4. Building New Spaces From Old

The use of figures is, above all, then, for the purpose of making known certain relations between the objects that we study, and these relations are those which occupy the branch of geometry that we have called Analysis Situs*, ...*

J. Henri Poincaré, 1895

Having introduced topologies on sets and continuous functions, we next apply set-theoretic constructions to form new topological spaces. The principal examples are:

1) the formation of subsets,
2) the formation of products, and
3) the formation of quotients by equivalence relations.

In later chapters, we will also introduce function spaces. In all cases we are guided by the need to construct natural continuous functions.

Subspaces

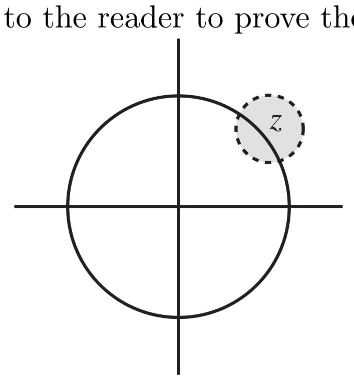
Many interesting mathematical objects are subsets of Euclidean space, which is a topological space—how are these subsets topological spaces? By restricting the metric to a subset, it becomes a metric space and so has a topology. However, this procedure does not generalize to all topological spaces. We need a more flexible approach.

For any subset A of a set X, we associate the function $i\colon A \to X$ given by $i(a) = a$ (*the inclusion*). Restriction of a function $f\colon X \to Y$ to the subset A becomes a composite $f|_A = f \circ i\colon A \to Y$. To topologize a subset A of X, a topological space, we want that restriction to A of a continuous function on X be continuous. This is accomplished by giving A a topology for which $i\colon A \to X$ is continuous.

Definition 4.1. *Let X be a topological space with topology $\mathcal{T}$ and A, a subset of X. The* **subspace topology** *on A is given by $\mathcal{T}_A = \{U \cap A \mid U \in \mathcal{T}\}$, also called the* **relative topology** *on A.*

Proposition 4.2. *The collection $\mathcal{T}_A$ is a topology on A and with this topology the inclusion $i\colon A \to X$ is continuous.*

Proof: If U is open in X, then $i^{-1}(U) = U \cap A$, which is open in A. The fact that $\mathcal{T}_A$ is a topology on A is easy to prove and, in fact, it is the smallest topology on A making $i\colon A \to X$ continuous. We leave it to the reader to prove these assertions. ◊

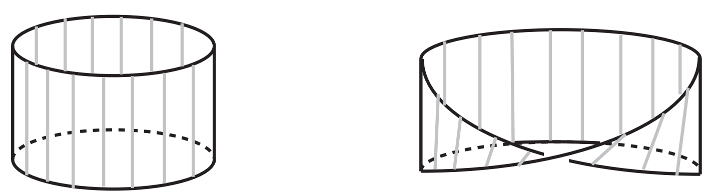


Example 1: Some interesting spaces are the **spheres** in $\mathbb{R}^n$, for $n \geq 1$. They are given by

$$S^{n-1} = \{\mathbf{x} \in \mathbb{R}^n \mid \|\mathbf{x}\| = 1\}.$$

Thus $S^0 = \{-1, 1\} \subset \mathbb{R}$, and $S^1 \subset \mathbb{R}^2$, etc. Open sets in S^1 are easily to picture: the intersection of an open ball in $\mathbb{R}^2$ with S^1 gives a sort of 'interval' in S^1. To be precise, take any point $z \in S^1$ with $z = (\cos\theta_0, \sin\theta_0)$, and let $w\colon (-\epsilon, \epsilon) \longrightarrow S^1$ be the mapping $r \mapsto (\cos(\theta_0 + r), \sin(\theta_0 + r))$. Then let $\rho = d(z, (\cos(\theta_0 + \epsilon), \sin(\theta_0 + \epsilon)))$. For small ϵ, we get $w^{-1}(B(z, \rho)) = (-\epsilon, \epsilon)$ and the mapping w is a homeomorphism. Thus each point of S^1 has a neighborhood around it homeomorphic to an open set in $\mathbb{R}$. This condition is special and characterizes S^1 as a 1-dimensional manifold. More on this later.

Example 2: Some interesting subspaces of $\mathbb{R}^3$ are pictured here: they are the cylinder and the Möbius band. (Are they homeomorphic?)



If a space X has a topological property, does a subset A of X as a subspace share it? Such a property is called **hereditary**.

PROPOSITION 4.3. *Metrizability is a hereditary property. The Hausdorff condition is also hereditary.*

Proof: That metrizability is hereditary is left to the reader to prove. To see how the Hausdorff condition is hereditary, suppose $a, b \in A$. Then a, b are also in X, which is Hausdorff. So there are open sets U, V in X with $a \in U$, $b \in V$, and $U \cap V = \emptyset$. Consider $U \cap A$ and $V \cap A$. Since these are non-empty, disjoint, open sets in A with $a \in U \cap A$ and $b \in V \cap A$, we have that A is Hausdorff. $\diamondsuit$

Reversing the notion of a hereditary property, we consider properties that, when they hold on a subspace, can be seen to hold on the whole space. For example, one can build continuous mappings this way:

THEOREM 4.4. *Suppose $X = A \cup B$ is a space, A, B, open subsets of X, and $f\colon A \to Y$, $g\colon B \to Y$ are continuous functions (where A and B have the subspace topologies). If $f(x) = g(x)$ for all $x \in A \cap B$, then $F = f \cup g\colon X \to Y$ is a continuous functions where F is defined by*

$$F(x) = \begin{cases} f(x), & \text{if } x \in A, \\ g(x), & \text{if } x \in B. \end{cases}$$

Proof: The condition that f and g agree on $A \cap B$ implies that F is well-defined. Let U be open in Y and consider $F^{-1}(U) = (f^{-1}(U) \cap A) \cup (g^{-1}(U) \cap B)$. The subset $f^{-1}(U) \cap A$ is open in A so it equals $V \cap A$ where V is open in X. But since A is open, $V \cap A$ is open in X, so $f^{-1}(U) \cap A$ is open in X. Similarly $g^{-1}(U) \cap B$ is open in X and their union is $F^{-1}(U)$. Thus F is continuous. $\diamondsuit$

If a space breaks up into disjoint open pieces, then continuity of a function defined on the whole space is determined by continuity on each piece.

There is a similar characterization for A, B closed in X. A subset $K \subset A$ is closed in A if there is an $L \subset X$ closed in X with $K = L \cap A$. To see this write $A - K = A \cap (X - L)$.

More generally, when A is a subspace of X and $f: A \to Y$ is a continuous function, is there an extension of f to all of X, $\hat{f}: X \to Y$, that is continuous, for which $f = \hat{f} \circ i$? This problem is called the **extension problem** and it is a common formulation of many problems in topology. An example where it is known to fail is the inclusion

$$i: S^{n-1} \to e^n = \text{cls } B(\mathbf{0}, 1) = \{\mathbf{x} \in \mathbb{R}^n \mid \|\mathbf{x}\| \leq 1\} \subset \mathbb{R}^n,$$

with respect to the mapping $\text{id}: S^{n-1} \to S^{n-1}$ (Brouwer Fixed Point Theorem in Chapter 11). The corollaries of this failure are numerous.

An extension problem with a positive solution is the following result.

Tietze Extension Theorem. *Any continuous function $f: A \to \mathbb{R}$ from a closed subspace A of a metric space (X, d) has an extension $g: X \to \mathbb{R}$ that is also continuous.*

We first prove a couple of lemmas:

Lemma 4.5. *For A a closed subset of (X, d), a metric space, let $d(x, A) = \inf\{d(x, a) \mid a \in A\}$. Then the function $x \mapsto d(x, A)$ is continuous on X.*

This is left to the reader to prove.

Lemma 4.6. *If A and B are disjoint closed subsets of (X, d), there is a real-valued continuous function in X with value 1 on A, -1 on B and values in $(-1, 1) \subset \mathbb{R}$ on $X - (A \cup B)$.*

Proof: Consider the function

$$g(x) = \frac{d(x, B) - d(x, A)}{d(x, A) + d(x, B)}.$$

Because A and B are disjoint and closed, $d(x, A) + d(x, B) > 0$ and $g(x)$ is well-defined. By Lemma 4.5 and the usual theorems of real analysis, $g(x)$ is continuous, and it is rigged to satisfy the statement of the lemma. ♢

Proof of Tietze's Theorem: ([Munkres, p. 212]) We first suppose $|f(x)| \leq M$ for all $x \in A$. Define

$$A_1 = \{x \in A \mid f(x) \geq M/3\}, \quad B_1 = \{x \in A \mid f(x) \leq -M/3\};$$

A_1 and B_1 are closed in A and hence in X. By Lemma 4.6, there is a continuous mapping, $g_1: X \to [-M/3, M/3]$ with $g_1(a) = M/3$ for $a \in A_1$, $g_1(b) = -M/3$ for $b \in B_1$ and taking values in $(-M/3, M/3)$ on $X - (A_1 \cup B_1)$. Since $|f(x)| \leq M$, $|f(x) - g_1(x)| \leq 2M/3$ for $x \in A$.

Next consider $f(x) - g_1(x)$ on A and define

$$A_2 = \{x \in A \mid f(x) - g_1(x) \geq 2M/9\}, \quad B_2 = \{x \in A \mid f(x) - g_1(x) \leq -2M/9\}.$$

As above A_2, B_2 are closed and disjoint and so there is a continuous function $g_2: X \to [-2M/9, 2M/9]$ with $g_2(a) = 2M/9$ for $a \in A_2$, $g_2(b) = -2M/9$ for $b \in B_2$ and taking values in $(-2M/9, 2M/9)$ on $x \in X - (A_2 \cup B_2)$. Notice, for $x \in A$, $|f(x) - g_1(x) - g_2(x)| \leq 4M/9$.

Iterate this process to get $g_n: X \to [-2^{n-1}M/3^n, 2^{n-1}M/3^n]$ such that

i) $|f(x) - g_1(x) - g_2(x) - \cdots - g_n(x)| \leq 2^n M/3^n$ on A

ii) $|g_n(x)| < 2^{n-1}M/3^n$ on $X - A$.
For all $x \in X - A$, the infinite series satisfies

$$\left|\sum_{n=1}^{\infty} g_n(x)\right| \leq \sum_{n=1}^{\infty} |g_n(x)| \leq M \sum_{n=1}^{\infty} 2^{n-1}/3^n = M,$$

and so $g(x) = \sum_{n=1}^{\infty} g_n(x)$ converges absolutely and hence converges, defining g on $X - A$. Furthermore, $g(x) = f(x)$ for $x \in A$, and so $g(x)$ is defined for all $x \in X$; also, $|g(x)| < M$ on X and g is bounded.

To show that g is continuous, let $x_0 \in X$. We show that for any $\epsilon > 0$ there is a $\delta > 0$ such that whenever $d(x_0, x) < \delta$, then $|g(x_0) - g(x)| < \epsilon$. Define $s_n(x) = \sum_{k=1}^{n} g_k(x)$, the nth partial sum of $g(x)$. Since, for all $x \in X - A$,

$$|g(x) - s_n(x)| = \left|\sum_{k=n+1}^{\infty} g_k(x)\right| \leq \sum_{k=n+1}^{\infty} |g_k(x)| \leq \sum_{k=n+1}^{\infty} 2^{k-1}M/3^k = M(2/3)^n,$$

then there is an N for which $|g(x) - s_n(x)| < \epsilon/3$ for $n \geq N$. On A, $|g(a) - s_n(a)| = |f(a) - s_n(a)| < 2^n M/3^n$, and so there is an N' with $|f(a) - s_n(a)| < \epsilon/3$ for $n \geq N'$. Let $N_1 = \max\{N, N'\}$.

Since $s_n(x)$ is a finite sum of continuous functions, for each n, there is a $\delta_n > 0$ for which $|s_n(x_0) - s_n(y)| < \epsilon/3$ whenever $d(x_0, y) < \delta_n$. Suppose that $L > N_1$. Then, for all $y \in X$ with $d(x_0, y) < \delta_L$, we have

$$\begin{aligned} |g(x_0) - g(y)| &= |g(x_0) - s_L(x_0) + s_L(x_0) - s_L(y) + s_L(y) - g(y)| \\ &\leq |g(x_0) - s_L(x_0)| + |s_L(x_0) - s_L(y)| + |g(y) - s_L(y)| < \epsilon. \end{aligned}$$

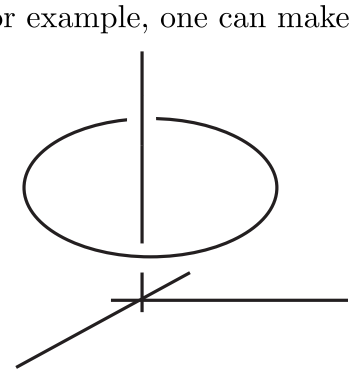
Thus, for any $x_0 \in X$, g is continuous at x_0, and so g is continuous.

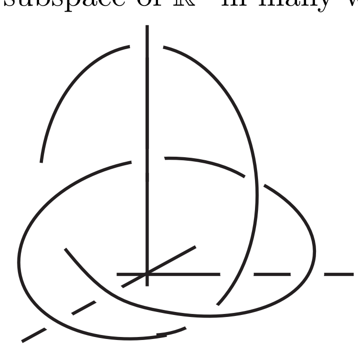
For an unbounded mapping $f\colon A \to \mathbb{R}$, apply the invertible mapping $h\colon \mathbb{R} \to (-1, 1)$ given by $h(r) = (2/\pi)\arctan(r)$. Let $F = h \circ f$. Then F is bounded and we can carry out the argument for F as in the bounded case to get G on X, with codomain $(-1, 1)$. Let $g = h^{-1} \circ G$. On A,

$$g = h^{-1} \circ G = h^{-1} \circ F = h^{-1} \circ h \circ f = f,$$

so g extends f to all of X. ♢

The manner in which a subspace sits inside a larger space determines new things about the space. For example, one can make a circle a subspace of $\mathbb{R}^3$ in many ways:





The study of such embeddings is another important part of topology called *knot theory* (see [Adams], [Burde-Zieschang]).

One way to focus on a subspace within a space is through the continuous functions.

DEFINITION 4.7. *A* **topological pair** *is a space X together with a subspace A, written (X, A). A* **mapping of pairs** *(a continuous function of pairs), $f\colon (X, A) \to (Y, B)$, is a continuous function $f\colon X \to Y$ satisfying the additional property $f(A) \subset B$.*

A composite of mappings of pairs gives a mapping of pairs and the identity mapping on a pair is a mapping of pairs. Two pairs are *homeomorphic* if there is a mapping of pairs $f\colon (X, A) \to (Y, B)$ with $f\colon X \to Y$ a homeomorphism and $f|_A\colon A \to B$ another homeomorphism. The notion of equivalence of knots reduces to whether there is a homeomorphism of pairs $(\mathbb{R}^3, K) \to (\mathbb{R}^3, K')$ where K and K' are knots, the images of homeomorphisms of S^1 with subspaces of $\mathbb{R}^3$.

A particular example of a topological pair is a pointed space.

DEFINITION 4.8. *Given a space X, a* **basepoint** *for X is a choice of point x_0 in X. We denote the pair $(X, \{x_0\}) = (X, x_0)$, and call (X, x_0) a* **pointed space**. *The mappings $f\colon (X, x_0) \to (Y, y_0)$ of such pairs, are called* **pointed maps**.

Example: Let $[0, 1] \subset \mathbb{R}$ with the usual topology denote the *unit interval*. A **path** in a space X is a continuous function $f\colon [0, 1] \to X$. Choose $0 \in [0, 1]$ as basepoint and define the set

$$PX = \mathrm{Hom}(([0, 1], 0), (X, x_0)) = \{f\colon [0, 1] \to X \mid f(0) = x_0, f \text{ continuous}\},$$

the *set of all paths in X beginning at* x_0. We can also consider the set of mappings of pairs $\Omega(X, x_0) = \mathrm{Hom}(([0, 1], \{0, 1\}), (X, x_0))$, the *set of all paths in X beginning and ending at* x_0, also called the **loops** on X based at x_0. The loops could be described equally well as $\mathrm{Hom}((S^1, 1), (X, x_0))$ where S^1 is the circle in $\mathbb{R}^2 = \mathbb{C}$ and $1 = e^{i\cdot 0} = 1 + 0i$ is chosen as basepoint for S^1. More on this set in Chapter 7.

PRODUCTS

Take a pair of topological spaces, X, Y, and form their cartesian product

$$X \times Y = \{(x, y) \mid x \in X, y \in Y\}.$$

How can this set be topologized to get a new space? Such a topology should make the associated projection functions continuous, namely,

$$\mathrm{pr}_1\colon X \times Y \longrightarrow X, \qquad \mathrm{pr}_2\colon X \times Y \longrightarrow Y.$$

If U is open in X then $\mathrm{pr}_1^{-1}(U) = U \times Y$. Similarly, if V is open in Y, then $\mathrm{pr}_2^{-1}(V) = X \times V$. At the very least, we need the collection

$$\mathcal{S} = \{U \times Y, X \times V \mid U \text{ open in } X,\ V \text{ open in } Y\}$$

to lie in our topology on $X \times Y$. In the exercises to Chapter 2, we identified collections like $\mathcal{S}$ called *subbases* for which the collection

$$\mathcal{B} = \{S_1 \cap \ldots \cap S_n \mid n \geq 1, S_i \in \mathcal{S}\}$$

forms a basis for a topology on $X \times Y$.

DEFINITION 4.9. *The* **product topology** *on* $X \times Y$ *is the topology generated by the basis* $\mathcal{B} = \{U \times V \mid U \text{ open in } X,\ V \text{ open in } Y\}$.

To see that we have the same basis as generated by the subbasis $\mathcal{S}$ observe that $(U \times Y) \cap (X \times V) = U \times V$. Thus the projections are continuous with the product topology on $X \times Y$. More can be said:

PROPOSITION 4.10. *Given three topological spaces* X, Y, *and* Z, *and a function* $f\colon Z \to X \times Y$, *then* f *is continuous if and only if* $\mathrm{pr}_1 \circ f\colon Z \to X$ *and* $\mathrm{pr}_2 \circ f\colon Z \to Y$ *are continuous.*

Proof: Certainly f being continuous implies $\mathrm{pr}_1 \circ f$ and $\mathrm{pr}_2 \circ f$ are continuous. To prove the converse, suppose W is an open set in $X \times Y$. Then W is a union of $U_i \times V_i$ with each U_i open in X, V_i open in Y. Since $f^{-1}(\bigcup(U_i \times V_i)) = \bigcup f^{-1}(U_i \times V_i)$, we can restrict our attention to a basis open set. The subsets $(\mathrm{pr}_1 \circ f)^{-1}(U_i)$ and $(\mathrm{pr}_2 \circ f)^{-1}(V_i)$ are both open in Z by the hypotheses. The proof reduces to proving

$$f^{-1}(U_i \times V_i) = (\mathrm{pr}_1 \circ f)^{-1}(U_i) \cap (\mathrm{pr}_2 \circ f)^{-1}(V_i) :$$

If z is in $f^{-1}(U_i \times V_i)$, then $f(z) \in U_i \times V_i$ and $\mathrm{pr}_1 \circ f(z) \in U_i$, $\mathrm{pr}_2 \circ f(z) \in V_i$. Thus $f^{-1}(U_i \times V_i) \subset (\mathrm{pr}_1 \circ f)^{-1}(U_i) \cap (\mathrm{pr}_2 \circ f)^{-1}(V_i)$. If $z \in (\mathrm{pr}_1 \circ f)^{-1}(U_i) \cap (\mathrm{pr}_2 \circ f)^{-1}(V_i)$, then $f(z) \in \mathrm{pr}_1^{-1}(U_i) \cap \mathrm{pr}_2^{-1}(V_i) = U_i \times V_i$. ♢

By induction, we can endow a finite product $X_1 \times X_2 \times \cdots \times X_n$ with a topology for which the projections $\mathrm{pr}_i\colon X_1 \times X_2 \times \cdots \times X_n \to X_i$, $\mathrm{pr}_i(x_1, \ldots, x_n) = x_i$, are continuous. Proposition 4.10 generalizes for functions $f\colon Z \to X_1 \times X_2 \times \cdots \times X_n$ that are continuous if and only if all the compositions $\mathrm{pr}_i \circ f$ are continuous. This generalizes the fact from classical analysis that a function $f\colon Z \to \mathbb{R}^n$ is continuous if and only if the coordinate functions expressing f are continuous.

We had hereditary properties for subspaces, are there topological properties that go over to products when they hold for each factor? We give an example:

PROPOSITION 4.11. *If* X *and* Y *are separable spaces, so is* $X \times Y$.

Proof: Let $A \subset X$ and $B \subset Y$ be countable dense subsets. Then $A \times B \subset X \times Y$ is also countable. To see that it is dense, suppose $(x, y) \in X \times Y$ and $(x, y) \notin A \times B$, and W is an open set in $X \times Y$ with $(x, y) \in W$. Then there is a basis open set $U \times V$ with $(x, y) \in U \times V \subset W$. Since A is dense in X, there is an $a \in A$ with $a \neq x$ and $a \in U$. Similarly there is a $b \in B$, $b \in V$ and $b \neq y$. Thus $(a, b) \in W$ with $(a, b) \neq (x, y)$. Hence (x, y) is a limit point of $A \times B$, and $\mathrm{cls}\,(A \times B) = X \times Y$. ♢

Many other properties act analogously, for example, the Hausdorff condition, or second countability, and others.

We can extend the notion of product to infinite products and then extend the product topology to them; this requires care.

DEFINITION 4.12. *Let* $\{X_\alpha \mid \alpha \in J\}$ *be any collection of nonempty sets. The* **product** *of the sets* $\prod_{\alpha \in J} X_\alpha$ *is the set of all functions* $c\colon J \to \bigcup_{\alpha \in J} X_\alpha$ *with* $c(\alpha) \in X_\alpha$ *for all* $\alpha \in J$. *For any* $\beta \in J$, *the* **projection** $\mathrm{pr}_\beta\colon \prod_{\alpha \in J} X_\alpha \to X_\beta$ *is given by evaluation of such a function* c *on* β, $c \mapsto c(\beta)$.

This structure describes products for any collection and generalizes finite products for which the indexing set is $\{1, 2, \ldots, n\}$. Why do we need such notions? Consider $\mathbb{R}^\omega = \{(r_1, r_2, r_3, \ldots)$ such that $r_i \in \mathbb{R}\}$, the countable product of $\mathbb{R}$ with itself. A nice example of a subspace of $\mathbb{R}^\omega$ is an important space in analysis that generalizes $\mathbb{R}^n$

$$l^2 = \{\text{ square summable sequences of } \mathbb{R}\,\} = \{(r_1, r_2, r_3, \ldots) \mid \sum_{i=1}^{\infty} r_i^2 < \infty\}.$$

The norm $\|(r_1, r_2, r_3, \ldots)\| = \sqrt{\sum_i r_i^2}$ provides a distance function and hence a metric space structure on l^2.

What is the infinite analogue of the product topology on $X \times Y$? Two alternatives are possible: let $\prod_{\alpha \in J} X_\alpha$ be a product of spaces $\{X_\alpha \mid \alpha \in J\}$,

i) $\mathcal{T}_{\text{box}}$ = the topology generated by the basis $\mathbb{B} = \{\prod_{\alpha \in J} U_\alpha \mid U_\alpha \subset X_\alpha$ for all α, each U_α open in $X_\alpha\}$.

ii) $\mathcal{T}_{\text{prod}}$ = the topology generated by the basis $\mathcal{B} = \{S_1 \cap S_2 \cap \cdots \cap S_n \mid n \geq 1, S_i \in \mathcal{S}\}$, where $\mathcal{S}$ is the subbasis of subsets $S = \prod_{\alpha \in J} V_\alpha$, where for each $\beta \in J$, V_β is open in X_β and $V_\gamma = X_\gamma$ for all but finitely many $\gamma \in J$.

Definition 4.13. *The topology $\mathcal{T}_{\text{box}}$ is called the* **box topology** *on $\prod_{\alpha \in J} X_\alpha$. The topology $\mathcal{T}_{\text{prod}}$ is called the* **product topology**.

In both cases it is easy to prove we have topologies. (Check this!) Furthermore, all of the projections $\text{pr}_{\alpha'} \colon \prod_{\alpha \in J} X_\alpha \to X_{\alpha'}$ are continuous in both topologies. To see the difference we observe the following: A subset W of $\prod_{\alpha \in J} X_\alpha$ is open in the product topology if it is a union of subsets of the form $\prod_{\alpha \in J} V_\alpha$ where $V_\alpha = X_\alpha$ for all but finitely many $\alpha \in J$. If J is infinite and only finitely many of the X_α are indiscrete spaces, then $\mathcal{T}_{\text{box}}$ is *strictly finer* than $\mathcal{T}_{\text{prod}}$.

An decisive difference appears when we form the product of a fixed space with itself over an index set.

Proposition 4.14. *Let X be a space and for all $\alpha \in J$, let $X_\alpha = X$. Define the function*

$$\Delta \colon X \to \prod_{\alpha \in J} X_\alpha$$

by $\Delta(x) \colon \alpha \mapsto x \in X_\alpha = X$. This function is continuous when $\prod_{\alpha \in J} X_\alpha$ has the product topology.

Proof: If $\prod_{\alpha \in J} V_\alpha$ is a basic open set, then $V_\beta = X$ for all but finitely many $\beta \in J$, say $\alpha_1, \alpha_2, \ldots, \alpha_n$. Then $\Delta^{-1}(\prod_{\alpha \in J} V_\alpha) = \bigcap_{\alpha \in J} V_\alpha = V_{\alpha_1} \cap \ldots \cap V_{\alpha_n}$, which is open in X.$\diamondsuit$

Compare $\Delta \colon (\mathbb{R}, \text{usual}) \to (\mathbb{R}^\omega, \mathcal{T}_{\text{box}})$. The open set

$$(-1, 1) \times (-1/2, 1/2) \times (-1/3, 1/3) \times \ldots = W$$

has $\Delta^{-1}(W) = \{0\}$ which is not open. Since the composites $\text{pr}_i \circ \Delta = \text{id}$, a desirable property of continuous functions on products fails. This example recommends the product topology over the box topology as *the* product topology.

Another nice property of the product topology is the preservation of certain properties: for example, a product of Hausdorff spaces is Hausdorff. However, an uncountable product of second countable spaces or separable spaces need not be second countable or separable.

When spaces are pointed, $(X_\alpha, x_{\alpha 0})$, we can construct some continuous functions of interest. The product $\prod_{\alpha\in J} X_\alpha$ is pointed with basepoint $(\alpha \mapsto x_{\alpha 0})_{\alpha\in J}$. Define the injections

$$i_\alpha\colon (X_\alpha, x_{\alpha 0}) \longrightarrow \left(\prod_{\beta\in J} X_\beta, (\beta \mapsto x_\beta)_{\beta\in J}\right)$$

given by $x \mapsto c$, where $c\colon J \to \bigcup_{j\in J} X_j$ is defined

$$c(j) = \begin{cases} x, & \text{if } j = \alpha, \\ x_{\alpha' 0}, & \text{if } j \neq \alpha,\ j = \alpha'. \end{cases}$$

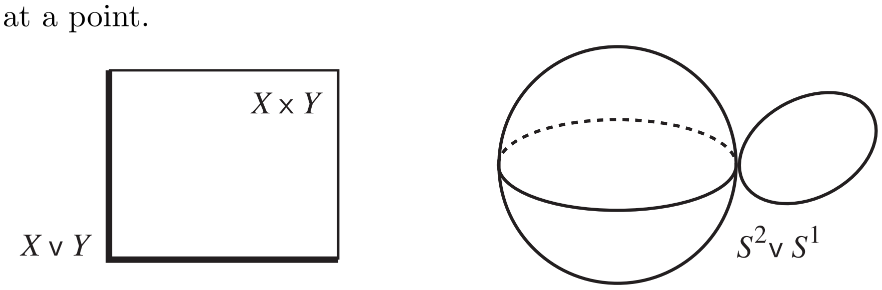
The pre-image under i_α of an open set is determined only by the open set in the coodinate α so each i_α is continuous. Notice, without the chosen basepoints, there is no obvious way to choose the other coordinates to define the inclusions i_α.

Next, notice $\mathrm{pr}_\alpha \circ i_\alpha = \mathrm{id}\colon X_\alpha \to X_\alpha$. Thus we can factor the identity through the pointed product space.

Finally, we mention an interesting subspace of $(X \times Y, (x_0, y_0))$.

Definition 4.15. *The* **one-point union of the pointed spaces** (X, x_0) *and* (Y, y_0), *denoted* $X \vee Y$ *is given by* $X \times \{y_0\} \cup \{x_0\} \times Y \subset X \times Y$.

One can think of $X \vee Y$ as the pair of axes in the product $X \times Y$ joined at the origin (x_0, y_0). A homeomorphic image of $S^2 \vee S^1$ can be pictured as a sphere with a circle touching it at a point.



There are canonical mappings $X \to X \vee Y \to X$ given by $x \mapsto (x, y_0) \mapsto x$. When $X = Y$, the extension problem posed by taking $X \vee X \subset X \times X$ and the *fold map* $\mathrm{fold}\colon X \vee X \to X$ given by $\mathrm{fold}(x, x_0) = x = \mathrm{fold}(x_0, x)$ is solved by a continuous binary operation $\mu\colon X \times X \to X$ for which x_0 is an identity element. Spaces like this are called *H-spaces* (or *Hopf spaces*). They are generalizations of groups and they play an important role in topology.

Quotients

Another method for building new spaces starts with a space X and an equivalence relation $\sim$ on X. The space X maps to the set of equivalence classes $[X]$ via the *canonical surjection* $\mathrm{pr}\colon X \to [X]$, $x \mapsto [x]$, the equivalence class of x. We want to introduce a topology on $[X]$ which makes the canonical surjection continuous. We take the most direct course.

Definition 4.16. *A subset $V \subset [X]$ is open in the* **quotient topology** *on $[X]$ if $\mathrm{pr}^{-1}(V)$ is open in X. The space $[X]$ with this topology is called a* **quotient space** *of X.*

Notice that the quotient topology is the *finest* topology making $\mathrm{pr}: X \to [X]$ continuous: anything larger would have open sets whose pre-image would not be open. We characterize the relation between the quotient topology and the canonical surjection.

Definition 4.17. *An onto map $f: X \to Y$ is called a* **quotient map** *when V is open in Y if and only if $f^{-1}(V)$ is open in X.*

Observation. Some continuous functions $f: X \to Y$ enjoy a more unlikely property; $f(U) \subset Y$ is open when U is open in X. Such continuous mappings are called **open mappings**; there is also the analogous notion of a closed mapping. A homeomorphism is open as is a canonical projection.

Theorem 4.18. (1) *If $f: X \to Y$ is an onto, continuous mapping, then f is a quotient map if it is an open mapping.* (2) *If $f: X \to Y$ is a quotient map, then a function $g: Y \to Z$ is continuous if and only if the composite $g \circ f: X \to Z$ is continuous.* (3) *Suppose $f: X \to Y$ is a quotient map. Suppose $\sim$ is the equivalence relation defined on X by $x \sim x'$ if $f(x) = f(x')$. Then the quotient space $[X]$ is homeomorphic to Y.*

Proof: (1) We need to show that f an open mapping implies f is a quotient map. Suppose V is any subset in Y. Then, if $f^{-1}(V)$ is open in X, $f(f^{-1}(V)) = V$ is open in Y since f is an onto, open mapping. Hence f is a quotient map.

(2) We need to show that $g \circ f$ being continuous implies g is continuous. Suppose W is open in Z. Then $(g \circ f)^{-1}(W) = f^{-1}(g^{-1}(W))$ is open in X. Since f is a quotient map, $g^{-1}(W)$ is open in Y. Hence, g is continuous.

(3) By the definition of the equivalence relation, we have the diagram.

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \big\downarrow \mathrm{pr} & & \| \\ [X] & \xrightarrow[\hat{f}]{} & Y \end{array}$$

The lift $\hat{f}: [X] \to Y$ is given by $\hat{f}([x]) = f(x)$ and it is well-defined by the conditions of (3). Notice that $\hat{f} \circ \mathrm{pr} = f$. Both f and pr are quotient maps so $\hat{f}$ is continuous. We show that $\hat{f}$ is one-one, onto and $\hat{f}^{-1}$ is continuous, which implies that $\hat{f}$ is a homeomorphism. If $\hat{f}([x]) = \hat{f}([x'])$, then $f(x) = f(x')$ and so $x \sim x'$, that is, $[x] = [x']$, and $\hat{f}$ is one-one. If $y \in Y$, then $y = f(x)$ since f is onto and $\hat{f}([x]) = y$ so $\hat{f}$ is onto. To see that $\hat{f}^{-1}$ is continuous, observe that since f is a quotient map and pr is a quotient map, this shows $\mathrm{pr} = \hat{f}^{-1} \circ f$ and (2) implies that $\hat{f}^{-1}$ is continuous. ♢

Part (3) of Theorem 4.18 allows useful comparisons. Let's consider an example:

Example: Let $\sim$ be the equivalence relation on $\mathbb{R}$ given by $r \sim s$ if $s - r$ is an integer. Give $\mathbb{R}$ the usual topology and consider $[\mathbb{R}]$. Intuitively we have identified two real numbers whenever they differ by an integer and so only $[0, 1]$ would be in $[\mathbb{R}]$ with $0 \sim 1$. That is, *form the space from* $[0,1]$ *by joining* 0 *to* 1. This ought to be a circle! Consider the mapping

$$f\colon \mathbb{R} \longrightarrow S^1, \qquad f(r) = (\cos(2\pi r), \sin(2\pi r)).$$

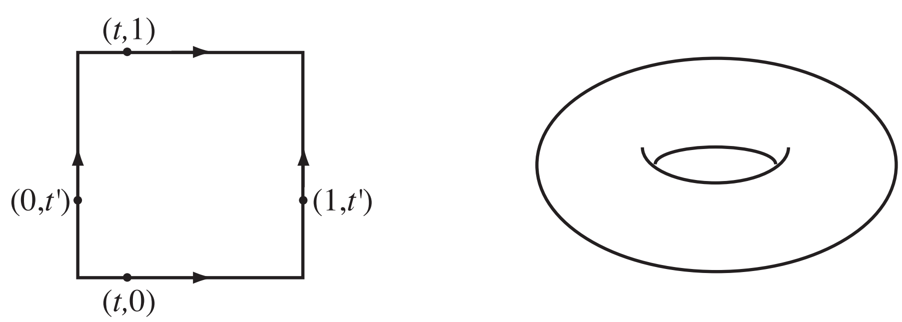
If $r \sim s$, then $f(r) = f(s)$ so we get a function $\hat{f}\colon [\mathbb{R}] \to S^1$, such that the following diagram commutes:

$$\begin{array}{ccc} \mathbb{R} & \xrightarrow{f} & S^1 \\ \big\downarrow \text{pr} & & \| \\ [\mathbb{R}] & \xrightarrow[\hat{f}]{} & S^1 \end{array}$$

From calculus we know f is continuous and $f = \hat{f} \circ \text{pr}$ so by Theorem 4.18 (2) $\hat{f}$ is continuous. Furthermore $\hat{f}$ is one-one and onto, so we only need to know if $\hat{f}$ is open to see that it is a homeomorphism. We could apply (3) above more easily if f were open, so we check: let $(a,b) \subset \mathbb{R}$, $a < b$, be a basic open set. The image $f((a,b))$ = those points on S^1 of angle between $2\pi a$ and $2\pi b$, which is open in S^1. Thus f is open and $[\mathbb{R}] \cong S^1$.

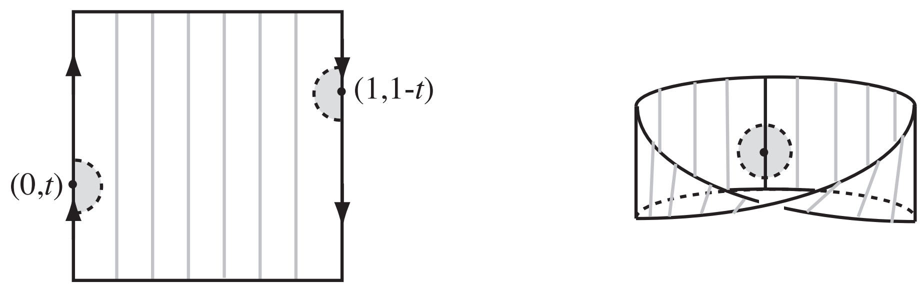
Quotient spaces let us make precise a construction called **glueing**. Suppose one has two subsets $A, B \subset X$ and a homeomorphism $h\colon A \to B$. We can define the equivalence relation $\sim_h$ on X by $x \sim_h x'$ if $x = x'$, $h(x) = x'$ or $h^{-1}(x) = x'$. This identifies points $a \in A$ with their counterpart $h(a) \in B$ and vice versa. This process 'glues' A to B according to h. Let's consider some specific examples.

(1) Let $I^2 = [0,1] \times [0,1]$ and define $A = \{0\} \times [0,1] \cup [0,1] \times \{0\}$ and $B = \{1\} \times [0,1] \cup [0,1] \times \{1\}$; then take the mapping $h\colon A \to B$ by $h((0,t)) = (1,t)$ and $h((t,0)) = (t,1)$. This glues the bottom of the box to the top and the sides to the sides. We get a *torus* in this fashion given as in the diagram:



Alternatively, the torus can be described as a circle rotated around a line outside it. Taking the coordinates of a point on the torus from the given circle and the rotation shows the torus $T^2 = S^1 \times S^1$. This description leads to a function $f\colon I^2 \to T^2$ given by $f(u,v) = (e^{2\pi iu}, e^{2\pi iv}) \in S^1 \times S^1$. Since $e^{2\pi i0} = e^{2\pi i1}$ we get $f(u,v) = f(\bar{u},\bar{v})$ if and only if $(u,v) \sim (\bar{u},\bar{v})$. Thus we get $\hat{f}\colon [I^2]_h \to T^2$ which is a homeomorphism in the same way as in the argument for the circle.

(2) The following famous quotient of a square was constructed in 1858 independently by Johann Listing, who introduced the word 'topology' for such studies, and Möbius for whom it is named. Let $X = [0,1] \times [0,1]$ and let $A = \{0\} \times [0,1]$, $B = \{1\} \times [0,1]$ with the homeomorphism $h(0,t) = (1, 1-t)$. Then $[X]_h$ represents the Möbius band, M. From a convenient representation of M in $\mathbb{R}^3$, the quotient map is evident.



Notice how an open set around a point on the line segment where it is glued has pre-image an open set (in two pieces) in X.

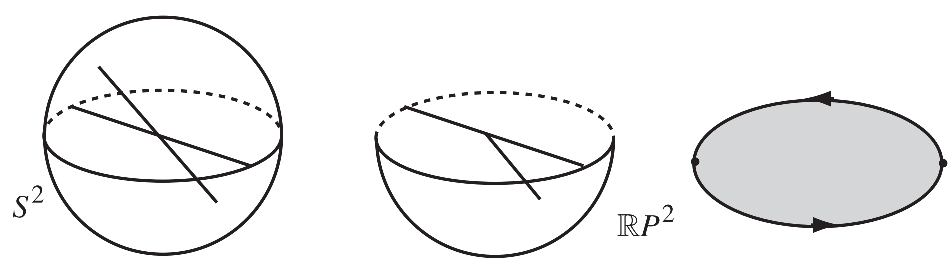
(3) One of the most important spaces in topology is the **projective plane**. Its formal definition is given as a set by

$$\mathbb{R}P^2 = \{\text{ lines through the origin in } \mathbb{R}^3 \}.$$

To 'tame' this description a bit, we introduce coordinates for a point in $\mathbb{R}P^2$. Suppose $(x,y,z) \in \mathbb{R}^3$ and $(x,y,z) \neq (0,0,0)$. Introduce the equivalence relation $(x,y,z) \sim (\lambda x, \lambda y, \lambda z)$ for $\lambda \in \mathbb{R} - \{0\}$. Then $\mathbb{R}P^2 = [\mathbb{R}^3 - \{\mathbf{0}\}]$ topologized as a quotient space.

The projective plane is the home for algebraic curves, defined as zero sets of homogeneous polynomials in two variables. The fact that such an algebraic curve lies in $\mathbb{R}P^2$ provides further geometry with which to study the curve. Also, projective geometry is modelled by the projective plane.

We construct a more easily described topological model for $\mathbb{R}P^2$: To each line in $\mathbb{R}^3$ through the origin, we can associate two points $\{\pm(x,y,z)\}$ in S^2 by taking the two points of intersection of the line with the sphere. The inclusion $S^2 \hookrightarrow \mathbb{R}^3 - \{\mathbf{0}\}$ composed with the canonical surjection $\mathrm{pr}\colon \mathbb{R}^3 - \{\mathbf{0}\} \to [\mathbb{R}^3 - \{\mathbf{0}\}]$ gives a mapping $S^2 \to \mathbb{R}P^2$ and we get the associated equivalence relation on S^2 as $(x,y,z) \sim (x',y',z')$ whenever $(x',y',z') = \pm(x,y,z)$. Thus $\mathbb{R}P^2 \cong [S^2]$, where we identify antipodal points together. A *projective line* is the image of the intersection of a plane through the origin with S^2 (a great circle) in $\mathbb{R}P^2$. Two points on $\mathbb{R}P^2$ determine a unique projective line by taking the plane spanned by the points and the origin in $\mathbb{R}^3$, and two projective lines meet in the line given by the intersection of the planes that determine them.



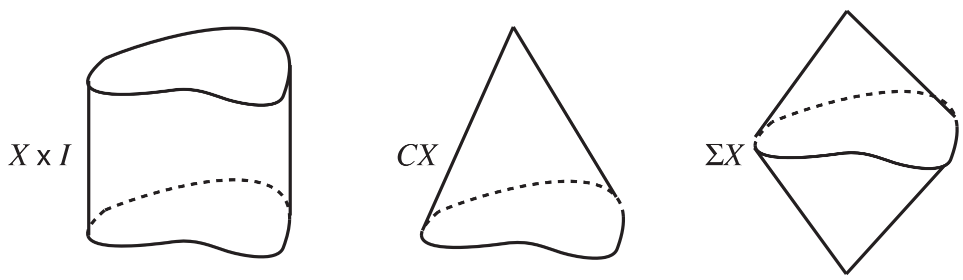
The hemisphere in the picture tells us how to represent $\mathbb{R}P^2$ as a quotient of a disk: On the rim of the hemisphere antipodal points are identified—this is the line at infinity in the projective plane. So let

$$e^2 = \{(x,y) \mid x^2 + y^2 \leq 1\} \subset \mathbb{R}^2$$

be the 2-disk. Let $A = \{(x,y) \mid x^2+y^2=1 \text{ and } x \geq 0\}$, $B = \{(x,y) \mid x^2+y^2=1 \text{ and } x \leq 0\}$ and define $h\colon A \to B$ by $h(x,y) = (-x,-y)$. The quotient space, $[e^2]_h$ is, once again, $\mathbb{R}P^2$.

All of this discussion generalizes to define $\mathbb{R}P^n$, the **n-dimensional projective space**, which is $[S^n]$ with equivalence relation $\mathbf{x} \sim \pm\mathbf{x}$. These spaces are the object of intense study in modern topology.

Here are some standard constructions that apply to any space X.



(4) The **cone** on X is given by $[X \times I]$ where $(x,t) \sim (x',t')$ if $(x,t) = (x',t')$ or $x, x' \in X$ and $t = t' = 0$. We write $CX = [X \times I]$ for the cone on X.

(5) The **suspension** of X, denoted ΣX, is the quotient of $X \times I$ where we identify the subsets $X \times \{0\}$ and $X \times \{1\}$ each to a point (two points here). Suspension gives a convenient construction of the spheres:

THEOREM 4.19. *The $(n+1)$-sphere S^{n+1} is homeomorphic to ΣS^n.*

Proof: Consider the function $\sigma\colon S^n \times [0,1] \longrightarrow S^{n+1}$ given by

$$\sigma(x_0,\ldots,x_n,t) = (\sqrt{1-(1-2t)^2}x_0,\ldots,\sqrt{1-(1-2t)^2}x_n, 1-2t).$$

This function is continuous as the calculus tells us. Notice that

$$\sigma(x_0,\ldots,x_n,0) = (0,0,\ldots,0,1), \quad \sigma(x_0,\ldots x_n,1) = (0,0,\ldots,0,-1).$$

Thus σ factors through $[S^n \times [0,1]] = \Sigma S^n$.

$$\begin{array}{ccc} S^n \times [0,1] & \xrightarrow{\sigma} & S^{n+1} \\ \big\downarrow \text{pr} & & \| \\ [S^n \times [0,1]] & \xrightarrow{\hat{\sigma}} & S^{n+1}. \end{array}$$

The function $\hat{\sigma}$ is one-one, onto away from the 'poles' $(0,\ldots,0,\pm 1)$. The classes remaining, $[S^n \times \{0\}]$ and $[S^n \times \{1\}]$ each go to the respective poles. To finish the proof we only need to show that σ is a quotient map. Let $S^n \times [0,1]$ get its topology as a subspace of $\mathbb{R}^{n+2}$. A basic open set in $S^n \times [0,1]$ takes the form $W = (S^n \times [0,1]) \cap [(a_1,b_1) \times \ldots \times (a_{n+2},b_{n+2})]$. Restricting (or extending) σ to W takes it to an open set and the image is easily determined to be the intersection of $\sigma(W)$ with S^{n+1}. Thus σ is open.

There are pointed versions of CX and ΣX: Given (X, x_0) a pointed space, then $(\tilde{C}X, Cx_0)$ is $[CX] = [X \times [0,1]]_{\approx}$ where $(x,t) \approx (x',t')$ if $(x,t) = (x',t')$, or $t = 0$, x,

$x' \in X$ or $x = x' = x_0$ and $t \in [0,1]$. The single class Cx_0 in $[X \times [0,1]]_{\approx}$ is given by the subset $\{(x,0), x \in X, (x_0,t), t \in [0,1]\}$.

The pointed suspension (SX, sx_0) has $[sx_0] = X \times \{0\} \cup X \times \{1\} \cup x_0 \times [0,1]$, and the rest of the equivalence classes the same as for ΣX. An extraordinary property of SX is the following

PROPOSITION 4.20. *There is a one-one correspondence of sets*

$$\mathrm{Hom}((SX, sx_0),(Y,y_0)) \cong \mathrm{Hom}((X,x_0), \mathrm{Hom}((S^1,1),(Y,y_0))).$$

Proof: Let $f\colon (SX, sx_0) \to (Y, y_0)$. Untangling the suspension coordinate we can write f in the composite

$$X \times I \xrightarrow{\mathrm{pr}} SX \xrightarrow{f} Y$$

and for each $x \in X$ associate the mapping $x \longmapsto \tilde{f}(t) = f \circ \mathrm{pr}(x,t)$. It follows that $\tilde{f}(0) = \tilde{f}(1) = f(sx_0) = y_0$ by the definition of the canonical projection for the equivalence relation. The inverse is as follows: given $F\colon (X, x_0) \to \mathrm{Hom}((S^1,1),(Y,y_0))$, then define $\hat{F}\colon (SX, sx_0) \to (Y, y_0)$ by $\tilde{F}(x,t) = F(x)(e^{2\pi i t})$. An explicit calculation shows these processes to be inverses and the proposition is proved. $\diamondsuit$

Are certain topological properties respected by quotient maps? One must be careful. For example, we can partition $(\mathbb{R}, \text{usual})$ into three parts $A = (-\infty, 0)$, $B = \{0\}$, $C = (0, \infty)$. The associated quotient is a three-point set $X = \{a,b,c\}$ for the equivalence classes and topology $\{\emptyset, X, \{a,b\}, \{a\}, \{b\}\}$, where $a = [A]$, $b = [B]$, and $c = [C]$. However, this topology is *not* Hausdorff! More can be said however.

THEOREM 4.21. *Let $\sim$ be an equivalence relation in a space X that is Hausdorff. Then $[X]$ is Hausdorff if and only if the* **graph of** $\sim$*, $\{(x,y) \mid x \sim y, x, y \in X\}$ is closed in $X \times X$.*

Proof: Let $[x], [y] \in [X]$ and $[x] \neq [y]$. Then the point $(x,y) \in X \times X$ lies outside the graph of $\sim$ which is closed. Choose a basic open set $U \times V \subset X \times X$ with $x \in U$, $y \in V$ and $U \times V \subset X \times X - \mathrm{graph}(\sim)$. Consider $\mathrm{pr}(U) \subset [X]$. Then $[x] \in \mathrm{pr}(U)$ and similarly $[y] \in \mathrm{pr}(V)$. We claim that $\mathrm{pr}(U)$ and $\mathrm{pr}(V)$ are open and disjoint. Openess follows from the fact that pr is an open mapping. Suppose $[w] \in \mathrm{pr}(U) \cap \mathrm{pr}(V)$. Then there is a point v, with $v \sim w$ and a point $v' \sim w$ with $v \in U$, $v' \in V$. But then $(v,v') \in U \times V$ and so $U \times V \cap \mathrm{graph}(\sim) \neq \emptyset$; a contradiction. This shows $[X]$ is Hausdorff. The converse is left to the reader. $\diamondsuit$

Exercises

1. Show that a space X is Hausdorff if and only if the subset $\Delta(X) = \{(x,x) \mid x \in X\}$ is a closed subset of the product space $X \times X$. Suppose X and Y are Hausdorff spaces. Show that $X \times Y$ is also Hausdorff. Finish the proof of Theorem 4.12.

2. Suppose $X = A_1 \cup A_2 \cup \cdots$ where A_n is open in X for all n. If $f\colon X \to Y$ is a function such that, for each n, $f|_{A_n}\colon A_n \to Y$ is continuous with respect to the subspace

topology on A_n, show that f is itself continuous. What is the analogous statement when X is a union of closed sets?

3. Suppose that we have two pointed spaces (X, x_0) and (Y, y_0). Show that the mappings, $X \to X \times Y$, given by $x \mapsto (x, y_0)$ and $Y \to X \times Y$, $y \mapsto (x_0, y)$ are each continuous, and have continuous **sections** (a function $f\colon U \to V$ has a section, g, if the function $g\colon V \to U$ is such that $g \circ f\colon U \to U$ is the identity mapping. This need not be a strict inverse as in the case above. Notice that f will be one-one, but not necessarily onto.)

4. Consider the subspace of $\mathbb{R}^2$ given by

$$S^1 = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\}.$$

   This is the unit circle. The mapping

$$w\colon [0, 1) \to S^1 \text{ given by } w(r) = (\cos(2\pi r), \sin(2\pi r))$$

   is one-one and onto. Show that it is continuous if you give S^1 the subspace topology from $\mathbb{R}^2$, but that the inverse function is *not* continuous.

5. A **topological group** is a group that is a Hausdorff topological space and the binary operation $\mu\colon G \times G \to G$, and the mapping $x \mapsto x^{-1}$ are continuous.
   i) Prove that a group G is a topological group if and only if it is a Hausdorff topological space and the mapping $G \times G \to G$ given by $(x, y) \mapsto x^{-1} \cdot y$ is continuous.
   ii) Let g_0 be an element of a topological group G. Show that the mappings $R_{g_0}\colon G \to G$ and $L_{g_0}\colon G \to G$ given by $R_{g_0}(h) = \mu(h, g_0)$ and $L_{g_0}(h) = \mu(g_0, h)$ are homeomorphisms of G with itself.
   iii) Prove that the reals with addition is a topological group, and the nonzero reals with multiplication form a topological group. This amounts to showing that $+$ and $\times$ are continuous on ($\mathbb{R}$,usual). Do this in detail.
6. Recall that the projective plane is defined to be the set of lines in $\mathbb{R}^3$ through the origin. There is also a representation of $\mathbb{R}P^2$ as a quotient of the 2-sphere by identifying antipodal points:
   i) Let $S^2 \cong D^+ \cup C \cup D^-$ where D^+ is the part above and on the plane $z = \frac{1}{2}$; where D^- is the part on and below the plane given by $z = -\frac{1}{2}$ and C is the part in between. Let $p\colon S^2 \to \mathbb{R}P^2$ be the quotient map. Verify that $D^+ \cong e^2 = \{\mathbf{x} \in \mathbb{R}^2 \mid \|\mathbf{x}\| \leq 1\}$. Verify that $C \cong S^1 \times [0, 1]$. And verify by cutting and glueing that $p(C)$ is homeomorphic to a Möbius band embedded in $\mathbb{R}P^2$.
   ii) Verify that $p(D^+) \cup p(C) = \mathbb{R}P^2$ and that $p(C) \cap p(D^+) \cong S^1$. This shows that the projective plane can be obtained from attaching a disk to the Möbius band along its edge.

7. Suppose that $A \subset X$ is a nonempty closed subset of a space X that is Hausdorff, and further X satisfies the property that if $x \in X$ and $x \notin A$, then there are open sets U and V with $x \in U$, $A \subset V$ and $U \cap V = \emptyset$. Define the relation $x \sim y$ if $x = y$ or x and $y \in A$. Show that this relation is an equivalence relation. The quotient topology on $[X]$ is denoted by the space X/A. Show that the quotient space X/A is Hausdorff. A space that has this separation property for every closed proper subset A is said to satisfy the T_3 axiom. Show that being T_3 is a topological property.