(X)$ has exponent less than or equal to p^r.

Let r go to infinity to obtain the short exact sequence

$$0 \to {}^{H_n(X)}/_{(pH_n(X) + p\text{-torsion})} \to B_n^\infty \to \nabla_{n-1}^{\infty,p} \to 0,$$

where $\nabla_{n-1}^{\infty,p}$ is the subgroup of $H_{n-1}(X)$ of infinitely p-divisible elements that vanish when multiplied by p. Because $H_{n-1}(X)$ is finitely generated, $\nabla_{n-1}^{\infty,p}$ is trivial and so

$$B_n^\infty \cong {}^{H_n(X)}/_{(pH_n(X) + p\text{-torsion})} \cong (H_n(X)/\text{torsion}) \otimes \mathbb{F}_p \qquad \square$$

Some immediate consequences of the existence and convergence of the Bockstein spectral sequence are the following inequalities. Suppose that X is a space of finite type. Then, in each dimension i, we have

$$\begin{aligned}\dim_{\mathbb{F}_p} H_i(X;\mathbb{F}_p) \geq & \text{ free rank } H_i(X) \\ & = \dim_{\mathbb{Q}} H_i(X;\mathbb{Q}) \\ & = \dim_{\mathbb{F}_p}((H_i(X)/\text{torsion}) \otimes \mathbb{F}_p).\end{aligned}$$

This follows from the Universal Coefficient theorem and the fact that $H_i(X)$ is finitely generated. Thus the Bockstein spectral sequence for X collapses at B^r if and only if $\dim_{\mathbb{F}_p} B_i^r(X) = \dim_{\mathbb{Q}} H_i(X;\mathbb{Q})$ for all i.

There is an alternate description of the differential that identifies the Bockstein homomorphism directly. Consider the short exact sequence of coefficients

$$0 \to \mathbb{Z}/p\mathbb{Z} \to \mathbb{Z}/p^2\mathbb{Z} \to \mathbb{Z}/p\mathbb{Z} \to 0$$

where we have written $\mathbb{Z}/p\mathbb{Z} \cong p\mathbb{Z}/p^2\mathbb{Z}$ as the kernel. The associated long exact sequence on homology for a space X is given by

$$\begin{aligned}\cdots \to H_n(X;\mathbb{Z}/p\mathbb{Z}) \xrightarrow{-\times p} & H_n(X;\mathbb{Z}/p^2\mathbb{Z}) \\ & \to H_n(X;\mathbb{Z}/p\mathbb{Z}) \xrightarrow{\beta} H_{n-1}(X;\mathbb{Z}/p\mathbb{Z}) \to \cdots\end{aligned}$$

and has $d^1 = \beta$, the connecting homomorphism. This can be seen by comparing the short exact sequences of coefficients

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathbb{Z} & \xrightarrow{-\times p} & \mathbb{Z} & \longrightarrow & \mathbb{Z}/p\mathbb{Z} & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \mathrm{red}_p} & & \downarrow{\scriptstyle \mathrm{red}_{p^2}} & & \| & & \\ 0 & \longrightarrow & \mathbb{Z}/p\mathbb{Z} & \xrightarrow[-\times p]{} & \mathbb{Z}/p^2\mathbb{Z} & \longrightarrow & \mathbb{Z}/p\mathbb{Z} & \longrightarrow & 0. \end{array}$$

The associated homomorphism of long exact sequences carries β to $\mathrm{red}_p \circ \partial$.

When we consider the short exact sequence of coefficients

$$0 \to \mathbb{Z}/p^r\mathbb{Z} \to \mathbb{Z}/p^{2r}\mathbb{Z} \to \mathbb{Z}/p^r\mathbb{Z} \to 0,$$

we obtain the r^{th} **order Bockstein operator** as connecting homomorphism. Taking all of the short exact sequences of coefficients for all $r \geq 1$, the following more refined picture of the Bockstein spectral sequence emerges.

Proposition 10.4. *B_n^r can be identified with the subgroup of $H_n(X;\mathbb{Z}/p^r\mathbb{Z})$ given by the image of $H_n(X;\mathbb{Z}/p^r\mathbb{Z}) \xrightarrow{-\times p^{r-1}} H_n(X;\mathbb{Z}/p^r\mathbb{Z})$ and $d^r\colon B_n^r \to B_{n-1}^r$ can be identified with the connecting homomorphism, the r^{th} order Bockstein homomorphism.*

PROOF: Write $G_n^r = \operatorname{im}(-\times p^{r-1}\colon H_n(X;\mathbb{Z}/p^r\mathbb{Z}) \to H_n(X;\mathbb{Z}/p^r\mathbb{Z}))$ and consider the sequence of homomorphisms

$$p^{r-1}H_n(X) \xrightarrow{-\times p} p^{r-1}H_n(X) \xrightarrow{\alpha} G_n^r \xrightarrow{\zeta} p^{r-1}H_{n-1}(X) \to p^{r-1}H_{n-1}(X)$$

defined by $\alpha\left(\left\{\sum_i p^{r-1}u_i\right\}\right) = \left\{\sum_i u_i \otimes (p^{r-1}+p^r\mathbb{Z})\right\} \in G_n^r$. This homomorphism is well-defined and has $\operatorname{im}(-\times p)$ as its kernel. If a homology class $\left\{\sum_i v_i \otimes (p^{r-1}+p^r\mathbb{Z})\right\} \in H_n(X;\mathbb{Z}/p^r\mathbb{Z})$ is in G_n^r, then define

$$\zeta\left(\left\{\sum_i v_i \otimes (p^{r-1}+p^r\mathbb{Z})\right\}\right) = \left\{\frac{1}{p}\sum_i \partial(v_i)\right\},$$

where ∂ is the chain boundary operator. Since $\partial\left(\sum v_i \otimes (p^{r-1}+p^r\mathbb{Z})\right) = 0$, it follows that $\sum \partial v_i = p\left(p^{r-1}\sum_j x_j\right)$ and so dividing by p determines a class in $p^{r-1}H_{n-1}(X)$. It is easy to see that $\ker\zeta = \operatorname{im}\alpha$ and we have exactness at G_n^r. We compare this sequence with the r^{th} derived couple

$$\begin{array}{ccccccccccc}
\longrightarrow & p^{r-1}H_n(X) & \xrightarrow{p} & p^{r-1}H_n(X) & \to & G_n^r & \to & p^{r-1}H_{n-1}(X) & \xrightarrow{p} & p^{r-1}H_{n-1}(X) & \longrightarrow \\
 & \| & & \| & & \downarrow & & \| & & \| & \\
\longrightarrow & p^{r-1}H_n(X) & \xrightarrow[p]{} & p^{r-1}H_n(X) & \to & B_n^r & \to & p^{r-1}H_{n-1}(X) & \xrightarrow[p]{} & p^{r-1}H_{n-1}(X) & \longrightarrow.
\end{array}$$

The Five-lemma implies $B_n^r \cong G_n^r$.

To identify the differential d^r with the higher Bockstein

$$\beta_r\colon H_n(X;\mathbb{Z}/p^r\mathbb{Z}) \to H_{n-1}(X;\mathbb{Z}/p^r\mathbb{Z})$$

it suffices to compare the connecting homomorphism that defines β_r with the definition of the homomorphism ζ. □

This representation of the terms in the Bockstein spectral sequence can be completed by embedding the data for all $r \geq 1$ into a Cartan-Eilenberg system, a general technique to construct a spectral sequence (also known as a **spectral system** in [Neisendorfer80] or a **coherent system of coalgebra/algebras/Lie algebras** in [Anick93]). The definition and relation between a Cartan-Eilenberg system and its associated spectral sequence are explored in Exercises 2.2 and 2.3. For a prime p and a pair (s,t) with $-\infty < s \leq t < \infty$ we define

$$H(s,t) = H_*(X; \mathbb{Z}/p^{t-s}\mathbb{Z}).$$

If $s \leq s'$ and $t \leq t'$, let $H(s,t) \to H(s',t')$ be the homomorphism induced by the map of coefficients, $\mathbb{Z}/p^{t-s}\mathbb{Z} \to \mathbb{Z}/p^{t'-s'}\mathbb{Z}$, that is determined by $1 \mapsto p^{t'-t}\colon H_*(X;\mathbb{Z}/p^{t-s}\mathbb{Z}) \to H_*(X;\mathbb{Z}/p^{t'-s'}\mathbb{Z})$. If $r \leq s \leq t$, then let $\partial\colon H(r,s) \to H(s,t)$ be the connecting homomorphism associated to the coefficient sequence

$$0 \to \mathbb{Z}/p^{t-s}\mathbb{Z} \to \mathbb{Z}/p^{t-r}\mathbb{Z} \to \mathbb{Z}/p^{s-r}\mathbb{Z} \to 0,$$

a homomorphism $H_*(X;\mathbb{Z}/p^{s-r}\mathbb{Z}) \to H_{*-1}(X;\mathbb{Z}/p^{t-s}\mathbb{Z})$. In this context the limit terms of the Cartan-Eilenberg system are given by $H(q) = H(q,q) = H_q(X)$ and $H(q,\infty) = H_q(X; \lim_{r\to\infty} \mathbb{Z}/p^r\mathbb{Z})$. The exact couple determined by the long exact sequence

$$\cdots \to H(q-1) \to H(q) \to H(q-1,q) \xrightarrow{\partial} H(q-1) \to H(q) \to \cdots$$

gives the Bockstein spectral sequence.

With this added structure the (co)multiplicative properties of the spectral sequence may be studied. We refer the reader to the work of [Neisendorfer80] and [Anick93] for more details.

Though we developed the Bockstein spectral sequence for homology, it is just as easy to make the same constructions and observations for cohomology. The Bockstein homomorphism for cohomology has degree 1,

$$\beta\colon H^n(X;\mathbb{F}_p) \to H^{n+1}(X;\mathbb{F}_p),$$

and is identified with the stable cohomology operation β in the Steenrod algebra $\mathcal{A}_p$, when p is odd, and Sq^1 in $\mathcal{A}_2$, when $p=2$. This leads to a spectral sequence of algebras since β is a derivation with respect to the cup product.

When X is an H-space

The naturality of the Bockstein spectral sequence applies to the diagonal mapping to give a morphism of spectral sequences $B_*^r(X) \to B_*^r(X\times X)$. When (X, x_0, μ) is an H-space, the multiplication mapping induces $B_*^r(\mu)\colon B_*^r(X\times X) \to B_*^r(X)$. Our goal in this section is to identify $B_*^r(X\times X)$ with $B_*^r(X)\otimes B_*^r(X)$ and so obtain a spectral sequence of coalgebras for the homology Bockstein spectral sequence. Dually, we obtain a spectral sequence of algebras for the cohomology Bockstein spectral sequence; and for H-spaces, a spectral sequence of Hopf algebras.

Following [Browder61], we introduce small models of chain complexes whose structure makes explicit the key features of the Bockstein spectral sequence. Suppose n and s are nonnegative integers. Define the chain complex $(A(n,s),d)$, free over $\mathbb{Z}$, where

$$A(n,s)_m = \begin{cases} \{0\}, & m \neq n, n+1, \\ \mathbb{Z} \cong \langle u \rangle, & m = n, \\ \mathbb{Z} \cong \langle v \rangle, & m = n+1 \ (= \{0\} \text{ if } s = 0). \end{cases}$$

The differential is given on generators by $d(v) = su$, and so $H_n(A(n,s),d) \cong \mathbb{Z}/s\mathbb{Z}$ and $H_r(A(n,s),d) = \{0\}$ for $r \neq n$. This chain complex can be realized cellularly by the **mod s Moore space** $P^{n+1}(s) = S^n \cup_s e^{n+1}$ where s here denotes the degree s map on S^n. The reduced integral homology of $P^{n+1}(s)$ is $H_*(A(n,s),d)$.

The *times* p map, denoted $- \times p$, on $A(n,s)$ fits into the short exact sequence

$$0 \to (A(n,s),d) \xrightarrow{-\times p} (A(n,s),d) \xrightarrow{\mathrm{red}_p} (A(n,s) \otimes \mathbb{F}_p, \bar{d}) \to 0,$$

where red_p denotes reduction mod p. The long exact sequence in homology is the Bockstein exact couple. We consider the Bockstein spectral sequence associated to this exact couple.

Proposition 10.5. *If* $\gcd(s,p) = 1$, *then* $H_*(A(n,s) \otimes \mathbb{F}_p, \bar{d}) = \{0\}$. *If* $s = 0$, *then* $B^1 \cong B^\infty \cong \mathbb{Z}/p\mathbb{Z}$ *in degree* n. *If* $s = ap^k$ *with* $k > 0$ *and* $\gcd(a,p) = 1$, *then* $B^1 \cong B^2 \cong \cdots \cong B^k$ *and* $B^{k+1} \cong B^\infty = \{0\}$.

PROOF: The first assertion follows from the Universal Coefficient theorem and the fact that $\mathbb{Z}/s\mathbb{Z} \otimes \mathbb{F}_p = \{0\}$. When $s = 0$, $A(n,0) \otimes \mathbb{F}_p$ is simply $\mathbb{F}_p$ concentrated in degree n and the spectral sequence collapses.

By the fundamental theorem for finitely generated abelian groups, we can split $\mathbb{Z}/ap^k\mathbb{Z}$ as $\mathbb{Z}/a\mathbb{Z} \oplus \mathbb{Z}/p^k\mathbb{Z}$. Since the contribution by $\mathbb{Z}/a\mathbb{Z}$ vanishes, we only need to consider the case $s = p^k$ with $k > 0$. Since $A(n,p^k) \otimes \mathbb{F}_p \cong A(n,p^k)/pA(n,p^k)$, we have that $\bar{d} = 0$ and so

$$H_r(A(n,p^k) \otimes \mathbb{F}_p, \bar{d}) \cong \begin{cases} \mathbb{F}_p, & \text{when } r = n \text{ or } n+1, \\ \{0\}, & \text{otherwise.} \end{cases}$$

We write $(u)_p$ and $(v)_p$ for the mod p reductions of u and v. The mapping $\partial\colon H_{n+1}(A(n,p^k) \otimes \mathbb{F}_p, \bar{d}) \to H_n(A(n,p^k),d)$ in the exact couple is given by $\partial((v)_p) = p^{k-1}u$ for reasons of exactness. We can peel away powers of p from $p^{k-1}u$ until it becomes the generator of $p^{k-1}(\mathbb{Z}/p^k\mathbb{Z}) \cong \mathbb{Z}/p\mathbb{Z}$, and so $d^1 = d^2 = \cdots = d^{k-1} = 0$. At B^k we have

$$\begin{array}{ccccc} B^k_{n+1} \cong \langle (v)_p \rangle & \xrightarrow{\partial} & p^{k-1}(\mathbb{Z}/p^k\mathbb{Z}) & \longrightarrow & B^k_n \cong \langle (u)_p \rangle \\ (v)_p & \mapsto & p^{k-1}u & \mapsto & (u)_p. \end{array}$$

Thus $B^{k+1} \cong B^{\infty} = \{0\}$. □

In fact, more can be deduced from the small complexes.

Lemma 10.6. *If* $s = ap^k$ *with* $k > 0$ *and* $\gcd(a,p) = 1$*, then there is an isomorphism of exact couples* $(q, \bar{q})$*:*

$$\begin{array}{ccccccc}
H(A(n,ap^k),d) & \xrightarrow{-\times p} & H(A(n,ap^k),d) & \xrightarrow{\mathrm{red}_{p*}} & H(A(n,ap^k)\otimes\mathbb{F}_p,\bar{d}) & \xrightarrow{\partial} \\
\downarrow q & & \downarrow q & & \downarrow \bar{q} & \\
pH(A(n,ap^{k+1}),d) & \xrightarrow{-\times p} & pH(A(n,ap^{k+1}),d) & \xrightarrow{\mathrm{red}_{p*}} & B^2(A(n,ap^{k+1})) & \xrightarrow{\partial'}
\end{array}$$

PROOF: Write $A = A(n, ap^{k+1})$ with generators u and v and $A' = A(n, ap^k)$ with generators U and V. Consider the mapping $q\colon A' \to A$ and its reduction $\bar{q}\colon A' \otimes \mathbb{F}_p \to A \otimes \mathbb{F}_p$ given by

$$\begin{aligned} q(U) &= pu, & \bar{q}((U)_p) &= (u)_p, \\ q(V) &= v, & \bar{q}((V)_p) &= (v)_p. \end{aligned}$$

By the linearity of the differentials, q is a chain map. By the definition of q, $q_* H(A(n,ap^k)) = pH(A(n,ap^{k+1}))$. If $k > 0$, then $\bar{q}_*$ is an isomorphism at $B^1(A) \cong B^2(A)$.

It is left to show that the mapping pair $(q_*, \bar{q}_*)$ is a morphism of exact couples. Since q is a chain map, it commutes with $- \times p$. The class $\{U\}$ generates $H_n(A')$. The mapping j on $H(A')$ is given by $\{U\} \mapsto (U)_p$, the reduction mod p of $\{U\}$. Therefore, $\bar{q}_* \circ j(\{U\}) = (u)_p$. By the definition of a derived couple and the fact that $j(\{u\}) = (u)_p$, we have $j' \circ q_*(\{U\}) = j'(p\{u\}) = j(\{u\})$. Thus $j' \circ q_* = \bar{q}_* \circ j$.

For dimensional reasons, $\partial((U)_p) = 0 = \partial'((u)_p)$. For $k > 0$, $(V)_p \neq 0$ and, by exactness, $\partial((V)_p) = \{ap^{k-1}U\}$ and $\partial'((v)_p) = \{ap^k u\}$. Since $q_*(\{U\}) = \{pu\}$, we have that $q_* \circ \partial = \partial' \circ q_*$ and so $(q_*, \bar{q}_*)$ is a morphism of exact couples. □

With this lemma, we prove a structure result.

Proposition 10.7. *Consider the Bockstein spectral sequence for* $C_1 \otimes C_2$ *where* $C_1 = (A(n,ap^k),d)$ *and* $C_2 = (A(m,bp^l),d)$*,* $k \geq l > 0$ *and* $\gcd(a,p) = 1 = \gcd(b,p)$*. Then* $B^2(C_1 \otimes C_2)$ *may be taken to be* $B^2(C_1) \otimes B^2(C_2)$*.*

PROOF: By Lemma 10.6 we can take $B^2(C_1) = H(A(n,ap^{k-1}) \otimes \mathbb{F}_p, \bar{d})$ and $B^2(C_2) = H(A(m,bp^{l-1}) \otimes \mathbb{F}_p, \bar{d})$. We write $B^2(C_i) = C_i'$; denote the generators of C_i by u_i, v_i, and the generators of C_i' by u_i', v_i' for $i = 1, 2$.

Assume that $k \geq l$ and let

$$
\begin{aligned}
\gamma &= \operatorname{lcm}(a,b) = ag = bh, \\
\delta &= \gcd(ap^{k-l}, b) = Nap^{k-l} + Mb, \\
x &= g(v_1 \otimes u_2) - (-1)^{\deg u_1} hp^{k-l}(u_1 \otimes v_2), \\
y &= N(v_1 \otimes u_2) + (-1)^{\deg u_1} M(u_1 \otimes v_2).
\end{aligned}
$$

It follows that $\{x, y\}$ is a basis for $(C_1 \otimes C_2)_{n+m+1}$. Putting primes on x, y, u_i and v_i, we get a basis $\{x', y'\}$ for $(C_1' \otimes C_2')_{n+m+1}$. By the definitions, $dx = 0 = dx'$, $dy = \delta p^l(u_1 \otimes u_2)$, and $dy' = \delta p^{l-1}(u_1' \otimes u_2')$. Define the morphism of exact couples by letting $q\colon C_1' \otimes C_2' \to C_1 \otimes C_2$ be given by

$$
q(u_1' \otimes u_2') = p(u_1 \otimes u_2), \qquad q(x') = px,
$$
$$
q(y') = y, \qquad q(v_1' \otimes v_2') = v_1 \otimes v_2.
$$

Then q is a chain map and $q_* H(C_1' \otimes C_2') = pH(C_1 \otimes C_2)$. On the reductions mod p, define the map $\bar{q}_i\colon C_i' \otimes \mathbb{F}_p \to C_i \otimes \mathbb{F}_p$ by $\bar{q}_i((u_i')_p) = (u_i)_p$ and $\bar{q}_i((v_i')_p) = (v_i)_p$. Let $\bar{q} = \bar{q}_1 \otimes \bar{q}_2$. Then

$$
\bar{q}_*\colon H(C_1' \otimes C_2' \otimes \mathbb{F}_p) \xrightarrow{\cong} H(C_1 \otimes C_2 \otimes \mathbb{F}_p) \cong B^2(C_1 \otimes C_2).
$$

The morphism $(q_*, \bar{q}_*)$ is a morphism of exact couples and, as in the proof of Lemma 10.6, an isomorphism. □

We put the small models to work after we state two results of [Browder61] that follow from the properties of free and torsion-free chain complexes. We leave the proofs to the reader.

Proposition 10.8. *Let (A, d) be a chain complex, free over $\mathbb{Z}$; let (A', d') be a torsion-free chain complex, and p, a prime. If $(\phi, \bar{\phi})$ is a morphism of the associated Bockstein exact couples,*

$$
\begin{array}{ccccccccc}
\longrightarrow & H_n(A) & \xrightarrow{-\times p} & H_n(A) & \xrightarrow{\mathrm{red}_{p*}} & H_n(A \otimes \mathbb{F}_p) & \xrightarrow{\partial} & H_{n-1}(A) & \longrightarrow \\
 & \downarrow{\scriptstyle \phi} & & \downarrow{\scriptstyle \phi} & & \downarrow{\scriptstyle \bar{\phi}} & & \downarrow{\scriptstyle \phi} & \\
\longrightarrow & H_n(A') & \xrightarrow[-\times p]{} & H_n(A') & \xrightarrow[\mathrm{red}_{p*}]{} & H_n(A' \otimes \mathbb{F}_p) & \xrightarrow[\partial']{} & H_{n-1}(A') & \longrightarrow.
\end{array}
$$

Then there is a chain map $f\colon (A, d) \to (A', d')$ such that $H(f) = \phi$ and $H(f \otimes \mathbb{F}_p) = \bar{\phi}$.

Lemma 10.9. *Let (A,d) and (A',d') be torsion-free chain complexes. Then, for all r, $B^r(A\oplus A', d+d')\cong B^r(A,d)\oplus B^r(A',d')$.*

Assume that (A',d') is a torsion-free chain complex whose homology groups are finitely generated in each dimension. Using Proposition 10.8 and Lemma 10.9 we can replace (A',d') with another complex (A,d) which is free and of the form $\bigoplus_i (A_i,d_i)$ with each (A_i,d_i) of the form $(A(n_i,a_ip^{k_i}),d)$. By Lemma 10.9, $B^r(A',d')\cong\bigoplus_i B^r(A_i,d_i)$.

Suppose X is a space of finite type. The homology Bockstein spectral sequence for X is the Bockstein spectral sequence for $(C_*(X),\partial)$ and this spectral sequence is functorial in X. The diagonal mapping on X gives a morphism of spectral sequences

$$B^r(\Delta)\colon B^r(X)\to B^r(X\times X).$$

Replacing the chains on X with a direct sum of small models, we can apply Proposition 10.7 to the Alexander-Whitney map to prove the following result.

Theorem 10.10. *For (X,x_0) a pointed space of finite type, the homology Bockstein spectral sequence is a spectral sequence of coalgebras.*

When X is an H-space of finite type, the same argument applied to the multiplication, along with the compatibility of the multiplication with the diagonal, gives the following key result.

Theorem 10.11. *For X, an H-space of finite type, the homology Bockstein spectral sequence for X is a spectral sequence of Hopf algebras.*

The cohomology Bockstein spectral sequence admits a dual analysis using the small complexes $\mathrm{Hom}(A(n,ap^k),\mathbb{Z})$. In fact, $\mathrm{Hom}(A(n,ap^k),\mathbb{Z})$ is simply $A(n,s)$ with the differential upside down. Its single nontrivial homology group is $H^{n+1}(A(n,ap^k),d^{\text{dual}})$. Using these complexes and carrying out the same kinds of arguments as for the homology Bockstein spectral sequence we obtain the theorem:

Theorem 10.12. *For (X,x_0) a pointed space of finite type, the cohomology Bockstein spectral sequence is a spectral sequence of algebras. Suppose (A_*,d) is a chain complex with homology of finite type. Let $\{B_r^*=B_*^r(\mathrm{Hom}(A_*,\mathbb{Z})),d_r\}$ denote the cohomology spectral sequence for the dual of (A_*,d). Then $B_r^*\cong\mathrm{Hom}(B_*^r(A_*),\mathbb{F}_p)$ and d_r is the differential adjoint to d^r. If X is an H-space of finite type, then the cohomology Bockstein spectral sequence for X, $B_r^*(X)=B_r^*(C^*(X),\delta)$, is a spectral sequence of Hopf algebras dual to the homology Hopf algebras $B_*^r(X)$.*

Having established these structural results, we turn to some examples. The universal examples for cohomology are the Eilenberg-Mac Lane spaces for which we have complete descriptions of the mod p cohomology according to the theorems of Cartan and Serre (Theorem 6.16). We reinterpret these known data to give a complete description of the Bockstein spectral sequence in a range of dimensions.

We note that the limit of the Bockstein spectral sequences for $K(\mathbb{Z}/p^k\mathbb{Z},n)$ has $B^\infty \cong \{0\}$. To see this, suppose $\tilde{H}_*(K(\mathbb{Z}/p^k\mathbb{Z},n))$ contained a torsion-free summand. Then $\tilde{H}_*(K(\mathbb{Z}/p^k\mathbb{Z},n);\mathbb{Q})$ would have a nonzero lowest degree generator. By the Hurewicz-Serre theorem over $\mathbb{Q}$ (Theorem 6.25), this would imply a torsion-free summand in the homotopy of $K(\mathbb{Z}/p^k\mathbb{Z},n)$ which does not happen. Hence $B^\infty \cong \{0\}$.

Suppose p is an odd prime. The cohomology of $K(\mathbb{Z}/p\mathbb{Z},n)$ with coefficients in the field $\mathbb{F}_p$ is a free graded commutative algebra (exterior on odd-dimensional classes, tensor polynomial on even-dimensional classes) generated by classes $St^I\imath_n$ where $I=(\varepsilon_0,s_1,\varepsilon_1,\ldots,s_m,\varepsilon_m)$ is an admissible sequence ($\varepsilon_i=0$ or 1, $s_i\geq ps_{i+1}+\varepsilon_i$, for $m>i\geq 1$; Definition 6.17) of excess less than or equal to n. Notice that the excess, $e(I)=2ps_1+2\varepsilon_0-|I|$, is such that, if $I=(1,s_1,\varepsilon_1,\ldots,s_m,\varepsilon_m)$ and $e(I)\leq n$, then $e(I')\leq n$ for $I'=(0,s_1,\varepsilon_1,\ldots,s_m,\varepsilon_m)$. Thus, the generators pair off. Since this pairing is given by $\beta St^{I'}\imath_n=St^I\imath_n$ and $d_1=\beta$, we are looking at two sorts of differential graded algebras:

$$\begin{aligned}&\Lambda(St^{I'}\imath_n)\otimes\mathbb{F}_p[St^I\imath_n],\ d_1(St^{I'}\imath_n)=St^I\imath_n,\quad \deg St^{I'}\imath_n \text{ odd},\\ &\mathbb{F}_p[St^{I'}\imath_n]\otimes\Lambda(St^I\imath_n),\ d_1(St^{I'}\imath_n)=St^I\imath_n,\quad \deg St^{I'}\imath_n \text{ even}.\end{aligned}$$

When $St^{I'}\imath_n$ has odd degree, the complex $\Lambda(St^{I'}\imath_n)\otimes\mathbb{F}_p[St^I\imath_n]$ has the same form as the Koszul complex for $\Lambda(x_{\mathrm{odd}})$ and so its homology is trivial. When $St^{I'}\imath_n$ has even degree, the complex has homology $H(\mathbb{F}_p[St^{I'}\imath_n]\otimes\Lambda(St^I\imath_n),d_1)\cong\Lambda(\{(St^{I'}\imath_n)^{p-1}\otimes St^I\imath_n\})\otimes\mathbb{F}_p[\{(St^{I'}\imath_n)^p\}]$, where $\{U\}$ denotes the homology class of U with respect to the the differential d_1. This follows because d_1 is a derivation and so $d_1((St^{I'}\imath_n)^p)=p(St^{I'}\imath_n)^{p-1}=0$. Notice how the class $\{(St^{I'}\imath_n)^{p-1}\otimes St^I\imath_n\}$ encodes the transpotence element that figures in Cartan's constructions and Kudo's transgression theorem (§6.2).

Suppose $n=2m$. Recall that $P^m\imath_{2m}=(\imath_{2m})^p$. In dimensions less than $2mp=\deg\imath_{2m}^p$, we find classes coming from the paired algebras:

$$\begin{aligned}(\mathbb{F}_p[\imath_{2m}]\otimes\Lambda(\beta\imath_{2m}))\otimes(\mathbb{F}_p[P^1\imath_{2m}]\otimes\Lambda(\beta P^1\imath_{2m}))\otimes\cdots\\ \otimes(\mathbb{F}_p[P^{m-1}\imath_{2m}]\otimes\Lambda(\beta P^{m-1}\imath_{2m})).\end{aligned}$$

Computing the homology of this product as a differential graded algebra with differential β, we are left with the first nonzero classes, $\{\imath_{2m}^{p-1}\otimes\beta\imath_{2m}\}\in$

B_2^{2mp-1} and $\{\imath_{2m}^p\} \in B_2^{2mp}$. The next indecomposable class in B_2^* corresponds to $\{(P^1\imath_{2m})^p\} \in B_2^{p(2m+2(p-1))}$. Thus , for $q < p(2m+2(p-1))$,

$$B_2^q(K(\mathbb{Z}/p\mathbb{Z}, 2m)) \cong (\,\Lambda(\{\beta\imath_{2m} \smile (\imath_{2m})^{p-1})\}) \otimes \mathbb{F}_p[\{\imath_{2m}^p\}]\,)^q.$$

The case of $K(\mathbb{Z}/p^k\mathbb{Z}, n)$ for $k > 1$ yields to a similar analysis of admissible sequences except in the lowest degrees. Here the contributing classes are $\imath_n$ and $\beta_k\imath_n$, the Bockstein of k^{th} order associated to the short exact sequence of coefficients $0 \to \mathbb{Z}/p\mathbb{Z} \to \mathbb{Z}/p^k\mathbb{Z} \to \mathbb{Z}/p^{k-1}\mathbb{Z} \to 0$. In dimensions $q < p(n+2(p-1))$ we have that $B_l^q(K(\mathbb{Z}/p^k\mathbb{Z}, n)) \cong$

$$\begin{cases} B_1^q, & \text{if } l \leq k, \\ \{0\}, & \text{if } l > k \text{ and } n \text{ is odd}, \\ (\Lambda(\{\beta_k\imath_n \smile (\imath_n)^{p-1}\}) \otimes \mathbb{F}_p[\{\imath_n^p\}])^q, & \text{if } l = k+1 \text{ and } n \text{ is even}. \end{cases}$$

We complete the analysis for the lower dimensions of the Bockstein spectral sequence when n is even. The input is part of the computation of [Cartan54] of the integral cohomology of the Eilenberg-Mac Lane spaces.

Proposition 10.13. *If p is any prime and $k \geq 1$, then $H_{2mp}(K(\mathbb{Z}/p^k\mathbb{Z}, 2m))$ contains a subgroup isomorphic to $\mathbb{Z}/p^{k+1}\mathbb{Z}$ as summand. Furthermore, there are no summands isomorphic to $\mathbb{Z}$ or $\mathbb{Z}/p^{k+j}\mathbb{Z}$ with $j > 1$.*

Corollary 10.14. *Suppose that p is an odd prime. Let $\imath_{2m}$ denote the fundamental class in $B_1^{2m}(K(\mathbb{Z}/p^k\mathbb{Z}, 2m))$. Then, for some $c \in \mathbb{F}_p$,*

$$d_{k+1}(\{\imath_{2m}^p\}) = c\{\beta_k\imath_{2m} \smile (\imath_{2m})^{p-1}\} \neq 0.$$

The proof of Proposition 10.13 is a direct computation using the method of constructions ([Cartan54]). This method applies integrally and so one can compute the desired homology group by hand and discover the p-torsion height.

The corollary follows from the convergence of the Bockstein spectral sequence. Since there are no other classes in the degree involved, the formula for $d_{k+1}(\{\imath_{2m}^p\})$ follows without choice. [Browder62, Theorem 5.11] gave a more general chain level computation that obtains the formula directly.

For the prime 2, a new phenomenon occurs in the Bockstein spectral sequence for $K(\mathbb{Z}/2\mathbb{Z}, n)$. [Serre53] showed that $H^*(K(\mathbb{Z}/2\mathbb{Z}, n); \mathbb{F}_2)$ is a polynomial algebra on generators $St^I\imath_n$ where I is an admissible sequence (mod 2) of excess less than or equal to n (Theorem 6.20). However, when $x = St^I\imath_n$ has odd degree $2m+1$, then

$$x^2 = Sq^{2m+1}x = Sq^1Sq^{2m}x = Sq^1St^{(2m,I)}\imath_n,$$

that is, the squares of certain classes are the image under the Bockstein of other generators. The pairing of classes that occurs in the case of odd primes does not occur here and new cycles are produced. We write $Sq^1\imath_{2m} = \eta_{2m+1}$. Because $Sq^1 = \beta$ is a derivation, we have

$$\eta_{2m+1}^2 = Sq^{2m+1}\imath_{2m} = Sq^1 Sq^{2m}\eta_{2m+1} = Sq^1(\imath_{2m}\eta_{2m+1}).$$

Thus $Sq^{2m}\eta_{2m+1} + \imath_{2m} \smile \eta_{2m+1}$ is a cycle under d_1. By the same analysis of the low degrees of $H^*(K(\mathbb{Z}/2\mathbb{Z}, 2m); \mathbb{F}_2)$ and Cartan's integral computation we have the following result.

Corollary 10.15. *Suppose that $p = 2$. Let $\imath_{2m} \in B_1^{2m}(K(\mathbb{Z}/2\mathbb{Z}, 2m))$ and $\eta_{2m+1} = Sq^1\imath_{2m}$. Then*

$$d_2(\{\imath_{2m}^2\}) = \{Sq^{2m}\eta_{2m+1} + \imath_{2m} \smile \eta_{2m+1}\}.$$

We leave the remaining case of $K(\mathbb{Z}/2^k\mathbb{Z}, n)$ for $k > 1$ to the reader. In this case, Corollary 10.14 for odd primes goes over analogously.

We next explore some of the consequences of these calculations.

Infinite implications and their consequences

The proof of Theorem 10.2 for fields of characteristic zero shows that the presence of a primitive element x of even degree implies the condition $x^n \neq 0$ for all n. For fields of characteristic $p > 0$, it can happen that a primitive element x of even degree can satisfy $x^{p^r} = 0$ for some r, and so the finiteness of the H-space need not be violated. For example, the exceptional Lie group F_4 has mod 3 cohomology given by

$$H^*(F_4; \mathbb{F}_3) \cong \mathbb{F}_3[x_8]/(x_8^3) \otimes \Lambda(x_3, x_7, x_{11}, x_{15}),$$

where x_8 is clearly primitive ([Borel54]). The rational cohomology is given by $H^*(F_4; \mathbb{Q}) \cong \Lambda(X_3, X_{11}, X_{15}, X_{23})$. The Bockstein spectral sequence mod 3 requires $\beta(x_7) = x_8$; subsequently the class X_{23} is represented by the product $[x_7 \smile x_8^2]$.

The E_∞-term of the Bockstein spectral sequence of a finite H-space is fixed by Hopf's theorem. The appearance of even-dimensional primitive elements in $H^*(X; \mathbb{F}_p)$ forces some nontrivial differentials in the Bockstein spectral sequence in order to realize this target. The consequences of such differentials are organized by the phenomenon of implications due to [Browder61].

Definition 10.16. *Let A_* denote a Hopf algebra of finite type over the finite field $\mathbb{F}_p$ and denote its dual by A^*. An element $x \in A_m$ is said to have r-**implications** if there are elements $x_i \in A_{mp^i}$, for $i = 0, 1, 2, \ldots, r$, with $x_0 = x$, $x_i \neq 0$ for all i, and either $x_{i+1} = x_i^p$ or there exists an element $\bar{x}_i \in A^{mp^i}$ such that $\bar{x}_i(x_i) \neq 0$ and $\bar{x}_i^p(x_{i+1}) \neq 0$. An element has ∞-**implications** if it has r-implications for all r.*

Lemma 10.17. *If A_* is a Hopf algebra over $\mathbb{F}_p$ that contains an element which has ∞-implications, then A_* is infinite dimensional as a vector space over $\mathbb{F}_p$.*

The Hopf algebras that we want to study are the terms of the Bockstein spectral sequence for an H-space which are, in fact, *differential* Hopf algebras. Before stating Browder's theorem on ∞-implications we collect a few basic lemmas about Hopf algebras and differential Hopf algebras.

Lemma 10.18. *Suppose (A_*, μ, Δ) is a Hopf algebra and $x \in A_{2m}$ is a primitive element. Then*

$$\Delta(x^n) = \sum\nolimits_{i=0}^{n} \binom{n}{i} x^{n-i} \otimes x^i.$$

This follows like the binomial theorem for the algebra $A_* \otimes A_*$ using the fact that the comultiplication Δ is an algebra map. (We do not need to assume associativity of μ if we define x^n inductively by $x^0 = 1$ and $x^n = x^{n-1} \cdot x$, and pay careful attention to parentheses.)

Lemma 10.19. *Suppose A_* is a Hopf algebra over a field k and A^* is its dual. If $x \in A_{2m}$ is a primitive element and $\bar{x} \in A^*$, then $\bar{x}^n(x^n) = n!(\bar{x}(x))^n$.*

PROOF: We compute

$$\begin{aligned}\bar{x}^n(x^n) &= \Delta^*(\bar{x}^{n-1} \otimes \bar{x})(x^n) = (\bar{x}^{n-1} \otimes \bar{x})(\Delta(x^n)) \\ &= (\bar{x}^{n-1} \otimes \bar{x}) \left(\sum\nolimits_i \binom{n}{i} x^{n-i} \otimes x^i \right) \\ &= (\bar{x}^{n-1} \otimes \bar{x})(n x^{n-1} \otimes x).\end{aligned}$$

Thus $\bar{x}^n(x^n) = n(\bar{x}^{n-1}(x^{n-1}) \cdot \bar{x}(x))$ and so, by induction, we get $\bar{x}^n(x^n) = n!(\bar{x}(x))^n$. □

Lemma 10.20. *Suppose that (A_*, μ, Δ, d) is a connected, differential graded Hopf algebra over the field $\mathbb{F}_p$, $x \in A_{2m}$ is primitive, $x = d(y)$ for some $y \in A_{2m+1}$, and $\bar{x} \in A^{2m}$ satisfies $\bar{x}(x) \neq 0$. Set $\bar{y} = d^*(\bar{x})$ where d^* is the dual differential on A^*. Then $(\bar{x}^{p-1} \cdot \bar{y})(x^{p-1} \cdot y) \neq 0$.*

PROOF: First notice that $\bar{y}(y) = (d^*(\bar{x}))(y) = \bar{x}(d(y)) = \bar{x}(x) \neq 0$ and so $\bar{y} \neq 0$. We next compute

$$(\bar{x}^{p-1} \cdot \bar{y})(x^{p-1} \cdot y) = \Delta^*(\bar{x}^{p-1} \otimes \bar{y})(x^{p-1} \cdot y) = (\bar{x}^{p-1} \otimes \bar{y})(\Delta(x^{p-1})\Delta(y))$$

By Lemma 10.18, we can write

$$\begin{aligned}\Delta(x^{p-1})\Delta(y) &= \\ &= \left(\sum_{i=0}^{p-1}\binom{p-1}{i}x^{p-1-i}\otimes x^i\right)\left(y\otimes 1+1\otimes y+\sum_j y'_j\otimes y''_j\right)\\ &= \left(x^{p-1}\otimes y+\sum_{\dim y''_j=1}(p-1)x^{p-2}y'_j\otimes xy''_j+\text{stuff}\right),\end{aligned}$$

where the "stuff" is a sum of tensor products of classes $u\otimes v$ where $\deg u\neq(p-1)\deg\bar{x}$ or $\deg v\neq\deg\bar{y}$. Since $(\bar{x}^{p-1}\otimes\bar{y})(x^{p-1}\otimes y)=\bar{x}^{p-1}(x^{p-1})\cdot\bar{y}(y)\neq 0$, it suffices to show that $\bar{y}(xy''_j)=0$ for $y''_j\in A_1$. Consider

$$\bar{y}(xy''_j)=(d^*(\bar{x}))(xy''_j)=\bar{x}(d(x)y''_j+xd(y''_j)).$$

Since $x=d(y)$, $d(x)=0$. Thus $d(xy''_j)=xd(y''_j)$. If $d(y''_j)\neq 0$, then there is an element $w_j\in A^0$ with $d^*(w_j)\neq 0$. Since A_* is taken to be connected, $w_j=\alpha_j\cdot 1$ for some $\alpha_j\neq 0\in\mathbb{F}_p$. But $d^*(1)=d^*(1\cdot 1)=d^*(1)\cdot 1+1\cdot d^*(1)=2d^*(1)$ and so $d^*(1)=0$. Thus $d^*(w_j)=d^*(\alpha_j\cdot 1)=\alpha_j d^*(1)=0$. This implies that $\bar{y}(xy''_j)=0$ for all j. □

Lemma 10.21. *If A_* is a differential graded Hopf algebra and $x\in H(A_*)$ satisfies $x^p\neq 0$, then for any $y\in A_*$ with $\{y\}=x$, we have $y^p\neq 0$. If x has r-implications in $H(A_*)$ for some $r\leq\infty$, then y has r-implications in A_*.*

PROOF: Since $x^p=\{y\}^p=\{y^p\}\neq 0$, then $y^p\neq 0$. We can apply this argument at each power of p. Thus, if x has ∞-implications in $H(A_*)$, then y has ∞-implications in A_*. □

Lemma 10.22. *Suppose that A_* is a differential graded Hopf algebra over $\mathbb{F}_p$. Suppose further that $x\in A_{2m}$ is primitive, that $x^p=0$, and there is an element y with $d(y)=x$. If $\{x^{p-1}y\}\neq 0$ in $H(A_*)$, then it is primitive.*

PROOF: By definition, $H(\Delta)(\{x^{p-1}y\})=\{\Delta(x^{p-1}y)\}$. By assumption we have $d(\Delta(y))=\Delta(d(y))=\Delta(x)=1\otimes x+x\otimes 1$. This implies that $d(\Delta(y)-y\otimes 1-1\otimes y)=0$. Furthermore, $\Delta(x^{p-1})=\sum_{i=0}^{p-1}\binom{p-1}{i}x^{p-1-i}\otimes x^i$. From elementary number theory we know that $\binom{p-1}{i}\equiv(-1)^i \bmod p$, and so we can write

$$\begin{aligned}
&\Delta(x^{p-1}y) = \Delta(x^{p-1})\Delta(y) \\
&= \left(\sum_{i=0}^{p-1}(-1)^i x^{p-1-i}\otimes x^i\right)(y\otimes 1 + 1\otimes y + (\Delta(y) - 1\otimes y - y\otimes 1)) \\
&= x^{p-1}y\otimes 1 + 1\otimes x^{p-1}y + \sum_{i=0}^{p-2}(-1)^i x^{p-1-i}\otimes x^i y \\
&\qquad + \sum_{i=1}^{p-1}(-1)^i x^{p-1-i}y\otimes x^i + \Delta(x^{p-1})(\Delta(y) - 1\otimes y - y\otimes 1) \\
&= x^{p-1}y\otimes 1 + 1\otimes x^{p-1}y + d\left(\sum_{i=1}^{p-1}(-1)^{i+1}(x^{p-1-i}y\otimes x^{i-1}y)\right) \\
&\qquad + d(\Delta(x^{p-2}y)(\Delta(y) - 1\otimes y - y\otimes 1)).
\end{aligned}$$

It follows that $\{\Delta(x^{p-1}y)\} = \{x^{p-1}y\}\otimes 1 + 1\otimes\{x^{p-1}y\}$. □

The last lemma we need before we state and prove the main theorem of [Browder61] is a technical fact about the mod 2 Steenrod algebra and H-spaces. While the previous lemmas followed for purely algebraic reasons, this lemma requires that we are working with the mod 2 cohomology of an H-space.

Lemma 10.23. *If (X, x_0, μ) is an H-space, $x \in H_{2m}(X;\mathbb{F}_2)$ is a primitive element, $y \in H_{2m+1}(X;\mathbb{F}_2)$, and $\bar{z} \in H^{2m+1}(X;\mathbb{F}_2)$, then $(Sq^{2m}\bar{z})(xy) = 0$.*

PROOF: In terms of the induced operations we can write

$$(Sq^{2m}\bar{z})(xy) = (\mu^*(Sq^{2m}\bar{z}))(x\otimes y) = (Sq^{2m}(\mu^*\bar{z}))(x\otimes y).$$

We write $\mu^*(\bar{z}) = \sum_i \bar{z}_i'\otimes\bar{z}_i''$ and the Cartan formula gives

$$Sq^{2m}(\bar{z}_i'\otimes\bar{z}_i'') = \sum_{q+r=2m} Sq^q(\bar{z}_i')\otimes Sq^r(\bar{z}_i'').$$

By the unstable axiom for the action of the Steenrod algebra, if $q > \dim\bar{z}_i'$, then $Sq^q(\bar{z}_i') = 0$, and similarly if $r > \dim\bar{z}_i''$. Let $c = \deg\bar{z}_i'$, $d = \deg\bar{z}_i''$. Then $c + d = 2m+1$ and it follows by examining the solutions to $q + r = 2m$ that

$$Sq^{2m}(\bar{z}_i'\otimes\bar{z}_i'') = Sq^c\bar{z}_i'\otimes Sq^{d-1}\bar{z}_i'' + Sq^{c-1}\bar{z}_i'\otimes Sq^d\bar{z}_i''.$$

Since $Sq^c\bar{z}_i' = (\bar{z}_i')^2$ and x is primitive, $(\bar{z}_i')^2(x) = 0$. It follows that $(Sq^c\bar{z}_i'\otimes Sq^{d-1}\bar{z}_i'')(x\otimes y) = 0$. Similarly, $Sq^d\bar{z}_i'' = (\bar{z}_i'')^2$, a class of even degree. Since y has odd degree, $(\bar{z}_i'')^2(y) = 0$ and so the lemma follows from $(Sq^{c-1}\bar{z}_i'\otimes Sq^d\bar{z}_i'')(x\otimes y) = 0$. □

Theorem 10.24. *Suppose (X, x_0, μ) is a connected, path-connected H-space of finite type and $\{B^r_*(X)\}$ is its homology Bockstein spectral sequence. If $x \in B^r_{2m}$ is a nonzero primitive element and, for some $y \neq 0$, $x = d^r(y)$, then x has ∞-implications.*

PROOF: We may assume that $x^p = 0$, for otherwise we can take $x_1 = x^p$, also a primitive, with $d^r(x^{p-1}y) = x^p$. Thus x_1 satisfies the hypotheses of the theorem, and if this process never stops, we have obtained the sequence of ∞-implications of x. Assuming $x^p = 0$, we will produce $x_1 \in B^r_{2mp}$ such that $\bar{x}^p(x_1) \neq 0$ for any $\bar{x} \in B^{2m}_r(X)$ for which $\bar{x}(x) \neq 0$. The x_1 produced will be neither primitive nor a boundary, but its homology class $\{x_1\} \in B^{r+1}_{2mp}$ will be both primitive and a boundary. By Lemma 10.21 it suffices to check that there is the 1-implication x_1 at the next stage of the Bockstein spectral sequence and then take a representative in B^r.

In the cohomology Bockstein spectral sequence suppose that $\bar{x} \in B^{2m}_r$ satisfies $\bar{x}(x) \neq 0$. Set $\bar{y} = d_r(\bar{x})$. Then

$$\bar{y}(y) = (d_r(\bar{x}))(y) = \bar{x}(d^r(y)) = \bar{x}(x) \neq 0.$$

It follows that $\bar{y} \neq 0$ and, by Lemma 10.20, that $(\bar{x}^{p-1}\bar{y})(x^{p-1}y) \neq 0$. Furthermore, if $p \neq 2$, $d_r(\bar{x}^{p-1}\bar{y}) = (p-1)\bar{x}^{p-2}\bar{y}^2 = 0$. If $p = 2$ and $r > 1$,

$$\bar{y}^2 = \{Sq^{2m+1}z\} = \{Sq^1 Sq^{2m}z\} = \{d_1(Sq^{2m}z)\} = 0$$

in B_2 where $z \in H^{2m+1}(X; \mathbb{F}_2)$ is such that $\{z\} = \bar{y}$. That is, squares of odd degree classes vanish in B_2. If $p = 2$ and $r = 1$, then

$$d_1(Sq^{2m}\bar{y} + \bar{x}\bar{y}) = \bar{y}^2 + \bar{y}^2 = 0$$

and, by Lemmas 10.20 and 10.23, $(Sq^{2m}\bar{y} + \bar{x}\bar{y})(xy) \neq 0$.

We check that the class $\{\bar{x}^{p-1}\bar{y}\}$ (or $\{Sq^{2m}\bar{y} + \bar{x}\bar{y}\}$ when $r = 1$ and $p = 2$) is nontrivial in B_{r+1}. Suppose that $\bar{x}^{p-1}\bar{y} = d_r(\bar{z})$. Then

$$\begin{aligned} 0 \neq (\bar{x}^{p-1}\bar{y})(x^{p-1}y) &= d_r(\bar{z})(x^{p-1}y) = \bar{z}(d^r(x^{p-1}y)) \\ &= \bar{z}(x^p) = \bar{z}(0) = 0, \end{aligned}$$

a contradiction. Thus $\bar{x}^{p-1}\bar{y} \neq d_r(\bar{z})$. Similarly, $(Sq^{2m}\bar{y} + \bar{x}\bar{y}) \neq d_1(\bar{z})$.

To complete the proof we show that the class $\{\bar{x}^{p-1}\bar{y}\} \in B_{r+1}$ satisfies $d_{r+1}(\{\bar{x}^p\}) = c\{\bar{x}^{p-1}\bar{y}\} \neq 0$ when $p \neq 2$ or $p = 2$ and $r > 1$. In the case $p = 2$ and $r = 1$, we show that the class $\{Sq^{2m}\bar{y} + \bar{x}\bar{y}\} \in B_2$ satisfies $d_2(\{\bar{x}^2\}) = \{Sq^{2m}\bar{y} + \bar{x}\bar{y}\}$. Recall $d_r(\bar{x}) = \bar{y}$. Then there is a class $\bar{u} \in H^{2m}(X; \mathbb{Z}/p^r\mathbb{Z})$ such that $\{\mathrm{red}_p^*\,\bar{u}\} = \bar{x} \in B_r$ where $\mathrm{red}_p\colon \mathbb{Z}/p^r\mathbb{Z} \to \mathbb{Z}/p\mathbb{Z}$ is reduction mod p. Let $f\colon X \to K(\mathbb{Z}/p^r\mathbb{Z}, 2m)$ represent $\bar{u}$, that is, $f^*(\imath_{2m}) =$

$\bar{u}$ where $\imath_{2m} \in H^{2m}(K(\mathbb{Z}/p^r\mathbb{Z}, 2m); \mathbb{Z}/p^r\mathbb{Z})$ is the fundamental class. Let $\bar{\imath} = \mathrm{red}_p^*(\imath_{2m})$. It follows that

$$f^*(\bar{\imath}) = f^*(\mathrm{red}_p^*\, \imath_{2m}) = \mathrm{red}_p^*(f^*(\imath_{2m})) = \mathrm{red}_p^*(\bar{u}) = \bar{x}.$$

Let $f_r^*\colon B_r(K(\mathbb{Z}/p^r\mathbb{Z}, 2m)) \longrightarrow B_r(X)$ denote the homomorphism induced by f on the cohomology Bockstein spectral sequences. If $\bar{\eta} = d_r(\bar{\imath})$, then we have

$$f_r^*(\bar{\eta}) = f_r^*(d_r(\bar{\imath})) = d_r(f_r^*(\bar{\imath})) = d_r(\bar{x}) = \bar{y}$$

and so $f_r^*(\bar{\imath}^{p-1}\bar{\eta}) = \bar{x}^{p-1}\bar{y}$. Since $\{\bar{x}^{p-1}\bar{y}\} \neq 0$ in B_{r+1}, $f_{r+1}^*(\{\bar{\imath}^{p-1}\bar{\eta}\}) = \{\bar{x}^{p-1}\bar{y}\}$. By naturality and the calculation of the cohomology Bockstein spectral sequence for $K(\mathbb{Z}/p^r\mathbb{Z}, 2m)$, $f_{r+1}^*(d_{r+1}(\{\bar{\imath}^p\})) \neq 0$ and $f_{r+1}^*(\{\bar{\imath}^p\}) = \{\bar{x}^p\} \neq 0$. Thus

$$d_{r+1}(\{\bar{x}^p\}) = d_{r+1}(f_{r+1}^*(\{\bar{\imath}^p\})) = f_{r+1}^*(c\{\bar{\imath}^{p-1}\bar{\eta}\}) = c\{\bar{x}^{p-1}\bar{y}\}.$$

The analogous argument mod 2 gives $d_2(\{\bar{x}^2\}) = \{Sq^{2m}\bar{y} + \bar{x}\bar{y}\}$.

In order to continue the argument, we show that there is an element $v \in B_{2mp}^{r+1}$ that is primitive with $\{\bar{x}^p\}(v) \neq 0$ and $v = d^{r+1}(w)$ for some w. Consider the element $w = \{x^{p-1}y\}$. We compute:

$$\{\bar{x}^p\}(d^{r+1}(\{x^{p-1}y\})) = c\{\bar{x}^{p-1}\bar{y}\}(\{x^{p-1}y\}) \neq 0.$$

By Lemma 10.22, w is primitive. Also, $v = d^{r+1}(w)$ is primitive. In the sequence of elements making up the ∞-implications of x we take x_1 to be a choice of representative of v in B^r. Then, $\bar{x}^p(x_1) = \{\bar{x}^p\}(v) \neq 0$, and, since $\bar{x}(x) \neq 0$, x_1 is the next element in the sequence making up the ∞-implications for x. To obtain x_2, either take x_1^p if nonzero, or repeat the argument using the primitive $v \in B_{2mp}^{r+1}$ with $v = d^{r+1}(w)$. □

Notice that if $x^p = 0$, then the choice of $\bar{x}$ with $\bar{x}(x) \neq 0$ was arbitrary in the construction. It follows from $\bar{x}^p(x_1) \neq 0$ that, if x is a primitive in B_{2m}^r with $0 \neq d^r(y) = x$ and $x^p = 0$, then $\bar{x}^p \neq 0$ for all $\bar{x} \in B_r^{2m}$ with $\bar{x}(x) \neq 0$.

We turn to applications of Theorem 10.24. A space X is said to be a **mod p finite H-space** if it is a connected, path-connected H-space of finite type for which the mod p homology ring is finite-dimensional over $\mathbb{F}_p$. By Theorem 10.2, for a mod p finite H-space X, $B_\infty(X)$ is an exterior algebra on finitely many odd-dimensional generators.

A shorthand statement of Theorem 10.24 is the expression for X, a mod p finite H-space,

$$\mathrm{Im}\, d^r \cap \mathrm{Prim}(H_{\mathrm{even}}(X; \mathbb{F}_p)) = \{0\}.$$

A dual formulation of Theorem 10.24 depends on a fundamental theorem due to [Milnor-Moore65]:

Theorem 10.25. *If (A, μ, Δ) is an associative, commutative, connected Hopf algebra over the field $\mathbb{F}_p$, then there is an exact sequence*

$$0 \to \mathrm{Prim}(\xi A) \to \mathrm{Prim}(A) \to Q(A)$$

where $\xi\colon A \to A$ is the **Frobenius homomorphism** $\xi(a) = a^p$.

SKETCH OF A PROOF: The reader can check that the theorem holds for A a monogenic Hopf algebra. For a finitely generated Hopf algebra A and A', a normal sub-Hopf algebra, there are short exact sequences:

$$\begin{array}{ccccccc} 0 \longrightarrow & \mathrm{Prim}(A') & \longrightarrow & \mathrm{Prim}(A) & \longrightarrow & \mathrm{Prim}(A/\!/A') \\ & \downarrow & & \downarrow & & \downarrow \\ & Q(A') & \longrightarrow & Q(A) & \longrightarrow & Q(A/\!/A'). \end{array}$$

We leave it to the reader to show that, if $A' = \xi(A)$, then the mapping $\mathrm{Prim}(A/\!/A') \to Q(A/\!/A')$ is injective. The theorem follows from the diagram of short exact sequences. □

Suppose X is a mod p finite H-space, $\bar{x} \in B_r^{2m}$ is a primitive element and $d_r(\bar{x}) = \bar{y} \neq 0$. Since $H^*(X; \mathbb{F}_p)$ is an associative, commutative connected Hopf algebra, Theorem 10.25 implies that $\bar{y}$, a primitive of odd degree, is not a p^{th} power ($\bar{y}^2 = 0$) and hence $\bar{y}$ is indecomposable. Thus there is an element y in B_{2m+1} with $\bar{y}(y) \neq 0$ and y primitive. Then $\bar{y}(y) = d_r(\bar{x})(y) = \bar{x}(d^r(y)) = \bar{x}(x) \neq 0$, and so $x \in B_{2m}$ is a primitive in the image of d^r. Since $H^*(X; \mathbb{F}_p)$ is a finite vector space, there cannot be ∞-implications, and so the assumption that there is an $\bar{x} \in B_r^{2m}$ with $d_r(\bar{x}) \neq 0$ must fail. Thus, the dual version of Theorem 10.24 for mod p finite H-spaces may be written

$$\mathrm{Im}\, d_r \cap \mathrm{Prim}(B_r^{2m+1}) = \{0\}, \text{ for all } m.$$

From the structure of an exact couple, an element in the image of the descending homomorphism is always a cycle (Proposition 2.9). In the case of the Bockstein spectral sequence, the descending homomorphism is reduction mod p. Thus, for a mod p finite H-space, the image of $\mathrm{red}_{p*}\colon H_*(X) \to H_*(X; \mathbb{F}_p)$ cannot contain an even-dimensional primitive element. If $x \in \mathrm{Im}\,\mathrm{red}_{p*} \cap \mathrm{Prim}(H_{\text{even}}(X; \mathbb{F}_p))$, then $d^r(x) = 0$ for all r and since x cannot persist to B^∞, then $x = d^s(y)$ for some s and y. But then x has ∞-implications and $H_*(X; \mathbb{F}_p)$ has infinite dimension over $\mathbb{F}_p$.

A consequence of this discussion is the theorem of [Browder61] generalizing the classical result of [Cartan, E36] that $\pi_2(G) = \{0\}$ for simply-connected Lie groups.

Theorem 10.26. *If X is a mod p finite H-space, then the least $m > 1$ for which $\pi_m(X) \otimes \mathbb{F}_p \neq \{0\}$ is odd.*

PROOF: Consider the mod p Hurewicz homomorphism $h \otimes \mathbb{F}_p \colon \pi_m(X) \otimes \mathbb{F}_p \to H_m(X) \otimes \mathbb{F}_p \to H_m(X; \mathbb{F}_p)$. This factors through red_{p*} and takes its image in the primitive elements. It follows that this mapping is trivial when m is even.

When X is simply-connected, the Hurewicz-Serre theorem for mod p coefficients (Theorem 6.25) implies that the first nonvanishing homology group $H_m(X; \mathbb{F}_p)$ is isomorphic via $h \otimes \mathbb{F}_p$ to the first nonvanishing homotopy $\mathbb{F}_p$-module $\pi_m(X) \otimes \mathbb{F}_p$. Since this must happen in an odd dimension, the theorem holds.

When X is not simply-connected we can argue using the universal cover $\tilde{X}$. [Browder59] showed that the universal cover of a mod p finite H-space is again a mod p finite H-space. Since $\pi_m(X) \cong \pi_m(\tilde{X})$ for $m > 1$, we reduce to the simply-connected case. □

In developments that grew out of the study of torsion in H-spaces, [Jeanneret92] and [Lin93] have shown that the first nonvanishing homotopy group of a mod 2 finite H-space, whose mod 2 homology ring is associative, must be in degree $1, 3$, or 7.

An H-space with the homotopy type of a finite CW-complex is called a **finite H-space**. The compact Lie groups offer a large class of examples of finite H-spaces. A guiding principle in the study of such spaces is that the topological properties of compact Lie groups have their origin at the homotopical level of structure. That is to say, what is true homotopically of a compact Lie group G ought to be true because G is a finite H-space. Hopf's theorem (10.2) and Browder's theorem (10.26) lend considerable support to this principle. That the class of finite H-spaces is larger than the examples of compact Lie groups is a result of the development of localization and the mixing of homotopy types due to [Zabrodsky72]. [Hilton-Roitberg71] used mixing to exhibit examples of finite H-spaces not of the homotopy type of any compact Lie group.

A major theme in the development of finite H-spaces is the application of the guiding principle to Bott's theorem (10.1)—if X is a simply-connected finite H-space, then $H_*(\Omega X)$ has no torsion.

Under the assumption that X is a simply-connected finite H-space and $H_*(\Omega X)$ has no torsion [Browder63] showed that $H_*(\Omega X) = H_{\mathrm{even}}(\Omega X)$, strengthening Bott's theorem considerably. This paper introduces a family of spectral sequences based on the natural filtrations on a Hopf algebra that interpolate between the terms in the Bockstein spectral sequence and enjoy a particularly nice algebraic expression.

[Kane77] applied work of [Browder63] and [Zabrodsky71] to obtain a necessary and sufficient condition that $H_*(\Omega X)$ have no p-torsion when X is a simply-connected finite H-space. The condition is given in terms of the action of the Steenrod algebra on the cohomology of the finite H-space:

$$Q(H^{\text{even}}(X;\mathbb{F}_p)) = \sum\nolimits_{m\geq 1} \beta P^m Q(H^{2m+1}(X;\mathbb{F}_p)).$$

Notice, in the case that $p = 2$, this condition holds only when $H^*(X;\mathbb{F}_2)$ has no even-dimensional indecomposables. When $p = 2$, $P^m = Sq^{2m}$ and $\beta Sq^{2m} = Sq^1 Sq^{2m} = Sq^{2m+1}$, which is the squaring map on H^{2m+1}. [Lin76, 78] established that $Q(H^{\text{even}}) = \sum_{m\geq 1} \beta P^m Q(H^{\text{odd}})$ holds for odd primes by extending work of [Zabrodsky71] on secondary operations.

Building on work of [Thomas63] on the action of the Steenrod algebra on the cohomology of an H-space, [Lin82] established the absence of 2-torsion in $H_*(\Omega X)$ when X is a mod 2 finite H-space and $H_*(X;\mathbb{F}_2)$ is an associative Hopf algebra. [Kane86] studied the presence of 2-torsion in $H_*(\Omega X)$ by using a version of the Bockstein spectral sequence for the extraordinary cohomology theory $k(n)^*$ introduced by [Morava85]. Putting together all of these developments, the goal of generalizing Bott's theorem was realized.

Theorem 10.27. *If X is a simply-connected finite H-space, then $H_*(\Omega X)$ has no torsion.*

The proof of Theorem 10.27 generated a number of powerful methods in algebraic topology. Accounts of these developments and much more can be found in [Kane88] and [Lin95].

Other applications of the Bockstein spectral sequence Ⓜ

Away from the study of H-spaces, the results of [Browder61] may be applied to obtain some general results about $H^*(\Omega X;\mathbb{F}_p)$. In particular, using ∞-implications, [McCleary87] proved a generalization of the results of [Serre51] (Proposition 5.16) and [Sullivan73] on the nontriviality of $H_*(\Omega X;k)$ for k a field.

Theorem 10.28. *Suppose M is a simply-connected compact finite-dimensional manifold and $\dim_k Q(\tilde{H}^*(M;k)) > 1$, then the set $\{\dim_k H^i(\Omega M;k) \mid i = 1,2,\dots\}$ is unbounded.*

The assumption that $\dim_{\mathbb{F}_p} Q(\tilde{H}^*(X;\mathbb{F}_p)) > 1$ together with the results over $\mathbb{Q}$ of [Sullivan73] force the existence of ∞-implications on two elements. The intertwining of the ∞-implications of these elements in a Hopf algebra gives a subspace of $H^*(\Omega M;\mathbb{F}_p)$ that is isomorphic as a vector space to a polynomial algebra on two generators. The vector space $\mathbb{F}_p[x,y]$ has subspaces $\mathbb{F}_p\{x^{lm}, x^{(l-1)m}y^n, \dots, x^{(l-i)m}y^{in}, \dots, x^m y^{(l-1)n}, y^n\}$ where $m\deg x = n\deg y = \operatorname{lcm}(\deg x, \deg y)$. This subspace has dimension $l+1$ and so grows unbounded with l.

This theorem, like Proposition 5.16, implies geometric results about the geodesics on the manifold M. Under the assumptions of the theorem, the number of geodesics on M joining two nonconjugate points of length less than λ grows at least quadratically in λ.

Another place where p-torsion makes a key appearance is in the Adams spectral sequence. Following the discussion in §9.3, the *times* p map is detected in the Adams spectral sequence by multiplication by a class $a_0 \in \mathrm{Ext}^{1,1}_{\mathcal{A}_p}(\mathbb{F}_p, \mathbb{F}_p)$. For an $\mathcal{A}_p$-module M, [May-Milgram81] say that an element $x \in \mathrm{Ext}_{\mathcal{A}_p}(M, \mathbb{F}_p)$ **generates a spike** if $x \neq a_0 x'$ and $a_0^i x \neq 0$ for all i. There is a single spike in $\mathrm{Ext}_{\mathcal{A}_2}(\mathbb{F}_2, \mathbb{F}_2)$ as the charts (pp. 443-444) in Chapter 9 show—the picture explains the terminology.

[Adams69] wrote of the Adams spectral sequence, "Whenever a chance has arisen to show that a differential d_r is non-zero, the experts have fallen on it with shouts of joy—'Here is an interesting phenomenon! Here is a chance to do some nice, clean research!'—and they have solved the problem in short order." The Bockstein spectral sequence interacts with the Adams spectral sequence to produce differentials that form a coherent pattern. The function $T(s)$ used in the statement of the following theorem refers to Lemma 9.45: When p is odd, then $T(s) = (2p-1)s - 1$; when $p = 2$, then $T(s)$ is defined by $T(4s) = 12s$, $T(4s+1) = 12s+2$, $T(4s+2) = 12s+4$, and $T(4s+3) = 12s+7$.

Theorem 10.29. *Suppose X is an $(n-1)$-connected space of finite type. For $r \geq 1$, suppose that C_r is a basis for $B^r_*(X)$, the homology Bockstein spectral sequence. Assume that C_r is chosen so that $C_r = D_r \cup \beta_r D_r \cup C_{r+1}$ where D_r, $\beta_r D_r$, and C_{r+1} are disjoint, linearly independent subsets of $B^r_*(X)$ such that $\beta_r D_r = \{\beta_r w \mid w \in D_r\}$ and C_{r+1} is a set of cycles with respect to β_r that projects onto the chosen basis for $B^{r+1}_*(X)$. Then*

(1) *The set of spikes in $E_r(X)$, $2 \leq r \leq \infty$, is in one-to-one correspondence with C_r. If $c \in C_r$ has degree q and $\gamma \in E^{s,t}_r(X)$ generates the corresponding spike, then $T(s) - s + n \leq q = t - s$.*
(2) *If $d \in D_r$ and $\delta \in E^{s,t}_r(X)$ and $\epsilon \in E^{u,v}_r(X)$, with $v - u = t - s - 1$, generate spikes corresponding to d and $\beta_r d$, then*

$$d_r(a_0^i \delta) = a_0^{i+r+s-u} \epsilon$$

provided $n + T(i+s) \geq t$.

Proof: Since X is taken to be of finite type, $H_*(X)$ is a direct sum of torsion prime to p, summands of the form $\mathbb{Z}/p^k\mathbb{Z}$, and summands $\mathbb{Z}$ whose generators reduce mod p to the elements of C_∞. We may use this decomposition to construct mappings

$$\phi_i \colon X \longrightarrow K(H_i(X), i)$$

that induce isomorphisms on integral homology in degree i. Let Y denote the space $\bigvee_i K(H_i(X), i)$ and $\phi = \bigvee_i \phi_i \colon X \to Y$ denote the wedge product of all of these mappings. On homology with coefficients in $\mathbb{F}_p$, ϕ_* induces a monomorphism from $H_i(X)$ for all i. This gives rise to a short exact sequence

$$0 \to H_*(X;\mathbb{F}_p) \to H_*(Y;\mathbb{F}_p) \to M_* \to 0,$$

where M_* is seen to be $\bigoplus_{q \geq i+2} H_q(K(H_i(X), i); \mathbb{F}_p)$.

Ignoring the contribution to torsion at primes not equal to p, we know from theorems of Cartan and Serre that the dual of M_* is $A(0)$-free (§9.6), that is, the Bockstein homomorphism on M_*^{dual}, as a differential, is exact. We next examine the long exact sequence of Ext groups associated to the short exact sequence:

$$\to \operatorname{Ext}_{\mathcal{A}_p}^{s-1,t}(M_*^{\text{dual}}, \mathbb{F}_p) \to E_2^{s,t}(X) \to E_2^{s,t}(Y) \to \operatorname{Ext}_{\mathcal{A}_p}^{s,t}(M_*^{\text{dual}}, \mathbb{F}_p) \to .$$

Lemma 9.47 implies that $\operatorname{Ext}_{\mathcal{A}_p}^{s,t}(M_*^{\text{dual}}, \mathbb{F}_p) = \{0\}$ when $0 < s < t \leq n + T(s)$. It follows that $E_2^{s,t}(X) \to E_2^{s,t}(Y)$ is onto in this range and an isomorphism when $s \geq 2$ and $0 < s < t \leq n + T(s-1)$. By the naturality of the Adams spectral sequence, that it suffices to examine the case of Eilenberg-Mac Lane spaces to prove the theorem. We leave it to the reader to show that a factor of $K(\mathbb{Z}/p\mathbb{Z}, i)$ introduces a single copy of $\mathbb{F}_p$ that persists to E_∞; a factor of $K(\mathbb{Z}/p^k\mathbb{Z}, i)$ introduces a pair of spikes at E_2 on generators z and y with $d_k(a_0^i z) = a^{i+k} y$, leaving a basis of $\{a_0^i y \mid 0 \leq i \leq k\}$ at E_∞; finally, a factor of $K(\mathbb{Z}, i)$ introduces a permanent spike at E_2. □

This argument requires that spikes have the right Adams filtration to work. Spikes in $E_2(X)$ could be generated by elements lying in lower filtration degree than in the range of the isomorphism. Such generators might have nontrivial differentials earlier than predicted by the theorem. Such differentials could occur on the bottoms of spikes whose top parts survive to $E_\infty(X)$.

Plugging this argument into a dual setting via Spanier-Whitehead duality, [Meyer98] has used the resulting differentials to compute certain cohomotopy groups and these groups force Euler classes associated to geometric bundles to vanish. These data imply an estimate of certain numerical invariants of lens spaces. Let

$$v_{p,2}(m) = \min\{n \mid \text{ there is a } \mathbb{Z}/p\mathbb{Z}\text{-equivariant } f \colon L^{2m-1}(p) \to S^{2n-1}\}.$$

Here the action of $\mathbb{Z}/p\mathbb{Z}$ on $L^{2m-1}(p)$ is induced by the multiplication by a primitive root of unity of order p^2 on $\mathbb{C}^m$ and on S^{2n-1} by the standard action. The estimates of [Meyer, D98] generalize work of [Stolz89] at the prime 2.

10.2 Other Bockstein spectral sequences

Consider the cofibration sequence

$$S^{n-1} \xrightarrow{r} S^{n-1} \xrightarrow{\beta} P^n(r) \xrightarrow{\eta} S^n \xrightarrow{r} S^n$$

where $P^n(r) = S^{n-1} \cup_r e^n$ is a mod r Moore space and r denotes the degree r map on S^{n-1}. Following [Peterson56], these spaces may be used to define the **mod r homotopy groups**,

$$\pi_n(X; \mathbb{Z}/r\mathbb{Z}) = [P^n(r), X].$$

The properties of cofibration sequences lead to an exact couple

$$\begin{array}{ccc} \pi_*(X) & \xrightarrow{\quad r \quad} & \pi_*(X) \\ & {}_{\beta}\nwarrow \quad \swarrow_{\eta} & \\ & \pi_*(X; \mathbb{Z}/r\mathbb{Z}) & \end{array}$$

and hence a Bockstein spectral sequence, denoted by ${}_\pi B^r_*(X)$, with ${}_\pi B^1_*(X) \cong \pi_*(X; \mathbb{Z}/r\mathbb{Z})$. When $r = p$, a prime, the spectral sequence converges to $(\pi_*(X)/\text{torsion}) \otimes \mathbb{F}_p$ for X of finite type. (Some care has to be taken when $p = 2$ because $\pi_3(X; \mathbb{Z}/2\mathbb{Z})$ need not be abelian.) This spectral sequence was studied by [Araki-Toda65] for applications to generalized cohomology theories, by [Browder78] for applications to algebraic K-theory, and by [Neisendorfer72] for its relations to unstable homotopy theory.

Among the properties of the Moore spaces is the following result of [Neisendorfer72]. The proof requires careful bookkeeping in low dimensions (for details see the memoir of [Neisendorfer80]).

Proposition 10.30. *If m, $n \geq 2$ and r, s are positive integers for which $d = \gcd(r, s)$ is odd, then there is a homotopy equivalence:*

$$\alpha_{m,n} \colon P^{m+n}(d) \vee P^{m+n-1}(d) \to P^m(r) \wedge P^n(s).$$

When $r = s = p$, an odd prime, this homotopy equivalence may be used to define pairings on mod p homotopy groups. In particular, given $f \colon P^m(r) \to X$ and $g \colon P^n(s) \to Y$, we can use the canonical injection, $x \mapsto (x, *)$, $P^{m+n}(d) \to P^{m+n}(d) \vee P^{m+n-1}(d)$ to obtain a mapping $P^{m+n}(d) \to X \wedge Y$ as the composite

$$P^{m+n}(d) \to P^{m+n}(d) \vee P^{m+n-1}(d) \xrightarrow{\alpha_{m,n}} P^m(r) \wedge P^n(s) \xrightarrow{f \wedge g} X \wedge Y.$$

A mapping $\sigma\colon X\wedge Y\to Z$ induces a pairing $\pi_m(X;\mathbb{Z}/r\mathbb{Z})\otimes\pi_n(Y;\mathbb{Z}/s\mathbb{Z})\to\pi_{m+n}(Z;\mathbb{Z}/d\mathbb{Z})$ and this pairing for $X=Y=Z=B\mathrm{Gl}(\Lambda)^+$ was developed by [Browder78] to study the algebraic K-theory with coefficients of a ring Λ via the homotopy Bockstein spectral sequence.

When (G,μ,e) is a **grouplike space**, that is, G is a homotopy associative H-space with a homotopy inverse (for example, a based loop space ΩX), then the commutator mapping $[\ ,\ ]\colon G\times G\to G$, given by $(x,y)\mapsto(xy)(x^{-1}y^{-1})$, determines a mapping $[\ ,\ ]\colon G\wedge G\to G$, since, up to homotopy, the commutator mapping restricted to $G\vee G$ is homotopic to the constant mapping to e. This mapping may be applied to the homotopy groups of G with coefficients to define a pairing for $d=\gcd(r,s)$:

$$[\ ,\ ]\colon \pi_m(G;\mathbb{Z}/r\mathbb{Z})\otimes\pi_n(G;\mathbb{Z}/s\mathbb{Z})\to\pi_{m+n}(G;\mathbb{Z}/d\mathbb{Z}).$$

The pairing is given by the composite

$$P^{m+n}(d)\to P^{m+n}(d)\vee P^{m+n-1}(d)\to P^m(r)\wedge P^n(s)\xrightarrow{f\wedge g} G\wedge G\xrightarrow{[\ ,\ ]} G.$$

The pairing induced on homotopy groups by the commutator mapping is the Samelson product. The properties of the generalized Samelson product for homotopy groups with coefficients are extensively developed by [Neisendorfer80]. In particular, we have the following result.

Proposition 10.31. *If $r=s=d$, $\gcd(r,6)=1$, and G is a 2-connected, grouplike space, then $\pi_*(G;\mathbb{Z}/r\mathbb{Z})$ is a graded Lie algebra.*

When G is grouplike, $H_*(G;\mathbb{Z}/r\mathbb{Z})$ is an associative algebra and hence enjoys a Lie algebra structure given by $[z,w]=zw-(-1)^{|z|\cdot|w|}wz$. The Hurewicz map, $h_*\colon\pi_*(X;\mathbb{Z}/r\mathbb{Z})\to H_*(X;\mathbb{Z}/r\mathbb{Z})$, is induced by $h_*([f])=f_*(y)$, where $y\in H_m(P^m(r);\mathbb{Z}/r\mathbb{Z})$ is the canonical generator. This mapping for $r=p$, an odd prime, induces a mapping ${}_\pi B^1_*(X)\to B^1_*(X)$, where $\{B^s_*(X),d^s\}$ denotes the mod p homology Bockstein spectral sequence. [Neisendorfer72] showed that both the homotopy and homology Bockstein spectral sequences are spectral sequences of Lie algebras for $p>3$, and that the Hurewicz homomorphism induces a Lie algebra homomorphism on B^s-terms for all s.

It is possible to develop the properties of differential Lie algebras by analogy with the development of differential Hopf algebras for the Bockstein spectral sequence. This development makes up the first few sections of [Cohen-Moore-Neisendorfer79], especially applied to the case of free Lie algebras. These results may be used to study the spaces $\Omega P^n(p^r)$ and $\Omega F^n(p^r)$, where $F^n(p^r)$ is the homotopy fibre of the pinch map $P^n(p^r)\to S^n$, defined by collapsing the bottom cell. The main results of [Cohen-Moore-Neisendorfer79] are homotopy equivalences between the space $\Omega P^n(p^r)$ (suitably localized) and products of countable wedges of known spaces whose structure may be read off the behavior of Bockstein spectral sequence. A similar result holds for $\Omega F^n(p^r)$. The comparison of the homotopy and homology Bockstein spectral sequences via the Hurewicz homomorphism allows one to obtain representative mappings that go into the construction of the homotopy equivalences. Finally, the decompositions are used to establish the inductive argument that goes from the theorem of [Selick78], that p annihilates the p-component ${}_{(p)}\pi_k(S^3)$ for $k \neq 3$ and $p > 3$, to prove the following result.

Theorem 10.32. *If $p > 3$, and $n > 0$, then p^{n+1} annihilates ${}_{(p)}\pi_k(S^{2n+1})$, for all $k > 2n+1$.*

The final generalization of the Bockstein spectral sequence that we present is best framed in the language of spectra and generalized cohomology theories. If $\boldsymbol{X}$ is a spectrum and $f\colon \boldsymbol{X} \to \boldsymbol{X}$ is a selfmap of degree k, then we can form the cofibre of f in the category of spectra and obtain an exact couple:

$$\begin{array}{ccc} [W,\boldsymbol{X}] & \xrightarrow{\quad f^* \quad} & [W,\boldsymbol{X}] \\ & \nwarrow \qquad \swarrow & \\ & [W, \text{cofibre}(f)] & \end{array}$$

The mapping f may be thought of as a cohomology operation and $[W,\boldsymbol{X}] = \boldsymbol{X}^*(W)$ as the value of the associated generalized cohomology theory on W. If $\boldsymbol{X} = H\mathbb{Z}$, the Eilenberg-Mac Lane spectrum for integer coefficients, and f represents the *times* p map, then the cofibre represents the Eilenberg-Mac Lane spectrum $H\mathbb{F}_p$ and we obtain the usual Bockstein spectral sequence.

Let $k(n)^*(\ )$ denote the generalized cohomology functor known as **connective Morava K-theory** (see the work of [Würgler77] for the definition and properties). This theory has certain remarkable properties:

(1) $k(n)^*(\text{point}) \cong \mathbb{F}_p[v_n]$ where v_n has degree $-2p^n+2$.
(2) $k(n)^*(W)$ has a direct sum decomposition into summands $\mathbb{F}_p[v_n]$ and $\mathbb{F}_p[v_n]/(v_n^s)$.

Property (2) is analogous to the result for a finitely generated abelian group modulo torsion away from a prime p where the summands are $\mathbb{Z}$ and $\mathbb{Z}/p^s\mathbb{Z}$. We choose the mapping of the representing spectrum for Morava K-theory that induces the *times* v_n map. The cofibre is represented by $H\mathbb{F}_p$ and the exact couple for a finite H-space X may be presented as

$$\begin{array}{ccc} k(n)^*(X) & \xrightarrow{\quad -\times v_n \quad} & k(n)^*(X) \\ & \nwarrow \qquad \swarrow {\scriptstyle \rho_n} & \\ & H^*(X;\mathbb{F}_p) & \end{array}$$

where ρ_n is mod v_n reduction. The Bockstein spectral sequence in this case has $B_1^* = H^*(X;\mathbb{F}_p)$ and the first differential d^1 is identifiable with Q_n, the Milnor primitive in $\mathcal{A}_p^{\text{dual}}$ ([Milnor58], [Kane86]). The limit term, B_∞, is given by $(k^*(n)(X)/v_n\text{-torsion}) \otimes_{\mathbb{F}_p[v_n]} \mathbb{F}_p$. The v_n-torsion subgroup of $k(n)^*(X)$ consists of elements annihilated by some power of v_n. [Johnson, D73] identified this spectral sequence with an Atiyah-Hirzebruch spectral sequence (Theorem 11.16). It follows from this observation that the spectral sequence supports a commutative and associative multiplication. [Kane86] developed many properties of this spectral sequence for the prime 2 including a notion of infinite implications that played a key role in a proof of Theorem 10.27. [Kane86] conjectured that, for a mod 2 finite H-space (X,μ,e), the Bockstein spectral sequence for Morava K-theory should satisfy the following two properties:

(1) The even degree algebra generators of $H^*(X;\mathbb{F}_2)$ can be chosen to be permanent cycles in B_r.
(2) In degrees greater than or equal to 2^{n+1}, the even degree generators can be chosen to be boundaries in B_r.

If these conjectures were to hold, a simple proof of the absence of 2-torsion in $H_*(\Omega X)$ for a mod 2 finite H-space (X,μ,e) would be possible (as outlined by [Kane86]).

Exercises

10.1. Show that the condition, $H^{\text{odd}}(\Omega G;k) = \{0\}$ for all fields k, implies that $H^*(\Omega G)$ is torsion-free.

10.2. Prove that a commutative, associative Hopf algebra over a field of characteristic zero that is generated by odd-dimensional generators is an exterior algebra.

10.3. From the structure of $H^*(\mathbb{R}P^n;\mathbb{F}_2)$ as a module over the Steenrod algebra, determine completely the mod 2 Bockstein spectral sequence for $\mathbb{R}P^n$.

10.4. The mod 2 cohomology of the exceptional Lie group G_2 is given by

$$H^*(G_2;\mathbb{F}_2) \cong \mathbb{F}_2[x_3,x_5]/\langle x_3^4, x_5^2\rangle.$$

The rational cohomology of G_2 is given by $H^*(G_2;\mathbb{Q}) \cong \Lambda(X_3,X_{11})$. From these data determine the mod 2 Bockstein spectral sequence for G_2.

10.5. Prove Proposition 10.8 and Lemma 10.9.

10.6. Prove the analogue of Corollary 10.14 for $K(\mathbb{Z}/2^k\mathbb{Z},n)$.

10.7. Suppose X is an H-space and $\pi\colon \bar{X} \to X$ a covering space of X. Then $\bar{X}$ is an H-space and π a multiplicative mapping. Use the Cartan-Leray spectral sequence (Theorem 8^{bis}.9) which is a spectral sequence of Hopf algebras in this case to prove that if X is a mod p finite H-space, then $\bar{X}$ is a mod p finite H-space ([Browder59]).

10.8. Show that if A' is a normal sub-Hopf algebra of the Hopf algebra A, then there is a diagram of short exact sequences:

$$\begin{array}{ccccccc} 0 \longrightarrow & \mathrm{Prim}(A') & \longrightarrow & \mathrm{Prim}(A) & \longrightarrow & \mathrm{Prim}(A//A') \\ & \downarrow & & \downarrow & & \downarrow \\ & Q(A') & \longrightarrow & Q(A) & \longrightarrow & Q(A//A'). \end{array}$$

Use this fact to give a complete proof of Theorem 10.25.

10.9. Show that the universal examples of $K(\mathbb{Z}/p^k\mathbb{Z}, n)$, for $k > 0$, and $K(\mathbb{Z}, n)$ lead to the spikes and differentials in the Adams spectral sequence as predicted by Theorem 10.29.

10.10. Suppose that M is compact, closed manifold (or more generally a Poincaré duality space). If M has dimension $4m + 1$, then prove the following result due to [Browder62']: either (1) $H_{2m}(M) \cong F \oplus T \oplus T$, where F is a free abelian group and T is a torsion group, or (2) $H_{2m}(M) \cong F \oplus T \oplus T \oplus \mathbb{Z}/2\mathbb{Z}$ and in this case, $Sq^{2m}\colon H^{2m+1}(M; \mathbb{F}_2) \to H^{4m+1}(M; \mathbb{F}_2)$ is nonzero.