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RESEARCH CONTRIBUTION

Polynomial Splitting Measures and Cohomology of the Pure Braid Group

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Abstract We study for each n a one-parameter family of complex-valued measures on the symmetric group S_n, which interpolate the probability of a monic, degree n, square-free polynomial in $\mathbb{F}_q[x]$ having a given factorization type. For a fixed factorization type, indexed by a partition λ of n, the measure is known to be a Laurent polynomial. We express the coefficients of this polynomial in terms of characters associated to S_n-subrepresentations of the cohomology of the pure braid group $H^\bullet(P_n, \mathbb{Q})$. We deduce that the splitting measures for all parameter values $z = -\frac{1}{m}$ (resp. $z = \frac{1}{m}$), after rescaling, are characters of S_n-representations (resp. virtual S_n-representations).

Keywords Symmetric groups · Braid group · Configuration space

Mathematics Subject Classification Primary 11R09; Secondary 11R32 · 12E20 · 12E25

1 Introduction

The purpose of this paper is to study for each $n \geq 1$ a one-parameter family of complex-valued measures on the symmetric group S_n arising from a problem in number theory, and to exhibit an explicit representation-theoretic connection between these measures and the characters of the natural S_n-action on the rational cohomology of the pure braid group P_n.

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This family of measures, denoted $\nu^*_{n,z}$, was introduced by Lagarias and Weiss (2015), where they were called *z-splitting measures*, with parameter z. The measures interpolate from prime power values $z = q$ the probability of a monic, degree n, square-free polynomial in $\mathbb{F}_q[x]$ having a given factorization type. Square-free factorization types are indexed by partitions λ of n specifying the degrees of the irreducible factors. Each partition λ of n corresponds to a conjugacy class C_λ of the symmetric group S_n; distributing the probability of a factorization of type λ equally across the elements of C_λ defines a probability measure on S_n. A key property of the resulting probabilities is that their values for each fixed partition λ are described by a rational function in the size of the field $\mathbb{F}_q$ as q varies. This property permits interpolation from q to a parameter $z \in \mathbb{P}^1(\mathbb{C})$ on the Riemann sphere, to obtain a family of complex-valued measures $\nu^*_{n,z}$ on S_n given in Definition 2.3 below.

On the number theory side, these measures connect with problems on the splitting of ideals in S_n-number fields, which are degree n number fields formed by adjoining a root of a degree n polynomial over $\mathbb{Z}[x]$ whose splitting field has Galois group S_n. The paper (Lagarias and Weiss 2015, Theorem 2.6) observed that for primes $p < n$ these measures vanish on certain conjugacy classes, corresponding to the phenomenon of essential discriminant divisors of polynomials having Galois group S_n, first noted by Dedekind (1878). These measures converge to the uniform measure on the symmetric group as $z = p \to \infty$, and in this limit agree with a conjecture of Bhargava (2007, Conjecture 1.3) on the distribution of splitting types of the prime p in S_n-extensions of discriminant $|D| \leq B$ as the bound $B \to \infty$, conditioned on $(D, p) = 1$.

The second author subsequently studied these measures interpolated at the special value $z = 1$, viewed as representing splitting probabilities for polynomials over the (hypothetical) "field with one element $\mathbb{F}_1$" (Lagarias 2016). These measures, called 1-*splitting measures*, turn out to be signed measures for all $n \geq 3$. They are supported on a small set of conjugacy classes, the Springer regular elements of S_n which are those conjugacy classes C_λ for which λ has a rectangular Young diagram or a rectangle plus a single box. Treated as class functions on S_n, rather than as measures, they were found to have a representation-theoretic interpretation: after rescaling by $n!$, the 1-splitting measures are virtual characters of S_n corresponding to explicitly determined representations. As n varies, their values on conjugacy classes were observed to have arithmetic properties compatible with the multiplicative structure of n; letting $n = \prod_p p^{e_p}$ be the prime factorization of n, the value of the measure on each conjugacy class factors as a product of values on classes of smaller symmetric groups $S_{p^{e_p}}$. That paper also showed the rescaled z-splitting measures at $z = -1$ have a related representation-theoretic interpretation.

In this paper we extend the representation-theoretic interpretation to the entire family of z-splitting measures and relate it to the cohomology of the pure braid group. Our starting point is the observation made in Lagarias (2016, Lemma 2.5) that for a fixed conjugacy class the z-splitting measures are Laurent polynomials in z. They have degree at most $n - 1$, so may be written

$$\nu^*_{n,z}(C_\lambda) = \sum_{k=0}^{n-1} \alpha_n^k(C_\lambda) \left(\frac{1}{z}\right)^k,$$

with rational coefficients $\alpha_n^k(C_\lambda)$, where λ is a partition of n. We call the $\alpha_n^k(C_\lambda)$ *splitting measure coefficients.* A main observation of this paper is that each splitting measure coefficient $\alpha_n^k(C_\lambda)$, viewed as a function of λ, is a rescaled character χ_n^k of a certain S_n-subrepresentation A_n^k of the cohomology of the pure braid group $H^k(P_n, \mathbb{Q})$. The pure braid groups P_n and their cohomology, along with the subrepresentations A_n^k, are defined and discussed in Sect. 4. In Sect. 4.3 we identify the S_n-representation A_n^k with the cohomology of a complex manifold Y_n carrying an S_n-action. We deduce as a consequence a topological interpretation of the 1-splitting measure as a rescaled version of the S_n-equivariant Euler characteristic of Y_n. We also deduce that the rescaled z-splitting measure is a character of S_n at $z = -\frac{1}{m}$ and is a virtual character of S_n at $z = \frac{1}{m}$, for all integers $m \geq 1$.

The last result extends the representation-theoretic connection of Lagarias (2016) for $z = \pm 1$ to parameter values $z = \pm\frac{1}{m}$ for all $m \geq 1$.

1.1 Results

The *z-splitting measure* on a conjugacy class C_λ of S_n is the rational function of z

$$\nu_{n,z}^*(C_\lambda) := \frac{N_\lambda(z)}{z^n - z^{n-1}},$$

where $N_\lambda(z) \in \mathbb{Q}[z]$ denotes the *cycle polynomial* associated to a partition λ describing the cycle lengths of C_λ. Given $\lambda = \left(1^{m_1(\lambda)} 2^{m_2(\lambda)} \cdots n^{m_n(\lambda)}\right)$, the associated cycle polynomial is

$$N_\lambda(z) := \prod_{j \geq 1} \binom{M_j(z)}{m_j(\lambda)}, \tag{1.1}$$

where $M_j(z)$ denotes the jth necklace polynomial. The *necklace polynomial* $M_j(z)$ of order j is given by

$$M_j(z) := \frac{1}{j} \sum_{d|j} \mu(d) z^{j/d}.$$

where $\mu(d)$ is the Möbius function.

To avoid confusion we make a remark on values of measures. Given a class function f on S_n we write $f(C_\lambda)$ to mean the sum of the values of f on C_λ, and write $f(\lambda)$ to mean the value $f(g)$ taken at one element $g \in C_\lambda$; the latter notation is standard for characters. Thus $\nu_{n,z}^*(C_\lambda) = |C_\lambda| \nu_{n,z}^*(\lambda)$.

In Sect. 3 we express the coefficients of the family of cycle polynomials $N_\lambda(z)$ in terms of characters of the cohomology of the pure braid group P_n viewed as an S_n-representation.

Theorem 1.1 (Character interpretation of cycle polynomial coefficients) *Let* λ *be a partition of* n *and* $N_\lambda(z)$ *be a cycle polynomial. Then*

$$N_\lambda(z) = \frac{|C_\lambda|}{n!} \sum_{k=0}^{n} (-1)^k h_n^k(\lambda) z^{n-k}.$$

where h_n^k *is the character of the kth cohomology of the pure braid group* $H^k(P_n, \mathbb{Q})$*, viewed as an* S_n*-representation.*

Theorem 1.1 is a rescaled version of a result of Lehrer (1987, Theorem 5.5). Lehrer arrived at it from his study of the Poincaré polynomials associated to the elements of a Coxeter group acting on the complements of certain complex hyperplane arrangements. We arrived at it through a direct study of the cycle polynomial $N_\lambda(z)$ appearing in the definition of the z-splitting measure, relating it to representation stability using the twisted Grothendieck–Lefschetz formula of Church et al. (2014, Prop. 4.1). We include a proof of Theorem 1.1 (as Theorem 3.2); the method behind this proof also traces back to work of Lehrer (1992).

At the end of Sect. 3 we apply Theorem 1.1 together with the formula (1.1) for $N_\lambda(z)$ to obtain explicit expressions for various characters h_n^k showing number-theoretic structure, and to determine restrictions on the support of various h_n^k.

In Sect. 4 we review Arnol'd's presentation of the cohomology ring of the pure braid group. In Sect. 4.2 we use it derive an exact sequence determining certain S_n-subrepresentations A_n^k of $H^k(P_n, \mathbb{Q})$ which play the main role in our results. These subrepresentations lead to a direct sum decomposition $H^k(P_n, \mathbb{Q}) \simeq A_n^{k-1} \oplus A_n^k$, for each $k \geq 0$. In Sect. 4.3 we interpret the A_n^k as the cohomology of an $(n-1)$-dimensional complex manifold Y_n that carries an S_n-action. The manifold Y_n is the quotient of the pure configuration space $\mathrm{PConf}_n(\mathbb{C})$ of n distinct (labeled) points in $\mathbb{C}$ by a free action of $\mathbb{C}^\times$.

The main result of this paper, given in Sect. 5, expresses the z-splitting measures $\nu_{n,z}^*$ in terms of the characters χ_n^k of the S_n-representations A_n^k.

Theorem 1.2 (Character interpretation of splitting measure coefficients) *For each* $n \geq 1$ *and* $0 \leq k \leq n-1$ *there is an* S_n*-subrepresentation* A_n^k *of* $H^k(P_n, \mathbb{Q})$ (*constructed explicitly in Proposition* 4.2) *with character* χ_n^k *such that for each partition* λ *of* n,

$$\nu_{n,z}^*(C_\lambda) = \frac{|C_\lambda|}{n!} \sum_{k=0}^{n-1} \chi_n^k(\lambda) \left(-\frac{1}{z}\right)^k.$$

Thus the splitting measure coefficient $\alpha_n^k(C_\lambda) = |C_\lambda|\, \alpha_n^k(\lambda)$ *is given by*

$$\alpha_n^k(C_\lambda) = (-1)^k \frac{|C_\lambda|}{n!} \chi_n^k(\lambda).$$

In Sect. 5.2 we interpret this result in terms of cohomology of the manifold Y_n. On setting $t = -\frac{1}{z}$, we have that for each $g \in S_n$,

$$\nu^*_{n,z}(g) = \frac{1}{n!} \sum_{k=0}^{n-1} \mathrm{Trace}(g, H^k(Y_n, \mathbb{Q}))t^k,$$

which is a value of the equivariant Poincaré polynomial for Y_n with respect to the S_n-action (Theorem 5.2). In particular we obtain the following topological interpretation of the 1-splitting measure, as the special case $t = -1$.

Theorem 1.3 (Topological interpretation of 1-splitting measure) *Let Y_n denote the open complex manifold* $\mathrm{PConf}_n(\mathbb{C})/\mathbb{C}^\times$, *which carries an S_n-action under permutation of the n points. Then the rescaled* 1*-splitting measure $\nu^*_{n,1}(\cdot)$ evaluated at elements $g \in S_n$ is the equivariant Euler characteristic of Y_n,*

$$\nu^*_{n,1}(g) = \frac{1}{n!} \sum_{k=0}^{n-1} (-1)^k \mathrm{Trace}(g, H^k(Y_n, \mathbb{Q})),$$

with respect to its S_n-action.

In Sect. 5.3 we obtain another corollary of Theorem 1.2. For $z = -\frac{1}{m}$ with $m \geq 1$, the rescaled splitting measure $\frac{n!}{|C_\lambda|}\nu^k_{n,z}(C_\lambda)$ is the character of an S_n-representation, and when $z = \frac{1}{m}$ it is the character of a virtual S_n-representation (Theorem 5.3).

In Sect. 5.4 we deduce an interesting consequence concerning the S_n-action on the full cohomology ring $H^\bullet(P_n, \mathbb{Q})$. The structure of the cohomology ring of the pure braid group $H^\bullet(P_n, \mathbb{Q})$ as an S_n-module has an extensive literature. Orlik and Solomon (1980) noted that $H^\bullet(P_n, \mathbb{Q}) \simeq H^\bullet(M(\mathcal{A}_n), \mathbb{Q})$ as S_n-modules, where

$$M(\mathcal{A}_n) = \mathbb{C}^n \setminus \cup_{H \in \mathcal{A}_n} H$$

is the complement of the (complexified) braid arrangement $\mathcal{A}_n$, i.e. the arrangement of $n(n-1)/2$ hyperplanes $z_i = z_j$ in $\mathbb{C}^n$ where $1 \leq i < j \leq n$ are the coordinate functionals of $\mathbb{C}^n$. The structure of the cohomology groups $H^k(M(\mathcal{A}_n), \mathbb{C}) = H^k(M(\mathcal{A}_n), \mathbb{Q}) \otimes \mathbb{C}$ as S_n-representations was determined in 1986 by Lehrer and Solomon (1986, Theorem 4.5) in terms of induced representations $\mathrm{Ind}^{S_n}_{Z(C_\lambda)}(\xi_\lambda)$ for specific linear representations ξ_λ on the centralizers $Z(C_\lambda)$ of conjugacy classes C_λ having $n-k$ cycles. In 1987 Lehrer (1987, p. 276) noted that his results on Poincaré polynomials implied the "curious consequence" that the action of S_n on $\bigoplus_k H^k(M(\mathcal{A}_n, \mathbb{C}))$ is "almost" the regular representation in the sense that the dimension is $n!$ and the character $\theta(g)$ of this representation is 0 unless g is the identity element or a transposition, see also Lehrer (1987, Corollary (5.5)', Prop. (5.6)). where r is a reflection and 1 is the trivial representation. In Sect. 5.4 we apply Theorem 1.2 together with values of the (-1)-splitting measure computed in Lagarias (2016) to make a precise connection between the S_n-representation structure on pure braid group cohomology and the regular representation $\mathbb{Q}[S_n]$.

Theorem 1.4 *Let* $\mathbf{1}_n$, $\mathbf{Sgn}_n$, *and* $\mathbb{Q}[S_n]$ *be the trivial, sign, and regular representations of* S_n *respectively. Then there is an isomorphism of* S_n*-representations,*

$$\bigoplus_{k=0}^{n} H^k(P_n, \mathbb{Q}) \otimes \mathbf{Sgn}_n^{\otimes k} \cong \mathbb{Q}[S_n].$$

Here $\mathbf{Sgn}_n^{\otimes k} \cong \mathbf{1}_n$ *or* $\mathbf{Sgn}_n$ *according to whether* k *is even or odd.*

When combined with Lehrer (1987, Prop. 5.6 (i)) determination of the character θ as $2\,\mathrm{Ind}_{\langle\tau\rangle}^{S_n}(1)$, where τ is a transposition, this result implies that each of the characters of the S_n-representations acting on the even-dimensional cohomology, resp. odd-dimensional cohomology are supported on the identity element plus transpositions. We comment on other related work in Sect. 1.2.

In Sect. 6 we describe further interpretations of the representations A_n^k in terms of other combinatorial homology theories. For fixed k and varying n, the sequence of S_n-representations $H^k(P_n, \mathbb{Q})$ was one of the basic examples exhibiting *representation stability* in the sense of Church and Farb (2013), see Church et al. (2014, 2015). We show in Proposition 6.2 that the representations A_n^k are isomorphic to others appearing in the literature known to exhibit representation stability. Hersh and Reiner (2015, Corollary 5.4) determine the precise rate of stabilization of these representations, yielding the following result.

Theorem 1.5 (Representation stability for A_n^k) *For each fixed* $k \geq 1$, *the sequence of* S_n*-representations* A_n^k *with characters* χ_n^k *are representation stable, and stabilize sharply at* $n = 3k + 1$.

To summarize these results:

(i) We start from a construction in number theory: a set of probability measures on S_n that describe the distribution of degree n squarefree monic polynomial factorizations $\pmod p$ defined for a parameter z being a prime p. These measure values interpolate at each fixed $g \in S_n$ in the z-variable as polynomials in $1/z$ to define complex-valued measures on S_n.

(ii) We make a connection of the interpolated measures as functions of z to topology and representation theory: For fixed n the kth Laurent coefficients of the z-parametrization at $g \in S_n$ (rescaled by $n!$) coincide with the character of an S_n-subrepresentation A_n^k of the cohomology of the pure braid group P_n, which is an S_n-representation on the cohomology of the complex manifold $Y_n = \mathrm{PConf}_n(\mathbb{C})/\mathbb{C}^\times$. As n varies with k fixed these coefficients exhibit representation stability as $n \to \infty$.

(iii) We deduce that (rescaled) measure values at values $z = -\frac{1}{m}$ for $m \geq 1$ coincide with characters of certain S_n-representations; those at $z = \frac{1}{m}$ with $m \geq 1$ coincide with certain virtual S_n-representations. For each n these representations combine stable and unstable cohomology of P_n.

(iv) As a by-product we find a precise connection between the (total) cohomology of the pure braid group as an S_n-representation and the regular representation of S_n.

The main observation of this paper is the relation of these interpolation measures to representation theory. We demonstrate this relation by calculation, and leave open the problem of finding a deeper conceptual explanation for its existence.

1.2 Related work

The representations A_n^k have appeared in the literature in numerous places. In particular, a 1995 result of Getzler (1995, Corollary 3.10) permits an identification of A_n^k as an S_n-module with the kth cohomology group of the moduli space $\mathcal{M}_{0,n+1}$ of the Riemann sphere with $n+1$ marked points, viewed as an S_n-module, holding one point fixed. Getzler identifies this cohomology with the S^1-equivariant cohomology of $\mathrm{PConf}_n(\mathbb{C})$, which is the cohomology of Y_n given in Theorem 5.2. Some more recent occurrences of A_n^k are discussed in Sect. 6.

In connection with Theorem 1.4, in Gaiffi (1996) further explained Lehrer's formula $\theta = 2\,\mathrm{Ind}_{\langle\tau\rangle}^{S_n}(1)$ by showing that

$$H^\bullet(M(\mathcal{A}_{n-1}), \mathbb{C}) \simeq H^\bullet(M(d\mathcal{A}_{n-1}), \mathbb{C}) \otimes \left(\mathbb{C} \oplus \frac{\mathbb{C}[\varepsilon]}{\varepsilon^2}\right),$$

as S_n-modules, where $d\mathcal{A}_{n-1}$ is obtained by a deconing construction, while the class ε has degree 1 and carries the trivial S_n-action. (His space $M(\mathcal{A}_{n-1})$ lies in $\mathbb{C}^{n-1}$ and is obtained by restricting the braid arrangement on $\mathbb{C}^n$ to the hyperplane $x_1 + x_2 + \cdots + x_n = 0$ in $\mathbb{C}^n$, and the deconed configuration space $M(d\mathcal{A}_{n-1}) \subset \mathbb{C}^{n-2}$.) On comparison with our direct sum decomposition we have $H^k(d\mathcal{A}_{n-1}, \mathbb{C}) \simeq A_n^k$ as S_n-modules, showing that the deconed space $d\mathcal{A}_{n-1}$ has an isomorphic cohomology ring as the complex manifold Y_n with an appropriate S_n-module structure. Gaiffi and also Mathieu (1996) showed there is a "hidden" S_{n+1}-action on this cohomology ring. For more recent developments on the "hidden" action see Callegaro and Gaiffi (2015).

1.3 Plan of the Paper

In Sect. 2 we recall properties of the z-splitting measures from Lagarias and Weiss (2015). In Sect. 3 we use the twisted Grothendieck–Lefschetz formula to relate the coefficients of cycle polynomials to the characters of the S_n-representations $H^k(P_n, \mathbb{Q})$. In Sect. 4 we discuss the cohomology $H^k(P_n, \mathbb{Q})$ of the pure braid group P_n, and derive an exact sequence leading to the construction of the S_n-representations A_n^k. In Sect. 5 we express the splitting measure coefficients $\alpha_n^k(C_\lambda)$ in terms of the character χ_n^k of the representation A_n^k. In Sect. 6 we discuss representation stability and connect the S_n-representations A_n^k with others in the literature.

1.4 Notation

1. $q = p^f$ denotes a prime power.

2. The set of monic, degree n, square-free polynomials in $\mathbb{F}_q[x]$ is denoted $\mathrm{Conf}_n(\mathbb{F}_q)$.
3. We write partitions either as $\lambda = [\lambda_1, \lambda_2, \ldots, \lambda_\ell]$, with parts $\lambda_1 \geq \lambda_2 \geq \cdots$ eventually 0, or as $\lambda = (1^{m_1} 2^{m_2} \cdots)$ where $m_j = m_j(\lambda)$ is the number of parts of λ of size j. The length of λ is $\ell(\lambda) = \max\{r : \lambda_r \geq 1\}$, the size of λ is $|\lambda| = \sum_i \lambda_i = \sum_j j m_j$, and λ_i is the ith largest part of λ. (Compare Macdonald 1995.)
4. Each partition λ of n corresponds to a conjugacy class C_λ of S_n given by the common cycle structure of the elements in C_λ. We let Z_λ denote the centralizer of C_λ in S_n. The size of the centralizer and conjugacy class are
$$z_\lambda := |Z_\lambda| = \prod_{j \geq 1} j^{m_j(\lambda)} m_j(\lambda)! \qquad c_\lambda := |C_\lambda| = \frac{n!}{z_\lambda}$$
respectively. Note that $c_\lambda z_\lambda = n!$.
5. Following Stanley (1986), we let $\mathrm{Par}(n)$ denote the set of partitions of n and $\mathrm{Par} = \bigcup_n \mathrm{Par}(n)$ the set of all partitions. However in Sect. 6, we let Π_n denote the set of partitions of n, partially ordered by refinement.

2 Splitting Measures

We review the splitting measures introduced in Lagarias and Weiss (2015), summarize their properties, and introduce the normalized splitting measures.

2.1 Necklace Polynomials and Cycle Polynomials

Definition 2.1 For $j \geq 1$, the jth *necklace polynomial* $M_j(z) \in \frac{1}{j}\mathbb{Z}[z]$ is
$$M_j(z) := \frac{1}{j} \sum_{d|j} \mu(d) z^{j/d},$$
where $\mu(d)$ is the Möbius function.

Moreau (1872) noted in 1872 that for all integers $m \geq 1$, $M_j(m)$ is the number of distinct necklaces having j beads drawn from a set of m colors, up to cyclic permutation. This fact motivated Metropolis and Rota (1983) to name them *necklace polynomials*. Relevant to the present paper, $M_j(q)$ is the number of monic, degree j, irreducible polynomials in $\mathbb{F}_q[X]$ Rosen (2002, Prop. 2.1). The factorization type of a polynomial $f \in \mathrm{Conf}_n(\mathbb{F}_q)$ is the partition formed by the degrees of its irreducible factors, which we write $[f]$.

Definition 2.2 Given a partition λ of n, the *cycle polynomial* $N_\lambda(z) \in \frac{1}{z_\lambda}\mathbb{Z}[z]$ is
$$N_\lambda(z) := \prod_{j \geq 1} \binom{M_j(z)}{m_j(\lambda)},$$

where $\binom{\alpha}{m}$ is the usual extension of a binomial coefficient,

$$\binom{\alpha}{m} := \frac{1}{m!}\prod_{k=0}^{m-1}(\alpha - k).$$

The cycle polynomial $N_\lambda(z)$ has degree $n = |\lambda|$ and is integer valued for $z \in \mathbb{Z}$. The number of $f \in \mathrm{Conf}_n(\mathbb{F}_q)$ with $[f] = \lambda$ is $N_\lambda(q)$ (see Lagarias and Weiss 2015, Sect. 4).

2.2 z-Splitting Measures

If λ a partition of n, then the probability of a uniformly chosen $f \in \mathrm{Conf}_n(\mathbb{F}_q)$ having factorization type λ is

$$\mathrm{Prob}\{f \in \mathrm{Conf}_n(\mathbb{F}_q) : [f] = \lambda\} = \frac{N_\lambda(q)}{|\mathrm{Conf}_n(\mathbb{F}_q)|}.$$

When $n = 1$, $|\mathrm{Conf}_n(\mathbb{F}_q)| = q$ and for $n \geq 2$ we have $|\mathrm{Conf}_n(\mathbb{F}_q)| = q^n - q^{n-1}$. (See Rosen (2002, Prop. 2.3) for a proof via generating functions. A proof due to Zieve appears in Weiss (2013, Lem. 4.1).) Hence, the probability is a rational function in q. Replacing q by a complex-valued parameter z yields the z-splitting measure.

Definition 2.3 For $n \geq 2$ the *z-splitting measure* $\nu^*_{n,z}(C_\lambda) \in \mathbb{Q}(z)$ is given by

$$\nu^*_{n,z}(C_\lambda) := \frac{N_\lambda(z)}{z^n - z^{n-1}}.$$

Proposition 2.4 *For each partition λ of $n \geq 1$, the rational function $\nu^*_{n,z}(C_\lambda)$ is a polynomial in $\frac{1}{z}$ of degree at most $n-1$. Thus it may be written as*

$$\nu^*_{n,z}(C_\lambda) = \sum_{k=0}^{n-1} \alpha_n^k(C_\lambda)\left(\frac{1}{z}\right)^k.$$

The function $\nu^*_{1,z}(C_1) = 1$ is independent of z.

Proof The case $n = 1$ is clear. For $n \geq 2$ we have $N_\lambda(1) = 0$ by Lagarias (2016, Lemma 2.5), whence $\frac{N_\lambda(z)}{z-1}$ is a polynomial of degree at most $n-1$ in z. Therefore,

$$\nu^*_{n,z}(C_\lambda) = \frac{N_\lambda(z)}{z^n - z^{n-1}} = \frac{1}{z^{n-1}}\left(\frac{N_\lambda(z)}{z-1}\right)$$

is a polynomial in $\frac{1}{z}$ of degree at most $n-1$. □

For $n \geq 2$ the Laurent polynomial $\nu^*_{n,z}(C_\lambda)$ is of degree at most $n-2$ since $z \mid N_\lambda(z)$ (Lagarias and Weiss 2015, Lemma 4.3); that is, $\alpha_n^{n-1}(C_\lambda) = 0$. Tables 1 and 2 give $\nu^*_{n,z}(C_\lambda)$, exhibiting the splitting measure coefficients $\alpha_n^k(C_\lambda)$ for $n = 4$ and $n = 5$.

Table 1 Values of the z-splitting measures $\nu^*_{4,z}(C_\lambda)$ on partitions λ of $n = 4$

λ	$\lvert C_\lambda\rvert$	z_λ	$\nu^*_{4,z}(C_\lambda)$
[1, 1, 1, 1]	1	24	$\frac{1}{24}\left(1-\frac{5}{z}+\frac{6}{z^2}\right)$
[2, 1, 1]	6	4	$\frac{1}{4}\left(1-\frac{1}{z}\right)$
[2, 2]	3	8	$\frac{1}{8}\left(1-\frac{1}{z}-\frac{2}{z^2}\right)$
[3, 1]	8	3	$\frac{1}{3}\left(1+\frac{1}{z}\right)$
[4]	6	4	$\frac{1}{4}\left(1+\frac{1}{z}\right)$

Table 2 Values of the z-splitting measures $\nu^*_{5,z}(C_\lambda)$ on partitions λ of $n = 5$

λ	$\lvert C_\lambda\rvert$	z_λ	$\nu^*_{5,z}(C_\lambda)$
[1, 1, 1, 1, 1]	1	120	$\frac{1}{120}\left(1-\frac{9}{z}+\frac{26}{z^2}-\frac{24}{z^3}\right)$
[2, 1, 1, 1]	10	12	$\frac{1}{12}\left(1-\frac{3}{z}+\frac{2}{z^2}\right)$
[2, 2, 1]	15	8	$\frac{1}{8}\left(1-\frac{1}{z}-\frac{2}{z^2}\right)$
[3, 1, 1]	20	6	$\frac{1}{6}\left(1-\frac{1}{z^2}\right)$
[3, 2]	20	6	$\frac{1}{6}\left(1-\frac{1}{z^2}\right)$
[4, 1]	30	4	$\frac{1}{4}\left(1+\frac{1}{z}\right)$
[5]	24	5	$\frac{1}{5}\left(1+\frac{1}{z}+\frac{1}{z^2}+\frac{1}{z^3}\right)$

3 Interpretation of Cycle Polynomial Coefficients

In Sect. 2.1 we defined the cycle polynomials $N_\lambda(z) \in \frac{1}{z_\lambda}\mathbb{Z}[z]$ for each partition λ of n. In this section we express the coefficients of $N_\lambda(z)$ as a function of λ in terms of characters h_n^k of the cohomology of the pure braid group P_n viewed as an S_n-representation. We establish this connection using the twisted Grothendieck–Lefschetz formula of Church et al. (2014). Using explicit formulas for the cycle polynomials we obtain constraints on the support of h_n^k, and we compute $h_n^k(\lambda)$ for varying n in several examples.

3.1 Cohomology of the Pure Braid Group

Given a set X of n distinct points in 3-dimensional affine space, the *braid group* B_n consists of homotopy classes of simple, non-intersecting paths beginning and terminating in X, with concatenation as the group operation. Each element of B_n determines a permutation of X, giving a short exact sequence of groups

$$0 \to P_n \to B_n \xrightarrow{\pi} S_n \to 0.$$

Then $P_n := \ker \pi$ is called the *pure braid group*. P_n consists of homotopy classes of simple, non-intersecting *loops* based in X. The action of S_n on X induces an action on P_n by permuting the loops. Thus, for each k, the kth group cohomology $H^k(P_n, \mathbb{Q})$ carries an S_n-representation whose character we denote by h_n^k.

3.2 Twisted Grothendieck–Lefschetz Formula

A *character polynomial* is a polynomial $P(x) \in \mathbb{Q}[x_j : j \geq 1]$. Character polynomials induce functions $P : \mathrm{Par} \to \mathbb{Q}$ by

$$P(\lambda) := P\left(m_1(\lambda), m_2(\lambda), \ldots\right),$$

noting that $m_i(\lambda) = 0$ for all but finitely many i. For $f \in \mathrm{Conf}_n(\mathbb{F}_q)$ we let $P(f) := P([f])$. Given two $\mathbb{Q}$-valued functions F and G defined on S_n let

$$\langle F, G \rangle := \frac{1}{n!} \sum_{g \in S_n} F(g)G(g).$$

The following theorem is due to Church et al. (2014, Prop. 4.1).

Theorem 3.1 (Twisted Grothendieck–Lefschetz formula for PConf_n) *Given a prime power q, an integer $n \geq 1$, and a character polynomial P, we have*

$$\sum_{f \in \mathrm{Conf}_n(\mathbb{F}_q)} P(f) = \sum_{k=0}^{n} (-1)^k \langle P, h_n^k \rangle\, q^{n-k}, \tag{3.1}$$

where h_n^k is the character of the cohomology of the pure braid group $H^k(P_n, \mathbb{Q})$.

The classic Lefschetz trace formula counts the fixed points of an endomorphism f on a compact manifold M by the trace of the induced map on the singular cohomology of M. One may interpret the $\overline{\mathbb{F}}_q$ points on an algebraic variety V defined over $\mathbb{F}_q$ as the fixed points of the *geometric Frobenius endomorphism* of V. Using the machinery of ℓ-adic étale cohomology, Grothendieck (1963) generalized Lefschetz's formula to count the number of points in $V(\mathbb{F}_q)$ by the trace of Frobenius on the étale cohomology of V. For nice varieties V defined over $\mathbb{Z}$, there are comparison theorems relating the étale cohomology of $V(\overline{\mathbb{F}}_q)$ to the singular cohomology of $V(\mathbb{C})$. This connects the topology of a complex manifold to point counts of a variety over a finite field. For hyperplane complements the connection was made in 1992 by Lehrer (1992), and for equivariant actions of a finite group on varieties the equivariant Poincaré polynomials were determined by Kisin and Lehrer (2002).

Church et al. (2014) build upon Grothendieck's extension of the Lefschetz formula to relate point counts on natural subsets of $\mathrm{Conf}_n(\mathbb{F}_q)$ to the singular cohomology of the covering space $\mathrm{PConf}_n(\mathbb{C}) \to \mathrm{Conf}_n(\mathbb{C})$. $\mathrm{PConf}_n(\mathbb{C})$ is the space of n distinct, labelled points in $\mathbb{C}$. The space $\mathrm{PConf}_n(\mathbb{C})$ has fundamental group P_n, the pure braid group, and is a $K(\pi, 1)$ for this group. Hence, the singular cohomology of $\mathrm{PConf}_n(\mathbb{C})$

is the same as the group cohomology of P_n. This fact yields the connection between $\mathrm{Conf}_n(\mathbb{F}_q)$ on the left hand side of (3.1) and the character of the pure braid group cohomology.

3.3 Cycle Polynomials and Pure Braid Group Cohomology

We express the coefficients of the cycle polynomials $N_\lambda(z)$ in terms of the characters h_n^k as an application of Theorem 3.1. Theorem 3.2 is equivalent to Lehrer's (1987, Theorem 5.5) by comparing numerators and making a slight change of variables.

Theorem 3.2 *Let λ be a partition of n, then*

$$N_\lambda(z) = \frac{1}{z_\lambda} \sum_{k=0}^{n} (-1)^k h_n^k(\lambda) z^{n-k},$$

where h_n^k is the character of the S_n-representation $H^k(P_n, \mathbb{Q})$.

Proof Define the character polynomial $1_\lambda(x) \in \mathbb{Q}[x_j : j \geq 1]$ by

$$1_\lambda(x) = \prod_{j \geq 1} \binom{x_j}{m_j(\lambda)}.$$

Observe that for a partition $\mu \in \mathrm{Par}(n)$ we have

$$1_\lambda(\mu) = \begin{cases} 1 & \text{if } \mu = \lambda, \\ 0 & \text{otherwise.} \end{cases}$$

Therefore,

$$N_\lambda(q) = \sum_{f \in \mathrm{Conf}_n(\mathbb{F}_q)} 1_\lambda(f).$$

On the other hand, by Theorem 3.1 we have

$$\sum_{f \in \mathrm{Conf}_n(\mathbb{F}_q)} 1_\lambda(f) = \sum_{k=0}^{n} (-1)^k \langle 1_\lambda, h_n^k \rangle q^{n-k}.$$

If $g \in S_n$, let $[g] \in \mathrm{Par}(n)$ be the partition given by the cycle lengths of g. Thus,

$$\langle 1_\lambda, h_n^k \rangle = \frac{1}{n!} \sum_{g \in S_n} 1_\lambda(g) h_n^k(g) = \frac{1}{n!} \sum_{\substack{g \in S_n \\ [g]=\lambda}} h_n^k(g) = \frac{c_\lambda}{n!} h_n^k(\lambda) = \frac{1}{z_\lambda} h_n^k(\lambda).$$

Therefore the identity

$$N_\lambda(q) = \frac{1}{z_\lambda} \sum_{k=0}^{n} (-1)^k h_n^k(\lambda) q^{n-k}$$

holds for all prime powers q, giving the identity as polynomials in $\mathbb{Q}[z]$. □

Remark A recent result of Chen (2016, Theorem 1) also yields the identity in Theorem 3.2 by specializing at $t = 0$.

One can explicitly compute $h_n^k(\lambda)$ using Theorem 3.2 by expanding the formula (1.1) for $N_\lambda(z)$ and comparing coefficients. Lehrer (1987) derives several corollaries this way. Here we give further examples intended to explore possible connections with number theory. We obtain restrictions on the support of h_n^k in Proposition 3.3. Then we compute values of $h_n^k(\lambda)$ in Sects. 3.5 and 3.6. For any fixed k, the h_n^k are given by character polynomials, while h_n^{n-k} for $k < 2n/3$ exhibit interesting arithmetic structure.

3.4 Support Restrictions on Characters h_n^k

The character h_n^k is supported on partitions with at least one small part, while h_n^{n-k} is supported on partitions having at most k different parts. The latter are *multi-rectangular Young diagrams* having at most k steps, using the terminology of Dołęga et al. (2010, Sect. 1.7) and Śniady (2014).

Proposition 3.3 *Let* $0 \leq k \leq n$ *and* h_n^k *be the character of the* S_n*-representation* $H^k(P_n, \mathbb{Q})$, *then*

1. h_n^k *is supported on partitions having at least one part of size at most* $2k$. *The value* $h_n^k(\lambda)$ *is determined by* $m_j(\lambda)$ *for* $1 \leq j \leq 2k$.
2. h_n^{n-k} *is supported on multi-rectangular partitions* λ *having at most* k *distinct values of* j *with* $m_j(\lambda) > 0$.

Proof (1) Theorem 3.2 implies $h_n^k(\lambda)$ is nonzero iff the coefficient of z^{n-k} in $N_\lambda(z)$ is nonzero. The degree of $M_j(z) - \frac{1}{j} z^j$ is at most $\lfloor j/2 \rfloor$. Hence if $j > 2k$, then the coefficient of z^{n-k} in $\binom{M_j(z)}{m_j(\lambda)}$ is zero. Thus the only j contributing to the coefficient of z^{n-k} in $N_\lambda(z)$ in (1.1) are those with $1 \leq j \leq 2k$.

(2) Theorem 3.2 implies $h_n^{n-k}(\lambda)$ is nonzero iff the coefficient of z^k in $N_\lambda(z)$ is nonzero. If $m_j(\lambda) > 0$, then z divides $\binom{M_j(z)}{m_j(\lambda)}$. Hence if $m_j(\lambda) > 0$ for more than k values of j, then $h_n^{n-k}(\lambda) = 0$. □

Remark Property (1) is a manifestation of representation stability of h_n^k, which says that for fixed k and all sufficiently large n, the values of $h_n^k(\lambda)$ are described by a character polynomial in λ. A *character polynomial* for a partition $\lambda = (1^{m_1} 2^{m_2} \cdots n^{m_n})$ is a polynomial in the variables m_j, see Example 3.7. Farb (2014) raised the problem of explicitly determining such character polynomials. Proposition 3.3 bounds which variables m_j may occur in the character polynomial for h_n^k. A known sharp representation stability property of h_n^k is that it equals such a character polynomial for all $n \geq 3k + 1$, as shown in Hersh and Reiner (2015, Theorem 1.1), taking dimension $d = 2$.

3.5 Character Values $h_n^k(\lambda)$ for Fixed λ and Varying k

We give special cases of explicit determinations for $h_n^k(\lambda)$ for various fixed λ and varying k by directly expanding the cycle polynomial $N_\lambda(z)$.

Example 3.4 (Dimensions of cohomology) The dimension of $H^k(P_n, \mathbb{Q})$ is the value of h_n^k at the identity element, corresponding to the partition (1^n). Since $M_1(z) = z$ and the centralizer of the identity in S_n has order $z_{(1^n)} = n!$, we have

$$N_{(1^n)}(z) = \binom{z}{n} = \frac{1}{n!}\prod_{i=0}^{n-1}(z - i) = \frac{1}{n!}\sum_{k=0}^{n}(-1)^k \begin{bmatrix} n \\ n-k \end{bmatrix} z^{n-k},$$

where $\left[{n \atop n-k}\right]$ is an *unsigned Stirling number of the first kind*. Theorem 3.2 says

$$N_{(1^n)}(z) = \frac{1}{n!}\sum_{k=0}^{n}(-1)^k h_n^k\left((1^n)\right) z^{n-k}.$$

Comparing coefficients recovers the well-known formula due to Arnol'd (1969) for the dimension of the pure braid group cohomology:

$$\dim H^k(P_n, \mathbb{Q}) = h_n^k\left((1^n)\right) = \begin{bmatrix} n \\ n-k \end{bmatrix}.$$

These values are given in Table 3.

Example 3.5 The partition $\lambda = [n]$ corresponds to an n-cycle in S_n. The centralizer of an n-cycle has order $z_{[n]} = n$ and

$$N_{[n]}(z) = \binom{M_n(z)}{1} = M_n(z) = \frac{1}{n}\sum_{d|n}\mu(d) z^{n/d}. \tag{3.2}$$

Theorem 3.2 gives us

$$N_{[n]}(z) = \frac{1}{n}\sum_{k=0}^{n}(-1)^k h_n^k\left([n]\right) z^{n-k}. \tag{3.3}$$

Table 3 Betti numbers of pure braid group cohomology $H^k(P_n, \mathbb{Q})$

$n\backslash k$	0	1	2	3	4	5	6	7	8
1	1	0	0	0	0	0	0	0	0
2	1	1	0	0	0	0	0	0	0
3	1	3	2	0	0	0	0	0	0
4	1	6	11	6	0	0	0	0	0
5	1	10	35	50	24	0	0	0	0
6	1	15	85	225	274	120	0	0	0
7	1	21	175	735	1624	1764	720	0	0
8	1	28	322	1960	6769	13,132	13,068	5040	0
9	1	36	546	4536	22,449	67,284	118,124	109,584	40,320

Comparing coefficients, we find that

$$h_n^{n-k}([n]) = \begin{cases} (-1)^{n-k}\mu(\frac{n}{k}) & \text{if } k \mid n, \\ 0 & \text{if } k \nmid n. \end{cases}$$

3.6 Character Values $h_n^k(\lambda)$ for Fixed k and Varying λ

We now compute $h_n^k(\lambda)$ for fixed k and varying λ.

Example 3.6 (Computing h_n^0 and h_n^n) The cases $k = 0$ and n are both constant: $h_n^0 = 1$ and $h_n^n = 0$. The leading coefficient of $N_\lambda(z)$ is $1/z_\lambda$, hence Theorem 3.2 tells us $h_n^0(\lambda) = 1$ for all λ. For $j \geq 1$, we have $z \mid M_j(z)$, from which it follows that $z \mid N_\lambda(z)$ for all partitions λ of $n \geq 1$. In other words, for all $m_j \geq 1$

$$\frac{1}{z_\lambda}(-1)^n h_n^n(\lambda) = N_\lambda(0) = 0.$$

Thus $h_n^n(\lambda) = 0$ for all λ, and $H^n(P_n, \mathbb{Q}) = 0$.

Example 3.7 (Computing h_n^1 and h_n^2) Taking $\lambda = (1^{m_1}2^{m_2}\cdots)$, a careful analysis of the z^{n-1} and z^{n-2} coefficients in $N_\lambda(z)$ and Theorem 3.2 yields the following formulas

$$h_n^1(\lambda) = \binom{m_1}{2} + \binom{m_2}{1}$$
$$h_n^2(\lambda) = 2\binom{m_1}{3} + 3\binom{m_1}{4} + \binom{m_1}{2}\binom{m_2}{1} - \binom{m_2}{2} - \binom{m_3}{1} - \binom{m_4}{1},$$

where $m_j = m_j(\lambda)$. These formulas represent h_n^1 and h_n^2 as character polynomials, and they appear in Church et al. (2014, Lemma 4.8). Note that $h_n^1(\lambda) = h_n^2(\lambda) = 0$ for partitions λ having all parts larger than 2 and 4 respectively, illustrating Proposition 3.3(1).

Example 3.8 (Computing h_n^{n-1}) The z coefficient of $N_\lambda(z)$ determines the value of $h_n^{n-1}(\lambda)$. Since each j with $m_j(\lambda) > 0$ contributes a factor of z to $N_\lambda(z)$, h_n^{n-1} is supported on partitions of the form $\lambda = (j^m)$. Note that the z coefficient of the necklace polynomial $M_j(z)$ is $\mu(j)/j$. Let $\lambda = (j^m)$, then the z coefficient of

$$N_\lambda(z) = \binom{M_j(z)}{m} = \frac{M_j(z)(M_j(z)-1)\cdots(M_j(z)-m+1)}{m!}$$

is $(-1)^{m-1}\frac{\mu(j)}{jm}$. Since $z_\lambda = j^m m!$, we conclude

$$h_n^{n-1}(\lambda) = \begin{cases} (-1)^{m-n}\mu(j)j^{m-1}(m-1)! & \text{if } \lambda = (j^m), \\ 0 & \text{otherwise.} \end{cases}$$

By Lehrer (1987, Corollary (5.5)$''$) $h_n^{n-1} = \mathbf{Sgn}_n \otimes \mathrm{Ind}_{c_n}^{S_n}(\zeta_n)$, where c_n is a cyclic group of order n and ζ_n is a faithful character on it, noted earlier by Stanley (1982).

Example 3.9 (Computing h_n^{n-2}) The z^2 coefficient of $N_\lambda(z)$ determines $h_n^{n-2}(\lambda)$. Proposition 3.3(2) tells us that $h_n^{n-2}(\lambda) = 0$ when $m_j(\lambda) > 0$ for at least three j. We treat the two remaining cases $\lambda = (i^{m_i} j^{m_j})$ and $\lambda = (j^m)$ in turn. If $\lambda = (i^{m_i} j^{m_j})$, then the z coefficient of $\binom{M_i(z)}{m_i}$ is $(-1)^{m_i-1}\frac{\mu(i)}{im_i}$, and similarly for $\binom{M_j(z)}{m_j}$. We have $z_\lambda = (i^{m_i} m_i!)(j^{m_j} m_j!)$. Thus, by Theorem 3.2

$$\begin{aligned} h_n^{n-2}\left((i^{m_1} j^{m_j})\right) &= (-1)^{m_i+m_j-n} z_\lambda \frac{\mu(i)\mu(j)}{(im_i)(jm_j)} \\ &= (-1)^{m_i+m_j-n}\left(\mu(i)i^{m_i-1}(m_i-1)!\right)\left(\mu(j)j^{m_j-1}(m_j-1)!\right). \end{aligned}$$

If $\lambda = (j^m)$, then the z^2 coefficient of $N_\lambda(z)$ receives a contribution of $(-1)^{m-1}\frac{\mu(j/2)}{jm}$ from the quadratic term of $M_j(z)$ if j is even. The z coefficient of $\binom{M_j(z)}{m_j}/M_j(z)$ is

$$\frac{\mu(j)}{jm!}\left(\sum_{i=1}^{m-1}\frac{(-1)^{m-2}(m-1)!}{i}\right) = (-1)^m\frac{\mu(j)}{jm}H_{m-1},$$

where $H_{m-1} = \sum_{i=1}^{m-1}\frac{1}{i}$ denotes the $(m-1)$*th harmonic number*. The z coefficient of $M_j(z)$ is $\frac{\mu(j)}{j}$. Using the convention that the Möbius function $\mu(\alpha)$ vanishes at non-integral α, we arrive at the following expression for $h_n^{n-2}(\lambda)$:

$$\begin{aligned} h_n^{n-2}\left((j^m)\right) &= z_\lambda(-1)^{m-n}\frac{\left(\mu(j)^2 H_{m-1} - \mu(\frac{j}{2})\right)}{jm} \\ &= (-1)^{m-n}\left(\mu(j)^2 H_{m-1} - \mu(\tfrac{j}{2})\right) j^{m-1}(m-1)!. \end{aligned}$$

4 Submodules A_n^k of Pure Braid Group Cohomology

Starting from Arnol'd's presentation for the S_n-algebra $H^\bullet(P_n, \mathbb{Q})$ we obtain a decomposition $H^k(P_n, \mathbb{Q}) = A_n^{k-1} \oplus A_n^k$ of S_n-modules. The characters of the sequence A_n^k of S_n-modules determine the splitting measure coefficients $\alpha_n^k(C_\lambda)$. In Sect. 4.3 we interpret $A_n^\bullet$ as the cohomology of $\mathrm{PConf}_n(\mathbb{C})/\mathbb{C}^\times$, where $\mathbb{C}^\times$ acts freely on $\mathrm{PConf}_n(\mathbb{C})$ by scaling coordinates.

4.1 Presentation of Pure Braid Group Cohomology Ring

Arnol'd (1969) gave the following presentation of the cohomology ring $H^\bullet(P_n, \mathbb{Q})$ of the pure braid group P_n as an S_n-algebra.

Theorem 4.1 (Arnol'd) *There is an isomorphism of graded S_n-algebras*

$$H^\bullet(P_n, \mathbb{Q}) \cong \Lambda^\bullet[\omega_{i,j}]/\langle R_{i,j,k}\rangle,$$

where $1 \le i, j, k \le n$ are distinct, $\omega_{i,j} = \omega_{j,i}$ have degree 1, and

$$R_{i,j,k} = \omega_{i,j} \wedge \omega_{j,k} + \omega_{j,k} \wedge \omega_{k,i} + \omega_{k,i} \wedge \omega_{i,j}.$$

An element $g \in S_n$ acts on $\omega_{i,j}$ by $g \cdot \omega_{i,j} = \omega_{g(i),g(j)}$.

In what follows, we identify $H^\bullet(P_n, \mathbb{Q})$ with this presentation as a quotient of an exterior algebra. The ring $\Lambda^\bullet[\omega_{i,j}]/\langle R_{i,j,k}\rangle$ is an example of an *Orlik–Solomon algebra*, which arise as cohomology rings of complements of hyperplane arrangements (see Orlik and Solomon 1980; Dimca and Yuzvinsky 2010; Yuzvinsky 2001).

4.2 S_n-Modules A_n^k Inside Braid Group Cohomology

Let $\tau = \sum_{1 \le i < j \le n} \omega_{i,j} \in H^1(P_n, \mathbb{Q})$. The element τ generates a trivial S_n-subrepresentation of $H^1(P_n, \mathbb{Q})$. We define maps $d^k : H^k(P_n, \mathbb{Q}) \to H^{k+1}(P_n, \mathbb{Q})$ for each k by $\nu \mapsto \nu \wedge \tau$. This map is linear and S_n-equivariant, since

$$g \cdot d^k(\nu) = g \cdot (\nu \wedge \tau) = (g \cdot \nu) \wedge (g \cdot \tau) = (g \cdot \nu) \wedge \tau = d^k(g \cdot \nu).$$

From $d^{k+1} \circ d^k = 0$ we conclude that

$$0 \to H^0(P_n, \mathbb{Q}) \xrightarrow{d^0} H^1(P_n, \mathbb{Q}) \xrightarrow{d^1} \cdots \xrightarrow{d^{n-1}} H^n(P_n, \mathbb{Q}) \xrightarrow{d^n} 0$$

is a chain complex of S_n-representations. It follows from the general theory of Orlik-Solomon algebras that the above sequence is exact (Dimca and Yuzvinsky 2010, Thm. 5.2). We include a proof in this case for completeness.

Proposition 4.2 *In the above notation,*

$$0 \to H^0(P_n, \mathbb{Q}) \xrightarrow{d^0} H^1(P_n, \mathbb{Q}) \xrightarrow{d^1} \cdots \xrightarrow{d^{n-1}} H^n(P_n, \mathbb{Q}) \xrightarrow{d^n} 0 \tag{4.1}$$

is an exact sequence of S_n-representations. Set $A_n^k := \mathrm{Im}(d^k) \subset H^{k+1}(P_n, \mathbb{Q})$. Hence we have an isomorphism of S_n-representations for each k,

$$H^k(P_n, \mathbb{Q}) \cong A_n^{k-1} \oplus A_n^k.$$

Proof Arnol'd (1969, Cor. 3) describes an additive basis $\mathcal{B}_k$ for $H^k(P_n, \mathbb{Q})$ comprised of all simple wedge products

$$\omega_{i_1,j_1} \wedge \cdots \wedge \omega_{i_k,j_k} \text{ such that } i_s < j_s \text{ for each } s, \text{ and } j_1 < j_2 < \cdots < j_k.$$

Let

$$U_k = \{\omega_{i_1,j_1} \wedge \cdots \wedge \omega_{i_k,j_k} \in \mathcal{B}_k : (i_s, j_s) \neq (n-1, n)\},$$

for $k > 0$ and $U_0 = \{1\}$. Then set

$$\mathcal{C}_k = U_k \cup \{\omega \wedge \tau : \omega \in U_{k-1}\}.$$

Claim. *$\mathcal{C}_k$ is a basis of $H^k(P_n, \mathbb{Q})$.*

For example, we have

$$\mathcal{C}_1 = \{\omega_{i,j} : (i, j) \neq (n-1, n)\} \cup \{\tau\},$$

which is clearly a basis for $H^1(P_n, \mathbb{Q})$.

To prove the claim, since $|\mathcal{B}_k| = |\mathcal{C}_k|$, it suffices to show $\mathcal{C}_k$ spans $H^k(P_n, \mathbb{Q})$. Note that

$$\mathcal{B}_k = U_k \cup \{\omega \wedge \omega_{n-1,n} : \omega \in U_{k-1}\},$$

further reducing the problem to expressing $\omega \wedge \omega_{n-1,n}$ as a linear combination of $\mathcal{C}_k$ for each $\omega \in U_{k-1}$. Given $\omega = \omega_{i_1,j_1} \wedge \cdots \wedge \omega_{i_{k-1},j_{k-1}} \in U_{k-1}$, we use the relation

$$\omega_{i_s,j} \wedge \omega_{i,j} = \omega_{i_s,i} \wedge \omega_{i,j} - \omega_{i_s,i} \wedge \omega_{i_s,j}$$

to express $\omega \wedge \omega_{i,j}$ in terms of elements of U_k as follows:

$$\omega \wedge \omega_{i,j} = \begin{cases} \pm\omega_{i_1,j_1} \wedge \cdots \wedge \omega_{i_s,j_s} \wedge \omega_{i,j} \wedge \omega_{i_{s+1},j_{s+1}} \wedge \cdots \wedge \omega_{i_{k-1},j_{k-1}} \\ \qquad\qquad\qquad \text{for} \quad j_s < j < j_{s+1}, \\ \pm\omega_{i_1,j_1} \wedge \cdots \wedge (\omega_{i_s,i} \wedge \omega_{i,j} - \omega_{i_s,i} \wedge \omega_{i_s,j}) \wedge \cdots \wedge \omega_{i_{k-1},j_{k-1}} \\ \qquad\qquad\qquad \text{for} \quad j_s = j, i_s \neq i, \\ 0 \qquad\qquad\quad\;\, \text{for} \quad (i_s, j_s) = (i, j). \end{cases}$$

The first and third cases are easily seen to belong in the span of U_k. Since $i_s, i < j$ and j does not occur twice as a largest subscript in ω, we see inductively that the second case also belongs in the span of U_k. Therefore, $\omega \wedge \tau = \omega \wedge \omega_{n-1,n} + \nu$, where ν is in the span of U_k. Hence $\omega \wedge \omega_{n-1,n} = \omega \wedge \tau - \nu$ is in the span of $\mathcal{C}_k$ and we conclude that $\mathcal{C}_k$ is a basis, proving the claim.

We now show the sequence (4.1) is exact. Suppose $\nu \in \ker(d^k)$. Express ν in the basis $\mathcal{C}_k$ as

$$\nu = \sum_{\omega \in U_k} a_\omega \, \omega + \sum_{\omega \in U_{k-1}} b_\omega \, \omega \wedge \tau.$$

Then

$$0 = d^k(\nu) = \nu \wedge \tau = \sum_{\omega \in U_k} a_\omega \, \omega \wedge \tau.$$

Since $\omega \wedge \tau$ is an element of the basis $\mathcal{C}_{k+1}$ for each $\omega \in U_k$, we have $a_\omega = 0$. Hence, $\nu = \mu \wedge \tau = d^{k-1}(\mu)$ where

$$\mu = \sum_{\omega \in U_{k-1}} b_\omega \, \omega,$$

so $\ker(d^k) = \mathrm{Im}(d^{k-1})$. $\square$

Recall from Sect. 3.5 that the dimension of $H^k(P_n, \mathbb{Q})$ is given by an unsigned Stirling number of the first kind

$$\dim\left(H^k(P_n, \mathbb{Q})\right) = \begin{bmatrix} n \\ n-k \end{bmatrix},$$

where the unsigned Stirling numbers are determined by the identity $\prod_{k=0}^{n-1}(x+k) = \sum_{k=0}^{n-1} \begin{bmatrix} n \\ k \end{bmatrix} x^k$. The exact sequence in Proposition 4.2 shows the dimension of A_n^k is

$$\dim(A_n^k) = \sum_{j=0}^{k} (-1)^j \begin{bmatrix} n \\ n-k+j \end{bmatrix}.$$

Table 4 gives values of $\dim(A_n^k)$ for small n and k; here $\dim(A_n^{n-1}) = 0$ for $n \geq 2$.

4.3 A_n^k as Cohomology of a Complex Manifold with an S_n-Action

Recall from Sect. 3.2 that the pure configuration space $\mathrm{PConf}_n(\mathbb{C})$ is defined by

$$\mathrm{PConf}_n(\mathbb{C}) = \{(z_1, z_2, \ldots, z_n) \in \mathbb{C}^n : z_i \neq z_j \text{ when } i \neq j\}.$$

Table 4 $\dim(A_n^k)$

$n\backslash k$	0	1	2	3	4	5	6	7
1	1	0	0	0	0	0	0	0
2	1	0	0	0	0	0	0	0
3	1	2	0	0	0	0	0	0
4	1	5	6	0	0	0	0	0
5	1	9	26	24	0	0	0	0
6	1	14	71	154	120	0	0	0
7	1	20	155	580	1044	720	0	0
8	1	27	295	1665	5104	8028	5040	0
9	1	35	511	4025	18,424	48,860	69,264	40,320

It is an open complex manifold, and the symmetric group S_n acts on $\mathrm{PConf}_n(\mathbb{C})$ by permuting coordinates. There is also a free action of $\mathbb{C}^\times$ on $\mathrm{PConf}_n(\mathbb{C})$ defined by

$$c \cdot (z_1, z_2, \ldots, z_n) = (cz_1, cz_2, \ldots, cz_n).$$

This action commutes with the S_n-action, hence induces an action of S_n on the quotient complex manifold $\mathrm{PConf}_n(\mathbb{C})/\mathbb{C}^\times$. Therefore $H^\bullet(\mathrm{PConf}_n(\mathbb{C})/\mathbb{C}^\times, \mathbb{Q})$ is an S_n-algebra. We now relate the graded components $H^k(\mathrm{PConf}_n(\mathbb{C})/\mathbb{C}^\times, \mathbb{Q})$ to the S_n-submodules A_n^k of $H^k(\mathrm{PConf}_n(\mathbb{C}), \mathbb{Q}) = H^k(P_n, \mathbb{Q})$ constructed in Proposition 4.2.

Theorem 4.3 *Let* $\mathrm{PConf}_n(\mathbb{C})/\mathbb{C}^\times$ *be the quotient of pure configuration space by the free* $\mathbb{C}^\times$ *action. The symmetric group* S_n *acts on* $\mathrm{PConf}_n(\mathbb{C})/\mathbb{C}^\times$ *by permuting coordinates. Let* $A_n^\bullet$ *be the sequence of* S_n*-modules constructed in Proposition* 4.2*. Then for each* $k \geq 0$ *we have an isomorphism of* S_n*-modules*

$$H^k(\mathrm{PConf}_n(\mathbb{C})/\mathbb{C}^\times, \mathbb{Q}) \cong A_n^k.$$

Proof We regard $X_n := \mathrm{PConf}_n(\mathbb{C})$ as the total space of a $\mathbb{C}^\times$-bundle over the base space $Y_n := \mathrm{PConf}_n(\mathbb{C})/\mathbb{C}^\times$. As noted in Sect. 3.2 the cohomology of X_n is that of the pure braid group, with its S_n-action. Viewing $\mathbb{C}^\times$ as $\mathbb{R}^+ \times S^1$, we see that X_n is an $\mathbb{R}^+$-bundle over the base space $Z_n := \mathrm{PConf}_n(\mathbb{C})/\mathbb{R}^+$, such that Z_n is an S^1-bundle over Y_n. The space Z_n is a real-analytic manifold which inherits the S_n-action. For any $(z_1, z_2, \ldots, z_n) \in \mathrm{PConf}_n(\mathbb{C})$, let $[[z_1, z_2, \ldots, z_n]]$ denote its image in Z_n Since $z_1 \neq z_2$, we may rescale this vector by $c = \frac{1}{|z_1 - z_2|} \in \mathbb{C}^\times$ to get $(\tilde{z}_1, \tilde{z}_2, \ldots, \tilde{z}_n) = \frac{1}{|z_1 - z_2|}(z_1, \ldots, z_n)$, which comprise exactly the set of all $(\tilde{z}_1, \tilde{z}_2, \ldots, \tilde{z}_n) \in X_n$ satisfying the linear constraint $\tilde{z}_1 - \tilde{z}_2 \in U(1) = \{z \in \mathbb{C} : |z| = 1\}$. We obtain a global section $Z_n \to X_n$ by mapping $[[z_1, z_2, \ldots, z_n]] \mapsto \frac{1}{|z_1 - z_2|}(z_1, \ldots, z_n)$, so may regard $Z_n \subset X_n$, noting that it is invariant under the S_n-action. Under this embedding we see that Z_n is a strong deformation retract of X_n, so has the same homotopy type as X_n.

The retraction map is:

$$h_t(z_1, z_2, \ldots, z_n) := ((1-t)|z_1 - z_2| + t)\,\frac{1}{|z_1 - z_2|}(z_1, z_2, \ldots, z_n) \quad \text{for } 0 \le t \le 1.$$

Consequently $H^k(X_n, \mathbb{Q}) \cong H^k(Z_n, \mathbb{Q})$, for each $k \geq 0$ as S_n-modules.

For any $(z_1, z_2, \ldots, z_n) \in X_n$, let $[z_1, z_2, \ldots, z_n]$ denote its image in Y_n. Since $z_1 \neq z_2$, we may rescale this vector by $\frac{1}{z_1 - z_2} \in \mathbb{C}^\times$ to get $(\tilde{z}_1, \tilde{z}_2, \ldots, \tilde{z}_n) = \frac{1}{z_1 - z_2}(z_1, \ldots, z_n)$, which comprise exactly the set of all $(\tilde{z}_1, \tilde{z}_2, \ldots, \tilde{z}_n) \in X_n$ satisfying the linear constraint $\tilde{z}_1 - \tilde{z}_2 = 1$. These define a global coordinate system for Y_n, identifying it as an open complex manifold, and the map $Y_n \to X_n$ sending $[z_1, z_2 \ldots, z_n] \mapsto (\tilde{z}_1, \tilde{z}_2, \ldots, \tilde{z}_n)$ is a nowhere vanishing global section of this bundle, so we may view $Y_n \subset Z_n \subset X_n$. This map is a nowhere vanishing section of Y_n inside the S^1-bundle Z_n as well.

The Gysin long exact sequence for Z_n as an S^1-bundle over Y_n is

$$\xrightarrow{e_\wedge} H^k(Y_n, \mathbb{Q}) \to H^k(Z_n, \mathbb{Q}) \to H^{k-1}(Y_n, \mathbb{Q}) \xrightarrow{e_\wedge} H^{k+1}(Y_n, \mathbb{Q}) \to H^{k+1}(Z_n, \mathbb{Q}) \to$$

The Euler class $e \in H^2(Y_n, \mathbb{Q})$ of this is zero since the bundle has a nowhere vanishing global section in Z_n. Thus $e_\wedge$ is the zero map and the Gysin sequence splits into short exact sequences

$$0 \longrightarrow H^k(Y_n, \mathbb{Q}) \longrightarrow H^k(Z_n, \mathbb{Q}) \longrightarrow H^{k-1}(Y_n, \mathbb{Q}) \longrightarrow 0.$$

The maps are S_n-equivariant, since the Gysin sequence is functorial. It follows from Maschke's theorem that

$$H^k(X_n, \mathbb{Q}) \cong H^k(Z_n, \mathbb{Q}) \cong H^{k-1}(Y_n, \mathbb{Q}) \oplus H^k(Y_n, \mathbb{Q}) \tag{4.2}$$

as S_n-modules. Since $H^{-1}(Y_n, \mathbb{Q}) = A_n^{-1} = 0$ by convention, we have $H^0(Y_n, \mathbb{Q}) \cong A_n^0 \cong H^0(Z_n, \mathbb{Q}) \cong H^0(X_n, \mathbb{Q})$. It then follows inductively from (4.2) and

$$H^k(X_n, \mathbb{Q}) \cong A_n^{k-1} \oplus A_n^k,$$

that $H^k(Y_n, \mathbb{Q}) \cong A_n^k$ as S_n-modules for all $k \geq 0$. □

Remark The configuration space $\mathrm{PConf}(\mathbb{C})$ is a hyperplane complement as treated in the book of Orlik and Terao (1992). It equals

$$M(\mathcal{A}_n) := \mathbb{C}^n \setminus \bigcup_{H_{i,j} \in \mathcal{A}_n} H_{i,j},$$

where $\mathcal{A}_n := \{H_{i,j} : 1 \leq i < j \leq n\}$ denotes the *braid arrangement* of hyperplanes $H_{i,j} : z_i = z_j$ for $1 \leq i < j \leq n$.

5 Polynomial Splitting Measures and Characters

We now express the splitting measure coefficients $\alpha_n^k(C_\lambda)$ in terms of the character values $\chi_n^k(\lambda)$ where χ_n^k is the character of the S_n-representation A_n^k constructed in Proposition 4.2. As a corollary we deduce that the rescaled z-splitting measures are characters when $z = -\frac{1}{m}$ and virtual characters when $z = \frac{1}{m}$, generalizing results from Lagarias (2016).

5.1 Expressing Splitting Measure Coefficients by Characters

Recall,

$$\nu_{n,z}^*(C_\lambda) = \frac{N_\lambda(z)}{z^n - z^{n-1}} = \sum_{k=0}^{n-1} \alpha_n^k(C_\lambda)\left(\frac{1}{z}\right)^k.$$

We now express the splitting measure coefficient $\alpha_n^k(C_\lambda)$ in terms of the character value $\chi_n^k(\lambda)$.

Theorem 5.1 *Let $n \geq 2$ and λ be a partition of n, then*

$$\nu_{n,z}^*(C_\lambda) = \frac{1}{z_\lambda}\sum_{k=0}^{n-1} (-1)^k \chi_n^k(\lambda)\left(\frac{1}{z}\right)^k,$$

where χ_n^k is the character of the S_n-representation A_n^k defined in Proposition 4.2. Thus,

$$\alpha_n^k(C_\lambda) = \frac{1}{z_\lambda}(-1)^k \chi_n^k(\lambda).$$

Proof In Theorem 3.2 we showed

$$N_\lambda(z) = \frac{1}{z_\lambda}\sum_{k=0}^{n} (-1)^k h_n^k(\lambda) z^{n-k},$$

where h_n^k is the character of $H^k(P_n, \mathbb{Q})$. The S_n-representations A_n^k were defined in Proposition 4.2 where we showed that

$$H^k(P_n, \mathbb{Q}) \cong A_n^{k-1} \oplus A_n^k. \tag{5.1}$$

Taking characters in (5.1) gives

$$h_n^k = \chi_n^{k-1} + \chi_n^k.$$

We compute

$$\begin{aligned}\frac{N_\lambda(z)}{z^n} &= \frac{1}{z_\lambda}\sum_{k=0}^{n}(-1)^k h_n^k(\lambda)\left(\frac{1}{z}\right)^k \\ &= \frac{1}{z_\lambda}\sum_{k=0}^{n}(-1)^k\left(\chi_n^{k-1}(\lambda)+\chi_n^k(\lambda)\right)\left(\frac{1}{z}\right)^k \\ &= \left(1-\frac{1}{z}\right)\frac{1}{z_\lambda}\sum_{k=0}^{n-1}(-1)^k\chi_n^k(\lambda)\left(\frac{1}{z}\right)^k.\end{aligned}$$

Dividing both sides by $\left(1-\frac{1}{z}\right)$ yields

$$\nu_{n,z}^*(C_\lambda) = \frac{N_\lambda(z)}{\left(1-\frac{1}{z}\right)z^n} = \frac{1}{z_\lambda}\sum_{k=0}^{n-1}(-1)^k\chi_n^k(\lambda)\left(\frac{1}{z}\right)^k.$$

Comparing coefficients in the two expressions for $\nu_{n,z}^*(C_\lambda)$ we find

$$\alpha_n^k(C_\lambda) = \frac{1}{z_\lambda}(-1)^k\chi_n^k(\lambda).$$

□

5.2 Cycle Polynomial and Splitting Measure as Equivariant Poincaré Polynomials

Given a complex manifold X, the *Poincaré polynomial of* X is defined by

$$P(X,t) = \sum_{k\geq 0}\dim H^k(X,\mathbb{Q})t^k.$$

If a finite group G acts on X, then the cohomology $H^k(X,\mathbb{Q})$ is a $\mathbb{Q}$-representation of G with character h_X^k, and the *equivariant Poincaré polynomial of* X *at* $g\in G$ is defined by

$$P_g(X,t) = \sum_{k\geq 0}\mathrm{Trace}(g, H^k(X,\mathbb{Q})t^k = \sum_{k\geq 0}h_X^k(g)t^k.$$

Note that if $g=1$ is the identity of G, then $h_X^k(1)=\dim H^k(X,\mathbb{Q})$ and $P_1(X,t)=P(X,t)$.

Under the change of variables $z=-\frac{1}{t}$, the work of Lehrer (1987, Theorem 5.5) identifies (rescaled) cycle polynomials with equivariant Poincaré polynomials of

$\mathrm{PConf}_n(\mathbb{C})$, for $g \in S_n$, as

$$\frac{1}{z^n} N_{[g]}(z) = \frac{|C_\lambda|}{n!} \sum_{k \geq 0} h_n^k(g) t^k = \frac{1}{z_\lambda} P_g(\mathrm{PConf}_n(\mathbb{C}), t)$$

Using the result of Sect. 4.3 we obtain a similar interpretation of the splitting measure values.

Theorem 5.2 *Let $Y_n = \mathrm{PConf}_n(\mathbb{C})/\mathbb{C}^\times$. Setting $t = -\frac{1}{z}$, for each $g \in S_n$ the z-splitting measure is given by the scaled equivariant Poincaré polynomial*

$$\nu_{n,z}^*(g) = \frac{1}{n!} \sum_{k=0}^{n-1} \mathrm{Trace}(g : H^k(Y_n, \mathbb{Q})) t^k,$$

attached to the complex manifold Y_n, where g acts as a permutation of the coordinates.

Proof This formula follows from Theorem 5.1, using also the identification of $A_n^k = H^k(Y_n, \mathbb{Q})$ as an S_n-module in Theorem 4.3. Since we evaluate the character on a single element $g \in S_n$, the prefactor becomes $\frac{1}{z_\lambda c_\lambda} = \frac{1}{n!}$. □

Remark In the theory of hyperplane arrangements treated in Orlik and Terao (1992) the change of variable $z = -\frac{1}{t}$ appears as an involution converting the Poincaré polynomial of a hyperplane complement (such as $\mathrm{PConf}_n(\mathbb{C})$) to another invariant, the *characteristic polynomial* of an arrangement, given in Orlik and Terao (1992, Defn. 2.52).

5.3 Splitting Measures for $z = \pm\frac{1}{m}$

Representation-theoretic interpretations of the rescaled z-splitting measures for $z = \pm 1$ were studied in Lagarias (2016, Sec. 5). Theorem 5.3 below generalizes those results to give representation-theoretic interpretations for $z = \pm\frac{1}{m}$ when $m \geq 1$ is an integer.

Theorem 5.3 *Let $n \geq 2$ and λ be a partition of n, then*

1. *For $z = -\frac{1}{m}$ with $m \geq 1$ an integer, we have*

$$\nu_{n,-\frac{1}{m}}^*(C_\lambda) = \frac{1}{z_\lambda} \sum_{k=0}^{n-1} \chi_n^k(\lambda) m^k.$$

The function $z_\lambda \nu_{n,-\frac{1}{m}}^*(C_\lambda)$ is therefore the character of the S_n-representation

$$B_{n,m} = \bigoplus_{k=0}^{n-1} \left(A_n^k\right)^{\oplus m^k},$$

with dimension

$$\dim B_{n,m} = \prod_{j=2}^{n-1} (1 + jm).$$

2. *For* $z = \frac{1}{m}$ *with* $m \geq 1$ *an integer, we have*

$$\nu^*_{n,\frac{1}{m}}(C_\lambda) = \frac{1}{z_\lambda} \sum_{k=0}^{n-1} (-1)^k \chi_n^k(\lambda) m^k.$$

The function $z_\lambda \nu^*_{n,\frac{1}{m}}(C_\lambda)$ *is a virtual character, the difference of characters of representations* $B^+_{n,m}$ *and* $B^-_{n,m}$,

$$B^+_{n,m} \cong \bigoplus_{2j<n} \left(A_n^{2j}\right)^{\oplus m^{2j}} \qquad B^-_{n,m} \cong \bigoplus_{2j+1<n} \left(A_n^{2j+1}\right)^{\oplus m^{2j+1}}.$$

These representations have dimensions

$$\dim B^\pm_{n,m} = \frac{1}{2}\left(\prod_{j=2}^{n-1}(1 + jm) \pm \prod_{j=2}^{n-1}(1 - jm)\right)$$

respectively.

Proof (1) The formula for the $(-\frac{1}{m})$-splitting measure follows by substituting $z = -\frac{1}{m}$ in Theorem 5.1. Arnol'd (1969, Cor. 2) shows the Poincaré polynomial $p(t)$ of the pure braid group P_n has the product form

$$p(t) = \prod_{j=1}^{n-1} (1 + jt) = \sum_{k=0}^{n} h_n^k \left((1^n)\right) t^k.$$

On the other hand, by Theorem 3.2 we have

$$n!(-1)^n t^n N_{(1^n)}(-t^{-1}) = \sum_{k=0}^{n} h_n^k \left((1^n)\right) t^k. \tag{5.2}$$

Dividing (5.2) by $1 + t$ we have

$$\prod_{j=2}^{n-1} (1 + jt) = n!(-1)^n t^n \frac{N_{(1^n)}(-t^{-1})}{1 + t} = \sum_{k=0}^{n-1} \chi_n^k \left((1^n)\right) t^k. \tag{5.3}$$

Substituting $t = m$ gives the dimension formula.

(2) Substituting $z = \frac{1}{m}$ in Theorem 5.1 gives the formula for the $\frac{1}{m}$-splitting measure. Separating the even and odd parts we have

$$z_\lambda \nu^*_{n,\frac{1}{m}}(C_\lambda) = \sum_{2j<n} \chi_n^{2j}(\lambda) m^{2j} - \sum_{2j+1<n} \chi_n^{2j+1}(\lambda) m^{2j+1}.$$

Hence $z_\lambda \nu^*_{n,\frac{1}{m}}(C_\lambda) = \chi^+_{n,m}(\lambda) - \chi^-_{n,m}(\lambda)$, where $\chi^\pm_{n,m}$ are characters of $B^\pm_{n,m}$ respectively. The dimension formulas follow from decomposing (5.3) into even and odd parts.

□

Remark Other results in Lagarias (2016, Theorems 3.2, 5.2 and 6.1) determine the values of the rescaled splitting measures for $z = \pm 1$, showing they are supported on remarkably few conjugacy classes; for $z = 1$ these were the Springer regular elements of S_n. Theorem 5.3 does not account for the small support of the characters for $z = \pm 1$. The characters h_n^k and χ_n^k have large support in general, hence cancellation must occur to explain the small support. It would be interesting to account for this phenomenon.

5.4 Cohomology of the Pure Braid Group and the Regular Representation

We use Theorem 5.1 together with the splitting measure values at $z = -1$ computed in Lagarias (2016) to determine a relation between the S_n-representation structure of the pure braid group cohomology and the regular representation of S_n. Let A_n^k be the S_n-subrepresentation constructed in Proposition 4.2, and define the S_n-representation

$$A_n := \bigoplus_{k=0}^{n-1} A_n^k.$$

Theorem 5.4 *Let $\mathbf{1}_n$, $\mathbf{Sgn}_n$, and $\mathbb{Q}[S_n]$ denote the trivial, sign, and regular representations of S_n respectively. Then there are isomorphisms of S_n-representations,*

$$\bigoplus_{k=0}^{n} H^k(P_n, \mathbb{Q}) \otimes \mathbf{Sgn}_n^{\otimes k} \cong \mathbb{Q}[S_n].$$

and

$$A_n \otimes \left(\mathbf{1}_n \oplus \mathbf{Sgn}_n\right) \cong \mathbb{Q}[S_n].$$

Proof We showed in Proposition 4.2 that $H^k(P_n, \mathbb{Q}) \cong A_n^{k-1} \oplus A_n^k$, with $A_n^{-1} = A_n^n = 0$. Therefore, summing over $0 \le k \le n$,

$$A_n \cong \bigoplus_{k \text{ even}} H^k(P_n, \mathbb{Q}) \cong \bigoplus_{k \text{ odd}} H^k(P_n, \mathbb{Q}).$$

Since $\mathbf{Sgn}_n^{\otimes 2} \cong \mathbf{1}_n$, we have

$$\begin{aligned}\bigoplus_{k=0}^{n} H^k(P_n, \mathbb{Q}) \otimes \mathbf{Sgn}_n^{\otimes k} &\cong \left(\bigoplus_{k \text{ even}} H^k(P_n, \mathbb{Q}) \otimes \mathbf{1}_n\right) \oplus \left(\bigoplus_{k \text{ odd}} H^k(P_n, \mathbb{Q}) \otimes \mathbf{Sgn}_n\right) \\ &\cong (A_n \otimes \mathbf{1}_n) \oplus (A_n \otimes \mathbf{Sgn}_n) \\ &\cong A_n \otimes (\mathbf{1}_n \oplus \mathbf{Sgn}_n).\end{aligned}$$

If χ_n is the character of A_n, then it follows from Theorem 1.4 that

$$\chi_n(\lambda) = \sum_{k=0}^{n-1} \chi_n^k(\lambda) = z_\lambda \nu_{n,-1}^*(C_\lambda),$$

so the values of χ_n are given by the rescaled (-1)-splitting measure.

Theorem 6.1 of Lagarias (2016) shows

$$\nu_{n,-1}^*(C_\lambda) = \begin{cases} \frac{1}{2} & \lambda = (1^n) \text{ or } (1^{n-2}\,2), \\ 0 & \text{otherwise.} \end{cases}$$

Now let $\rho = \chi_n \cdot (1_n + \mathrm{sgn}_n)$ be the character of $A_n \otimes (\mathbf{1}_n \oplus \mathbf{Sgn}_n)$. If $\lambda = (1^n)$, we compute

$$\rho(\lambda) = \chi_n(\lambda)\left(1 + \mathrm{sgn}_n(\lambda)\right) = n!\nu_{n,-1}^*(C_\lambda)(2) = n!.$$

If $\lambda = (1^{n-2}\,2)$, then $\left(1 + \mathrm{sgn}_n(\lambda)\right) = 0$, hence $\rho(\lambda) = 0$. If λ is any other partition, then $\nu_{n,-1}^*(C_\lambda) = 0$, hence $\rho(\lambda) = 0$. Therefore ρ agrees with the character of the regular representation, proving

$$\bigoplus_{k=0}^{n} H^k(P_n, \mathbb{Q}) \otimes \mathbf{Sgn}_n^{\otimes k} \cong A_n \otimes (\mathbf{1}_n \oplus \mathbf{Sgn}_n) \cong \mathbb{Q}[S_n].$$

□

6 Other Interpretations of A_n^k

Theorem 4.3 interprets the S_n-representation A_n^k geometrically as

$$A_n^k \cong H^k(\mathrm{PConf}_n(\mathbb{C})/\mathbb{C}^\times, \mathbb{Q}).$$

In this section we note two other interpretations of A_n^k, coming from combinatorial constructions previously studied in the literature. These interpretations imply that the A_n^k for fixed k exhibit representation stability in the sense of Church and Farb (2013) as $n \to \infty$.

Proposition 4.2 gave the following direct sum decomposition of the pure braid group cohomology,

$$H^k(P_n, \mathbb{Q}) \cong A_n^{k-1} \oplus A_n^k. \tag{6.1}$$

The isomorphisms (6.1) uniquely determine the A_n^k as S_n-representations up to isomorphism. Uniqueness holds since finite-dimensional representations are semisimple by Maschke's theorem, using the general result that if $0 = C^0, C^1, C^2, \ldots$ is any sequence of semisimple modules with submodules $B^k \subseteq C^k$, then isomorphisms

$$C^k \cong B^{k-1} \oplus B^k$$

for each k determine the B^k up to isomorphism.

Let Π_n denote the collection of partitions of a set with n elements, partially ordered by refinement (see Stanley 1986, Example 3.10.4)).

Hersh and Reiner (2015, Sec. 2) describe two other sequences of S_n-representations giving direct sum decompositions of $H^k(P_n, \mathbb{Q})$ coming from the Whitney and simplicial homology of the lattice Π_n.

Proposition 6.1 1. *There is an isomorphism of S_n-representations*

$$H^k(P_n, \mathbb{Q}) \cong WH_k(\Pi_n), \tag{6.2}$$

where $WH_k(\Pi_n)$ is the kth Whitney homology of the lattice Π_n.

2. *There is an isomorphism of S_n-representations*

$$WH_k(\Pi_n) \cong \beta_{[k-1]}(\Pi_n) \oplus \beta_{[k]}(\Pi_n)$$

where $\beta_{[k]}(\Pi_n)$ is the $[k] = \{1, 2, \ldots, k\}$-rank selected homology of the lattice Π_n.

3. *There is an isomorphism of S_n-representations*

$$\beta_{[k]}(\Pi_n) \cong \widetilde{H}_{k-1}\left(\Pi_n^k\right),$$

where Π_n^k is the sub-poset of $\lambda \in \Pi_n$ with $|\lambda| - \ell(\lambda) \leq k$ and $\widetilde{H}_{k-1}\left(\Pi_n^k\right)$ denotes its reduced simplicial homology.

Proof (1) This result is due to Sundaram and Welker (1997, Theorem 4.4 (iii)), cf. Hersh and Reiner (2015, Thm. 2.11, Sec. 2.3). (See Hersh and Reiner 2015, Sec. 2.4 for more on the Whitney homology of Π_n.)

(2) Sundaram (1994, Prop. 1.9) decomposes $WH_k(\Pi_n)$ as

$$WH_k(\Pi_n) \cong \beta_{[k-1]}(\Pi_n) \oplus \beta_{[k]}(\Pi_n), \tag{6.3}$$

where $[k] = \{1, 2, \ldots, k\}$ and $\beta_{[k]}(\Pi_n)$ is the $[k]$-*rank selected homology* of the lattice Π_n (Hersh and Reiner 2015, Prop. 2.17).

(3) Because the lattice Π_n is *Cohen–Macaulay*, (Hersh and Reiner 2015, Sec. 2.5) note the isomorphism

$$\beta_{[k]}(\Pi_n) \cong \widetilde{H}_{k-1}\big(\Pi_n^k\big), \tag{6.4}$$

where Π_n^k is the sub-poset of $\lambda \in \Pi_n$ with $|\lambda| - \ell(\lambda) \leq k$ and $\widetilde{H}_{k-1}\left(\Pi_n^k\right)$ is its *reduced simplicial homology*. □

The following proposition relates A_n^k, $\beta_{[k]}(\Pi_n)$, and $\widetilde{H}_{k-1}\left(\Pi_n^k\right)$ using (6.1).

Proposition 6.2 *Let Π_n be the lattice of partitions of an n-element set, and $\Pi_n^k \subseteq \Pi_n$ the sub-poset comprised of $\lambda \in \Pi_n$ with $|\lambda| - \ell(\lambda) \leq k$. Then we have the following isomorphisms of S_n-representations*

$$A_n^k \cong \beta_{[k]}(\Pi_n) \cong \widetilde{H}_{k-1}\big(\Pi_n^k\big).$$

Proof The isomorphisms (6.2) and (6.3) in Proposition 6.1 give the direct sum decompositions

$$H^k(P_n, \mathbb{Q}) \cong \beta_{[k-1]}(\Pi_n) \oplus \beta_{[k]}(\Pi_n)$$

for $0 \leq k \leq n$. By (6.1) we have that

$$H^k(P_n, \mathbb{Q}) \cong A_n^{k-1} \oplus A_n^k.$$

Since for $k = 0$,

$$\beta_{[-1]}(\Pi_n) \cong A_n^{-1} = \{0\},$$

we obtain by induction on $k \geq 1$ that

$$A_n^k \cong \beta_{[k]}(\Pi_n)$$

Combining this isomorphism with (6.4) finishes the proof. □

We deduce the representation stability of the characters χ_n^k from known results.

Proof of Theorem 1.5 The S_n-representations of the rank-selected homology $\beta_{[k-1]}(\Pi_n)$ were shown by Hersh and Reiner (2015, Corollary 5.4) to exhibit representation-stability for fixed k and varying n and to stabilize sharply at $n = 3k+1$. This fact combined with Proposition 6.2 proves Theorem 1.5. □

The following tables for A_n^1 and A_n^2 exhibit representation stability and the sharp stability phenomenon at $n = 3k + 1$. We give irreducible decompositions, with multiplicities, of $H^k(P_n, \mathbb{Q})$ and A_n^1 in Table 5 and for A_n^2 in Table 6. To read the tables, for example, the entry [4, 1, 1] denotes the isomorphism class of the irreducible rep-

Table 5 Irreducible S_n-module decompositions for $H^1(P_n, \mathbb{Q})$ and A_n^1

n	dim H^1	$H^1(P_n, \mathbb{Q})$	dim A_n^1	A_n^1
2	1	[2]	0	0
3	3	$[3] \oplus [2, 1]$	2	$[2, 1]$
4	6	$[4] \oplus [3, 1] \oplus [2, 2]$	5	$[3, 1] \oplus [2, 2]$
5	10	$[5] \oplus [4, 1] \oplus [3, 2]$	9	$[4, 1] \oplus [3, 2]$
$n \geq 4$	$\left[{n \atop n-1}\right]$	$[n] \oplus [n-1, 1] \oplus [n-2, 2]$	$\left[{n \atop n-1}\right] - 1$	$[n-1, 1] \oplus [n-2, 2]$

Here λ abbreviates the irreducible representation $\mathcal{S}^\lambda$

Table 6 Irreducible S_n-module decomposition for A_n^2

n	dim A_n^2	A_n^2
3	0	0
4	6	$[3, 1] \oplus [2, 1, 1]$
5	26	$[4, 1] \oplus [3, 2] \oplus 2[3, 1, 1] \oplus [2, 2, 1]$
6	71	$[5, 1] \oplus [4, 2] \oplus 2[4, 1, 1] \oplus [3, 3] \oplus 2[3, 2, 1]$
7	155	$[6, 1] \oplus [5, 2] \oplus 2[5, 1, 1] \oplus [4, 3] \oplus 2[4, 2, 1] \oplus [3, 3, 1]$
8	295	$[7, 1] \oplus [6, 2] \oplus 2[6, 1, 1] \oplus [5, 3] \oplus 2[5, 2, 1] \oplus [4, 3, 1]$
$n \geq 7$	$\left[{n \atop n-2}\right] - \left[{n \atop n-1}\right] + 1$	$[n-1, 1] \oplus [n-2, 2] \oplus 2[n-2, 1, 1] \oplus [n-3, 3]$ $\oplus 2[n-3, 2, 1] \oplus [n-4, 3, 1]$

resentation of S_6 associated to the Specht module of the partition [4, 1, 1] of $n = 6$, in the notation of Sagan (2013, Sec. 2.3), who gives a construction of the Specht module representatives of the irreducible isomorphism classes.

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