2. Metric and Topological Spaces

Topology begins where sets are implemented with some cohesive properties enabling one to define continuity.

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In order to forge a language of continuity, we begin with familiar examples. Recall from single-variable calculus that a function \( f: \mathbb{R} \to \mathbb{R} \), is continuous at a point \( x_0 \in \mathbb{R} \) if for every \( \epsilon > 0 \), there is a \( \delta > 0 \) so that, whenever \( |x - x_0| < \delta \), we have \( |f(x) - f(x_0)| < \epsilon \).

The route to generalization begins with the distance notion on the real line: the distance between the real numbers \( x \) and \( y \) is given by \( |x - y| \). The general properties of a distance are abstracted in the notion of a metric space, first introduced by MAURICE FRÉCHET (1878–1973) and named by Hausdorff.

**Definition 2.1.** A **metric space** is a set \( X \) together with a distance function \( d: X \times X \to \mathbb{R} \) satisfying

i) \( d(x, y) \geq 0 \) for all \( x, y \in X \) and \( d(x, y) = 0 \) if and only if \( x = y \).

ii) \( d(x, y) = d(y, x) \) for all \( x, y \in X \).

iii) The **Triangle Inequality**: \( d(x, y) + d(y, z) \geq d(x, z) \) for all \( x, y, z \in X \).

The open ball of radius \( \epsilon > 0 \) centered at a point \( x \) in a metric space \( (X, d) \) is given by

\[
B(x, \epsilon) = \{ y \in X \mid d(x, y) < \epsilon \},
\]

that is, the points in \( X \) within \( \epsilon \) in distance from \( x \).

The intuitive notion of ‘near’ can be made precise in a metric space: a point \( y \) is ‘near’ the point \( x \) if it is in \( B(x, \epsilon) \) for \( \epsilon \) suitably small.

**Examples:**

1) The most familiar example is \( \mathbb{R}^n \). If \( x = (x_1, \ldots, x_n) \) and \( y = (y_1, \ldots, y_n) \), then the Euclidean metric is given by

\[
d(x, y) = \|x - y\| = \sqrt{(x_1 - y_1)^2 + \cdots + (x_n - y_n)^2}.
\]

In fact, one can endow \( \mathbb{R}^n \) with other metrics, for example,

\[
d_1(x, y) = \max\{|x_1 - y_1|, \ldots, |x_n - y_n|\}
\]

The nonnegative, nondegenerate, and symmetric conditions are clear for \( d_1 \). The triangle inequality follows in the same way as the proof in the next example.

Notice that an open ball with this metric is an ‘open box’ as pictured here in \( \mathbb{R}^2 \).
2) Let $X = \text{Bdd}([0,1], \mathbb{R})$ denote the set of bounded functions $f: [0,1] \to \mathbb{R}$, that is, functions $f$ for which there is a real number $M(f)$ such that $|f(t)| < M(f)$ for all $t \in [0,1]$. Define the distance between two such functions to be

$$d(f,g) = \text{lub}_{t \in [0,1]} \{|f(t) - g(t)|\}.$$ 

Certainly $d(f,g) \geq 0$, and $d(f,g) = 0$ if and only if $f = g$. Furthermore, $d(f,g) = d(g,f)$.

The triangle inequality is more subtle:

$$d(f,h) = \text{lub}_{t \in [0,1]} \{|f(t) - h(t)|\} \leq \text{lub}_{t \in [0,1]} \{|f(t) - g(t)| + |g(t) - h(t)|\}$$

$$\leq \text{lub}_{t \in [0,1]} \{|f(t) - g(t)|\} + \text{lub}_{t \in [0,1]} \{|g(t) - h(t)|\}$$

$$= d(f,g) + d(g,h).$$

An open ball in this metric space, $B(f, \epsilon)$, consists of all functions defined on $[0,1]$ with graph in the stripe pictured:

3) Let $X$ be any set and define

$$d(x,y) = \begin{cases} 0, & \text{if } x = y, \\ 1, & \text{if } x \neq y. \end{cases}$$

This is a perfectly good distance function—open balls are funny, however—either they consist of one point or the whole space depending on whether $\epsilon \leq 1$ or $\epsilon > 1$. The resulting metric space is called the discrete metric space.

Using open balls, we can rewrite the definition of a continuous real-valued function $f: \mathbb{R} \to \mathbb{R}$ to say (see the appendix for the definition and properties of $f^{-1}(A)$, the preimage of a function):

A function $f: \mathbb{R} \to \mathbb{R}$ is continuous at $x_0 \in \mathbb{R}$ if for any $\epsilon > 0$, there is a $\delta > 0$ so that $B(x_0, \delta) \subset f^{-1}(B(f(x_0), \epsilon))$.

The step from this definition of continuity to a general definition of continuous mappings of metric spaces is clear.

**Definition 2.2.** Suppose $(X,d_X)$ and $(Y,d_Y)$ are two metric spaces and $f: X \to Y$ is a function. Then $f$ is **continuous** at $x_0 \in X$ if, for any $\epsilon > 0$, there is a $\delta > 0$ so that $B(x_0, \delta) \subset f^{-1}(B(f(x_0), \epsilon))$. The function $f$ is **continuous** if it is continuous at $x_0$ for all $x_0 \in X$. 2
For example, if \( X = Y = \mathbb{R}^n \) with the usual Euclidean metric \( d(x, y) = \|x - y\| \), then \( f: \mathbb{R}^n \to \mathbb{R}^n \) is continuous at \( x_0 \) if for any \( \epsilon > 0 \), there is \( \delta > 0 \) so that whenever \( x \in B(x_0, \delta) \), that is, \( \|x - x_0\| < \delta \), then \( x \in f^{-1}(B(f(x_0), \epsilon)) \), which is to say, \( f(x) \in B(f(x_0), \epsilon) \), or \( \|f(x) - f(x_0)\| < \epsilon \). Thus we recover the \( \epsilon-\delta \) definition of continuity. We develop the generalization further.

**Definition 2.3.** A subset \( U \) of a metric space \((X, d)\) is open if for any \( u \in U \) there is an \( \epsilon > 0 \) so that \( B(u, \epsilon) \subset U \).

We note the following properties of open subsets of metric spaces.

1) An open ball \( B(x, \epsilon) \) is an open set in \((X, d)\).

2) An arbitrary union of open subsets in a metric space is open.

3) The finite intersection of open subsets in a metric space is open.

Suppose \( y \in B(x, \epsilon) \). Let \( \delta = \epsilon - d(x, y) > 0 \). Consider the open ball \( B(y, \delta) \). If \( z \in B(y, \delta) \), then \( d(z, y) < \delta = \epsilon - d(x, y) \), or \( d(z, y) + d(y, x) < \epsilon \). By the triangle inequality \( d(z, x) \leq d(z, y) + d(y, x) \) and so \( d(z, x) < \epsilon \) and \( B(y, \delta) \subset B(x, \epsilon) \). Thus \( B(x, \epsilon) \) is open.

Suppose \( \{U_\alpha, \alpha \in I\} \) is a collection of open subsets of \( X \). If \( x \in \bigcup_{\alpha \in I} U_\alpha \), then \( x \in U_\beta \) for some \( \beta \in I \). But \( U_\beta \) is open so there is an \( \epsilon > 0 \) with \( B(x, \epsilon) \subset U_\beta \subset \bigcup_{\alpha \in I} U_\alpha \). Therefore, the union \( \bigcup_{\alpha \in I} U_\alpha \) is open.

Suppose \( U_1, U_2, \ldots, U_n \) are open in \( X \), and suppose \( x \in U_1 \cap U_2 \cap \ldots \cap U_n \). Then \( x \in U_i \) for \( i = 1, 2, \ldots, n \) and since each \( U_i \) is open there are \( \epsilon_1, \epsilon_2, \ldots, \epsilon_n > 0 \) with \( B(x, \epsilon_i) \subset U_i \). Let \( \epsilon = \min\{\epsilon_1, \epsilon_2, \ldots, \epsilon_n\} \). Then \( \epsilon > 0 \) and \( B(x, \epsilon) \subset B(x, \epsilon_i) \subset U_i \) for all \( i \), so \( B(x, \epsilon) \subset U_1 \cap \ldots \cap U_n \) and the intersection is open.

We can use the language of open sets to rephrase the definition of continuity for metric spaces.

**Theorem 2.4.** A function \( f: X \to Y \) between metric spaces \((X, d)\) and \((Y, d)\) is continuous if and only if for any open subset \( V \) of \( Y \), the subset \( f^{-1}(V) \) is open in \( X \).

**Proof:** Suppose \( x_0 \in X \) and \( \epsilon > 0 \). Then \( B(f(x_0), \epsilon) \) is an open set in \( Y \). By assumption, \( f^{-1}(B(f(x_0), \epsilon)) \) is an open subset of \( X \). Since \( x_0 \in f^{-1}(B(f(x_0), \epsilon)) \), there is \( \delta > 0 \) with \( B(x_0, \delta) \subset f^{-1}(B(f(x_0), \epsilon)) \) and so \( f \) is continuous at \( x_0 \).

Suppose that \( V \) is an open set in \( Y \), and that \( x \in f^{-1}(V) \). Then \( f(x) \in V \) and there is an \( \epsilon > 0 \) with \( B(f(x), \epsilon) \subset V \). Since \( f \) is continuous at \( x \), there is a \( \delta > 0 \) with
$B(x, \delta) \subset f^{-1}(B(f(x), \epsilon)) \subset f^{-1}(V)$. Thus, for each $x \in f^{-1}(V)$, there is a $\delta > 0$ with $B(x, \delta) \subset f^{-1}(V)$, that is, $f^{-1}(V)$ is open in $X$. 

It follows from this theorem that, for metric spaces, continuity may be described entirely in terms of open sets. To study continuity in general we take the next step and focus on the collection of open sets. The key features of the structure of open sets in metric spaces may be abstracted to the following definition, first given by Hausdorff in 1914 [Hausdorff].

**Definition 2.5.** Let $X$ be a set and ℰ a collection of subsets of $X$ called open sets. The collection ℰ is called a topology on $X$ if

1. We have that $\emptyset \in ℰ$ and $X \in ℰ$.
2. The union of an arbitrary collection of members of ℰ is in ℰ.
3. The finite intersection of members of ℰ is in ℰ.

The pair $(X, ℰ)$ is called a topological space.

It is important to note that open sets are basic and determine the topology. Open set does not always refer to the ‘open’ sets we are used to in $\mathbb{R}^n$. Let’s consider some examples.

**Examples:**

1) If $(X, d)$ is a metric space, we defined a subset $U$ of $X$ to be open if for any $x \in U$, there is an $\epsilon > 0$ with $B(x, \epsilon) \subset U$, as above. This collection of open sets defines a topology on $X$ called the metric topology.

2) For any set $X$, let $T_1 = \{X, \emptyset\}$. This collection trivially satisfies the criteria for being a topology and is called the indiscrete topology on $X$. Let $T_2 = \mathcal{P}(X)$ be the set of all subsets of $X$. This collection trivially satisfies the conditions to be a topology and is called the discrete topology on $X$. It has the same open sets as the metric topology in $X$ with the discrete metric. It is the largest topology possible on a set (the most open sets), while the indiscrete topology is the smallest topology.

3) For the set with only two elements $X = \{0, 1\}$ consider the collection of open sets given by $T_S = \{\emptyset, \{0\}, \{0, 1\}\}$. The reader can quickly check that $T_S$ is a topology. This topological space is called the Sierpinski 2-point space.

4) Let $X$ be an infinite set. Define $T_{FC} = \{U \subseteq X \mid U = \emptyset \text{ or } X - U \text{ is finite}\}$. We show that $T_{FC}$ is a topology:

   (1) The empty set is already in $T_{FC}$; $X$ is open since $X - X = \emptyset$, which is finite.
   (2) If $\{U_\alpha, \alpha \in J\}$ is an arbitrary collection of open sets, then
   
   $$X - \bigcup_{\alpha \in J} U_\alpha = \bigcap_{\alpha \in J} (X - U_\alpha)$$
   
   by DeMorgan’s Law. Each $X - U_\alpha$ is finite or all of $X$ so we have $X - \bigcup_{\alpha \in J} U_\alpha$ is finite or all of $X$ and so $\bigcup_{\alpha \in J} U_\alpha$ is open.
   (3) If $U_1, U_2, \ldots, U_n$ are open, then $X - (U_1 \cap \ldots \cap U_n) = (X - U_1) \cup \ldots \cup (X - U_n)$, again by DeMorgan’s Law. Either one gets all of $X$ or a finite union of finite sets and so an open set.
The collection $T_{FC}$ is called the finite-complement topology on the infinite set $X$. The finite-complement topology will offer an example later of how strange convergence properties can become in some topological spaces.

5) On a three-point set there are nine distinct topologies, where by distinct we mean up to renaming the points. The distinct topologies are shown in the following diagram.

Given two topologies $T$, $T'$ on a given set $X$ we say $T$ is finer than $T'$ if $T' \subset T$. Equivalently we say $T'$ is coarser than $T$. For example, on any set the indiscrete topology is coarser and the discrete topology is finer than any other topology. The finite-complement topology on $\mathbb{R}$ is strictly coarser than the metric topology. I have added a line joining comparable topologies in the diagram of the distinct topologies on a three-point set. Coarser is lower in this case, and the relation is transitive. As we will see later, the ordering of topologies plays a role in the continuity of functions.

On a given set $X$ it would be nice to have a way of generating topologies. One way is to use a basis for the topology:

**Definition 2.6.** A collection of subsets, $B$, of a set $X$ is a basis for a topology on $X$ if (1) for all $x \in X$, there is a $B \in B$ with $x \in B$, and (2) if $x \in B_1 \in B$ and $x \in B_2 \in B$, then there is some $B_3 \in B$ with $x \in B_3 \subset B_1 \cap B_2$.

**Proposition 2.7.** If $B$ is a basis for a topology on a set $X$, then the collection of subsets

$$
T_B = \{ \bigcup_{\alpha \in A} B_\alpha \mid A \text{ is any index set and } B_\alpha \in B \text{ for all } \alpha \in A \}
$$

is a topology on $X$ called the topology generated by the basis $B$.

**Proof:** We show that $T_B$ satisfies the axioms for a topology. By the definition of a basis, we can write $X = \bigcup_{B \in B} B$ and $\emptyset = \bigcup_{i \in \emptyset} U_i$; so $X$ and $\emptyset$ are in $T_B$. If $U_j$ is in $T_B$ for all
than the metric topology since

generates the

4) A nonstandard basis for a topology on X, and suppose w
is a basis for the metric topology in X. We say that a space is
second countable when it has a basis for its topology that is
countable as a set. Consider B(z, δ) and suppose w ∈ B(z, δ).
Then

\[ d(x, w) \leq d(x, z) + d(z, w) \]
\[ < d(x, z) + \delta \leq d(x, z) + \epsilon - d(x, z) = \epsilon. \]

Likewise, \( d(y, w) < \epsilon' \) and so \( B(z, \delta) \subset B(x, \epsilon) \cap B(y, \epsilon') \) as required.

4) A nonstandard basis for a topology on \( \mathbb{R} \) is given by \( B_{ho} = \{ (a, b) | a < b \} \). This basis
generates the half-open topology on \( \mathbb{R} \). Notice that the half-open topology is strictly finer
than the metric topology since

\[ (a, b) = \bigcup_{n=k}^{\infty} [a + (1/n), b) \]
for \( k \) large enough that \( a + (1/k) < b \). However, no subset \([a, b)\) is a union of open intervals.

**Proposition 2.8.** If \( B_1 \) and \( B_2 \) are bases for topologies in a set \( X \), and for all \( x \in X \) and \( x \in B_1 \subseteq B_1 \), there is a \( B_2 \) with \( x \in B_2 \subseteq B_2 \) and \( B_2 \subseteq B_2 \), then \( T_{B_2} \) is finer than \( T_{B_1} \).

The proof is left as an exercise. The proposition applies to metric spaces. Given two metrics on a space, when do they give the same topology? Let \( d_1 \) and \( d_2 \) denote the metrics and \( B_1(x, \epsilon), B_2(x, \epsilon) \) the open balls of radius \( \epsilon \) at \( x \) given by each metric, respectively. The proposition is satisfied if, for \( i = 2 \), \( j = 1 \) and again for \( i = 1 \), \( j = 2 \), for any \( y \in B_2(x, \epsilon) \), there is an \( \epsilon' > 0 \) with \( B_2(y, \epsilon') \subseteq B_1(x, \epsilon) \). Then the topologies are equivalent. For example, the two metrics defined on \( \mathbb{R}^m \),

\[
    d_1(x, y) = \sqrt{(x_1 - y_1)^2 + \cdots + (x_m - y_m)^2}, \quad d_2(x, y) = \max\{|x_i - y_i| \mid i = 1, \ldots, m\},
\]

give the same topology.

**Continuity**

Having identified the places where continuity can happen, namely, topological spaces, we define what it means to be a continuous function between spaces.

**Definition 2.9.** Let \((X, T)\) and \((Y, T')\) be topological spaces and \(f: X \rightarrow Y\) a function. We say that \(f\) is continuous if whenever \(V\) is open in \(Y\), \(f^{-1}(V)\) is open in \(X\).

This simple definition generalizes the definition of continuous function between metric spaces, and hence recovers the classical definition from the calculus.

The identity mapping, \(\text{id}: (X, T) \rightarrow (X, T)\) is always continuous. However, if we change the topology on the domain or codomain, this may not be true. For example, \(\text{id}: (\mathbb{R}, \text{usual}) \rightarrow (\mathbb{R}, \text{half-open})\) is not continuous since \(\text{id}^{-1}([0, 1)) = [0, 1)\), which is not open in the usual topology. The following proposition is an easy observation.

**Proposition 2.10.** If \(T\) and \(T'\) are topologies on a set \(X\), then the identity mapping \(\text{id}: (X, T) \rightarrow (X, T')\) is continuous if and only if \(T\) is finer than \(T'\).

With this formulation of continuity it is straightforward to give proofs of some of the properties of continuous functions.

**Theorem 2.11.** Given two continuous functions \(f: X \rightarrow Y\) and \(g: Y \rightarrow Z\), the composite function \(g \circ f: X \rightarrow Z\) is continuous.

**Proof:** If \(V\) is open in \(Z\), then \(g^{-1}(V) = U\) is open in \(Y\) and so \(f^{-1}(U)\) is open in \(X\). But \((g \circ f)^{-1}(V) = f^{-1}(g^{-1}(V)) = f^{-1}(U)\), so \((g \circ f)^{-1}(V)\) is open in \(X\) and \(g \circ f\) is continuous. \(\diamondsuit\)

We next give a key definition for topology—the means of comparison of spaces.

**Definition 2.12.** A function \(f: (X, T_X) \rightarrow (Y, T_Y)\) is a homeomorphism if \(f\) is continuous, one-one, onto and has a continuous inverse. We say \((X, T_X)\) and \((Y, T_Y)\) are homeomorphic topological spaces if there is a homeomorphism \(f: (X, T_X) \rightarrow (Y, T_Y)\). A property of a space \((X, T_X)\) is said to be a topological property if, whenever \((Y, T_Y)\) is homeomorphic to \((X, T_X)\), then the space \((Y, T_Y)\) also has the property.

**Examples:** 1) We may take all functions known from the calculus to be continuous functions as having been proved continuous in our language. For example, the mapping \(\text{arctan}: \mathbb{R} \rightarrow \mathbb{R}\)
$(-\pi/2, \pi/2)$ is a homeomorphism. Notice that the metric idea of a subset being of infinite extent is not a topological notion.

2) By the definition of the indiscrete and discrete topologies, any function $f: (X, \text{discrete}) \to (Y, T)$ is continuous as is any function $g: (X, T) \to (Y, \text{indiscrete})$. A partial order is obtained on topologies on a set $X$ by $T \leq T'$ if the identity mapping $\text{id}: (X, T) \to (X, T')$ is continuous. This order is the relation of fineness.

The definition of homeomorphism makes topology the geometry of topological properties in the sense of Klein’s Erlangen Program [Klein]. We treat a figure as a subset of a space $(X, T)$ and the homeomorphisms $f: X \to X$ are the transformations carrying a figure to a “congruent” figure.

The simplest topological property is the cardinality of the space, because a homeomorphism is a one-one correspondence. A more topological example is the notion of second countability.

Proposition 2.13. The property of being second countable is a topological property.

Proof: Suppose $(X, T)$ has a countable basis $\{U_i, i = 1, 2, \ldots\}$. Suppose that $f: (X, T_X) \to (Y, T_Y)$ is a homeomorphism. Write $g = f^{-1}: (Y, T_Y) \to (X, T_X)$ for the inverse homeomorphism. Let $V_i = g^{-1}(U_i)$. Then the proposition follows from a proof that $\{V_i : i = 1, 2, \ldots\}$ is a countable basis for $Y$. To prove this we take any open set $W \subset Y$ and show for all $w \in W$ there is some $j$ with $w \in V_j \subset W$. Let $O = f^{-1}(W)$ and $u = f^{-1}(w) = g(w)$ so that $u \in O \subset X$. Then there is some $j$ with $u \in U_j \subset O$. Apply $g^{-1}$ to get $w \in V_j = g^{-1}(U_j) \subset g^{-1}(O)$. But $g^{-1}(O) = W$ so $w \in V_j \subset W$ as desired, and $(Y, T_Y)$ is second countable.

Later chapters will be devoted to some of the most important topological properties.

Exercises

1. Prove Proposition 2.8.

2. Another way to generate a topology on a set $X$ is from a subbasis, which is a set $S$ of subsets of $X$ such that, for any $x \in X$, there is an element $S \in S$ with $x \in S$. Show that the collection $B_S = \{S_1 \cap \cdots \cap S_n \mid S_i \in S, n > 0\}$ is a basis for a topology on $X$. Show that the set $\{(-\infty, a), (b, \infty) \mid -\infty < a, b < \infty\}$ is a subbasis for the usual topology on $\mathbb{R}$.

3. Suppose that $X$ is an uncountable set and that $x_0$ is some given point in $X$. Let $T_F$ be the collection of subsets $T_F = \{U \subset X \mid X - U \text{ is finite or } x_0 \notin U\}$. Show that $T_F$ is a topology on $X$, called the **Fort topology**.

4. Suppose $X = \text{Bdd}([0, 1], \mathbb{R})$ is the metric space of bounded real-valued functions on $[0, 1]$. Let $F: X \to \mathbb{R}$ be defined by $F(f) = f(1)$. Show that this is a continuous function when $\mathbb{R}$ has the usual topology.
5. A space \((X, \mathcal{T})\) is said to have the **fixed point property** (FPP) if any continuous function \(f: (X, \mathcal{T}) \to (X, \mathcal{T})\) has a fixed point, that is, there is some \(x \in X\) with \(f(x) = x\). Show that the FPP is a topological property.

6. The **taxicab metric** on \(\mathbb{R}^n\) is given by
\[
d(x, y) = |x_1 - y_1| + \cdots + |x_n - y_n|.
\]
Prove that this is indeed a metric on \(\mathbb{R}^n\). Describe the open balls in the taxicab metric on \(\mathbb{R}^2\). How do the usual topology and the taxicab metric topology compare on \(\mathbb{R}^n\)?

7. A space \((X, \mathcal{T})\) is said to be a \(T_1\)-space if for any \(x \in X\), the complement of \(\{x\}\) is open in \(X\). Show that a metric space is \(T_1\). Which of the topologies on the three-point set are \(T_1\)? Show that being \(T_1\) is a topological property.

8. We displayed the nine distinct topologies on a three element set in this chapter. The sequence of integers
\[
t_n = \text{number of distinct topologies on a set of } n \text{ elements}
\]
may be found in Neil Sloane’s On-Line Encyclopedia of Integer Sequences with ID Number A001930. The first few values of \(t_n\), beginning with \(t_0\), are given by
\[
1, 1, 3, 9, 33, 139, 718, 4535, 35979, 363083, 4717687, 79501654, 1744252509
\]