A Little Bit of Math

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So what is mathematics?

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“What one gets to know” or “what one learns”

From *A Mathematician's Lament* by Paul Lockhart

"...nobody has the faintest idea what it is that mathematicians do."

Mathematics is an art.

G.H. Hardy: "A mathematician, like a painter or poet, is a maker of patterns. If his patterns are more permanent that theirs, it is because they are made with *ideas*.

The Pigeonhole Principle

Suppose there are 200 pigeonholes in a local post office for patrons' mail. On a particular Monday, after all the mail has been picked up over the weekend and the boxes are empty, a sack of mail comes in containing 201 letters. What can you conclude?

Now suppose we know only that there are more than 200 letters in the sack?

A mathematician likes to make an abstract version of her experience. So perhaps we just say there are N many pigeonholes and there are more than N letters that arrive in the sack. What can you conclude?

A little long division

Let's divide 7919 by 13. Using long division we get what we see on the board.

Other examples:

$$\frac{2}{3} = 0.6\overline{6}$$

$$\frac{1}{7} = 0.142857\overline{142857}$$

$$\frac{1}{11} = 0.09\overline{09}$$

A little long division

Proposition. A rational number is represented by an infinite repeating decimal.

Proof (a convincing argument): Suppose we divide q into p where p and q are positive integers. If q divides p, then $p = qd$ and $\frac{p}{q} = d.00\overline{0}$. Suppose there is a positive remainder $p = qd + r$. The remainder will be between 0 and q, $0 < r < q$. We bring down a 0 $(r \times 10)$ and divide again. We produce another remainder $0 \leq r_1 < q$. If we get 0's for the rest of the way. Notice that there are $q - 1$ possible nonzero remainders, 1, 2, 3, $\ldots$, $q - 1$. If we do the divisions bringing down zeroes q times, we have q nonzero remainders from among $q - 1$ possibilities. Some remainder must repeat! If it repeats, we get an infinite repeating decimal. By the Pigeonhole Principle, it has to repeat.

A little long division

The proposition above flows one way, a fraction has a repeating decimal. It could be that some repeating decimal represents a number that is NOT a rational number. That would lead to some confusion. Let"s see how an example works.

$$r = 1.234567\overline{567}.$$

What does this mean?

$$r = 1 + \frac{234}{1000} + \frac{567}{1000^2} + \frac{567}{1000^3} + \cdots$$

A little long division

Don't be fooled – a sum of infinitely many fractions need not be a fraction as we will soon see. Let's do some arithmetic:

$$r = 1 + \frac{234}{1000} + \frac{567}{1000^2}\left(1 + \frac{1}{1000} + \frac{1}{1000^2} + \cdots\right)$$

Suppose $s = 1 + \frac{1}{1000} + \frac{1}{1000^2} + \frac{1}{1000^3} + \cdots$. Then $1000s = 1000 + 1 + \frac{1}{1000} + \frac{1}{1000^2} + \cdots = 1000 + s$. Then $999s = 1000$ and $s = \frac{1000}{999}$, a fraction.

Therefore $r = 1 + \frac{234}{1000} + \frac{567}{1000^2} \cdot \frac{1000}{999}$, a finite sum of fractions, and hence a fraction.

Rational/irrational

This calculation leads from any infinite repeating decimal to some rational number. So we have proved:

Theorem. Numbers represented by an infinite repeating decimal are exactly the rational numbers.

This means that an infinite nonrepeating decimal does not represent a rational number. For example,

$$\begin{aligned} L &= 0.101001000100001000001\cdots \\ &= \frac{1}{10^1} + \frac{1}{10^{1+2}} + \frac{1}{10^{1+2+3}} + \frac{1}{10^{1+2+3+4}} + \cdots \end{aligned}$$

Because the strings of 0's grow bigger and bigger, if part repeated, it would eventually fall into a string of consecutive 0's and so be 0. But evidently a 1 appears every so often forever. This number is called *Liouville's number*, an irrational number.

Rational/irrational

Also, if a number is irrational for some reason, then it has a nonrepeating decimal representation.

Claim: $\sqrt{2}$ is irrational.

Proof (see Aristotle). Suppose $\sqrt{2}$ is rational. Then $\sqrt{2} = \frac{p}{q}$ with p and q whole numbers and $\frac{p}{q}$ is in lowest terms, meaning, if any number divides p it does not divide q, and vice versa. By squaring, $2 = \frac{p^2}{q^2}$ and $2q^2 = p^2$. Since the left hand side is even, p^2 is even. Now an even number squared is even, and an odd number squared is odd. So p is even. Say $p = 2r$, then $p^2 = (2r)^2 = 4r^2 = 2q^2$ and so $2r^2 = q^2$ and q is even. But then 2 divides both p and q, a contradiction. How did this happen? Working our way back through the argument, the error must have been made at the beginning and so $\sqrt{2}$ is not a rational number.

Rational/irrational

$\sqrt{2} = 1.41421355237\ldots$ is an infinite, nonrepeating decimal.

This argument generalizes to other square roots and higher roots of 2.

Infinities

Theorem of Georg Cantor. The infinity of real numbers is larger than the infinity of the counting numbers.

The counting numbers are 1, 2, 3, ... If we consider only the real numbers between 0 adn 1, then we can represent them as infinite decimals of the form $0.a_1a_2a_3\ldots$.

If we could count all the real numbers between 0 and 1, we would get an infinite list:

$$
\begin{aligned}
r_1 &= 0.a_{11}a_{12}a_{13}a_{14}\ldots \\
r_2 &= 0.a_{21}a_{22}a_{23}a_{24}\ldots \\
r_3 &= 0.a_{31}a_{32}a_{33}a_{34}\ldots \\
&\vdots \\
r_n &= 0.a_{n1}a_{n2}a_{n3}a_{n4}\ldots \\
&\vdots
\end{aligned}
$$

Infinities

If the infinity of the counting numbers and the real numbers between 0 and 1 were the same, then we could include every real number between 0 and 1 in the list. Consider the following procedure that gives a number, call it

$$R = 0.b_1 b_2 b_3 \ldots .$$

$$b_n = \begin{cases} 7, & \text{if } 0 \le a_{nn} \le 4, \\ 2, & \text{if } 5 \le a_{nn} \le 9. \end{cases}$$

Notice that $b_n \neq a_{nn}$ for any n and so R differs from every real number in the list. For example, if we look at r_{50}, then R differs from r_{50} at the 50th decimal place.

Infinities

Hence the list is incomplete. Even if we added R to the list by shifting the list, there would be another version of R that would be missed. Therefore, the collection of real numbers between 0 and 1 is larger than the collection of counting numbers, a bigger infinity.

(There are a few details that I have left out. They can be filled in nicely and any difficulties overcome.)

You might say, why not use the rational numbers to count the real numbers? There seem to be more of them than there are counting numbers. But, in fact, that is not correct. The infinity of the counting numbers is the same as the infinity of the rational numbers – because we can list them!

Infinities

$$
\begin{array}{ccccc}
\frac{1}{1} & \frac{2}{1} & \frac{3}{1} & \frac{4}{1} & \cdots \\
\frac{1}{2} & \frac{2}{2} & \frac{3}{2} & \frac{4}{2} & \cdots \\
\frac{1}{3} & \frac{2}{3} & \frac{3}{3} & \frac{4}{3} & \cdots \\
\frac{1}{4} & \frac{2}{4} & \frac{3}{4} & \frac{4}{4} & \cdots \\
\vdots & \vdots & \vdots & \vdots &
\end{array}
$$