A Wallis Product on Clovers

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Abstract. The m-clover is the plane curve defined by the polar equation $r^{m/2} = \cos\left(\frac{m}{2}\theta\right)$. In this article we extend a well-known derivation of the Wallis product to derive a generalized Wallis product for arc lengths of m-clovers.

1. INTRODUCTION. The Wallis product,

$$\pi = 4 \cdot \frac{2 \cdot 4}{3 \cdot 3} \cdot \frac{4 \cdot 6}{5 \cdot 5} \cdot \frac{6 \cdot 8}{7 \cdot 7} \cdot \frac{8 \cdot 10}{9 \cdot 9} \cdots$$
$$= 4 \prod_{n=1}^{\infty} \frac{2n(2n+2)}{(2n+1)^2}, \tag{1.1}$$

is a well-known infinite product expression for π by rational factors and was derived by John Wallis in his 1655 treatise [**10**]. One proof of (1.1) considers the sequence of definite integrals

$$I(n) = \int_0^{\pi} \sin(x)^n \, dx,$$

and computes the limit of $I(2k-1)/I(2k)$ in two ways. See [**2**] for a nice exposition of this argument. For a positive integer m, let $\{\varpi_m\}$ be the sequence of real numbers defined by

$$\varpi_m = 2 \int_0^1 \frac{dt}{\sqrt{1 - t^m}}. \tag{1.2}$$

The goal of this paper is to prove the generalized Wallis product formula

$$\varpi_m = \frac{2(m+2)}{m} \prod_{n=1}^{\infty} \frac{n\,(2(mn+1)+m)}{(mn+1)(2n+1)}. \tag{1.3}$$

In Section 2, we recall the theory of *clover curves* introduced in [**6**] and realize ϖ_m as an arc length on the *m-clover*. We define the *m-clover function* $\varphi_m(x)$ and develop its properties. Section 3 considers the sequence of definite integrals

$$I_m(n) = \int_0^{\varpi_m} \varphi_m(x)^n \, dx.$$

To complete the derivation, we compute the limit of $I_m(mk-1)/I_m(mk)$ in two ways. Section 4 relates our generalized Wallis product to the work of Euler and the product formula for the beta function.

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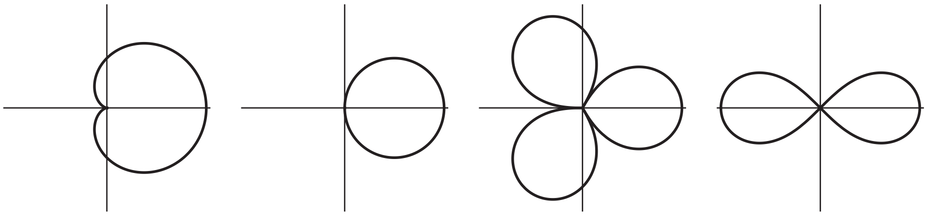
2. CLOVERS.

For a positive integer m, we define the *m-clover* to be the locus of the polar equation

$$r^{m/2} = \cos\left(\frac{m}{2}\theta\right).$$

Examples of m-clovers are displayed in Figure 1 for small m. The m-clover has m identical *leaves* for m odd and $\frac{m}{2}$ leaves for m even. The *principal leaf* is defined as the points on the m-clover satisfying $(r, \theta) \in [0, 1] \times [-\frac{\pi}{m}, \frac{\pi}{m}]$.



(a) Cardioid (b) Circle (c) Clover (d) Lemniscate

Figure 1. m-clovers for $m = 1, 2, 3, 4$

Consider the polar arc length integral for a segment of the principal leaf in the upper-half plane, beginning at the origin and terminating at the unique point with radial component $r \in [0, 1]$ (see Figure 2):

$$\lambda_m(r) = \int_0^r \frac{dt}{\sqrt{1 - t^m}}.$$

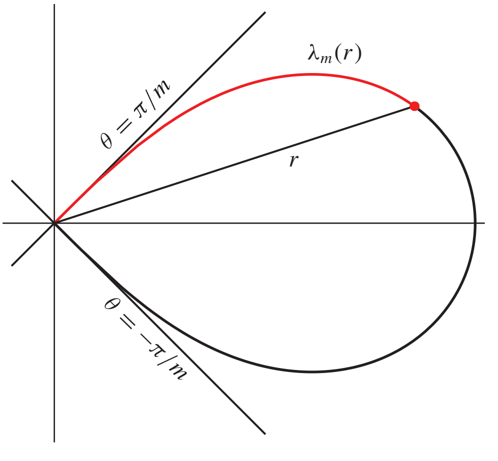


Figure 2. Principal leaf of the m-clover

A derivation of this identity from the definition of an arc length integral in polar coordinates may be found in [**6**, Prop. 1]. The constant ϖ_m defined in (1.2) may be expressed as

$$\varpi_m = 2\int_0^1 \frac{dt}{\sqrt{1 - t^m}} = 2\lambda_m(1).$$

Thus, ϖ_m denotes the arc length of an m-clover leaf. The integral $\lambda_m(r)$ has an elementary closed form only when $m = 1$ or 2:

$$\lambda_1(r) = 2\left(1 - \sqrt{1-r}\right),$$
$$\lambda_2(r) = \sin^{-1}(r).$$

Thus, it follows that $\sin(x) = \lambda_2^{-1}(x)$. Motivated by this example, we define the *m-clover function* $\varphi_m(x)$ for $x \in [0, \frac{1}{2}\varpi_m]$ by

$$\varphi_m(x) = \lambda_m^{-1}(x).$$

Symmetry suggests that we consider the functional equation $\varphi_m(\varpi_m - x) = \varphi_m(x)$, which allows us to extend the domain of φ_m to $[0, \varpi_m]$. There is a natural geometric interpretation of $\varphi_m(x)$. If $\mathcal{P}_x = (r, \theta)$ is the point at arc distance x from the origin along the principal leaf, then $\varphi_m(x) = r$ is the radial component of $\mathcal{P}_x$.

Note that $\varphi_m(x)$ is differentiable on $[0, \varpi_m]$. The m-clover function may also be characterized by the differential equation $\varphi_m(x)^m + \varphi_m'(x)^2 = 1$ and the initial values

$$\varphi_m(0) = \varphi_m(\varpi_m) = 0, \quad \text{and}$$
$$\varphi_m'(0) = -\varphi_m'(\varpi_m) = 1.$$

Proposition 2.1. *For all positive integers m and* $x \in [0, \varpi_m]$, *we have*

1. $\varphi_m'(x)^2 = 1 - \varphi_m(x)^m$, *and*
2. $\varphi_m(x)^{m-1} = -\frac{2}{m}\varphi_m''(x)$.

Proof. The first equation is obtained by differentiating the defining identity,

$$x = \int_0^{\varphi_m(x)} \frac{dt}{\sqrt{1-t^m}},$$

which is valid for all $x \in [0, \varpi_m]$. Then (2) follows from (1) by differentiation and the observation that $\varphi_m'(x)$ only vanishes at $x = \frac{1}{2}\varpi_m$ in $[0, \varpi_m]$. ∎

When $m = 2$, the m-clover is the circle $r = \cos(\theta)$ with $\varphi_2(x) = \sin(x)$, $\varpi_2 = \pi$, and hence (1.3) reduces to the classic Wallis product. When $m = 4$, the m-clover is the lemniscate $r^2 = \cos(2\theta)$ and $\varphi_4(x)$ is Abel's lemniscate function. The number

$$\varpi_4 = 2.6220575549\ldots$$

is known as the *lemniscate constant*. More information on the lemniscate, including its connections to number theory, may be found in [**3**], [**4**, Chap. 15], [**5**], and [**9**].

3. A SEQUENCE OF INTEGRALS. In this section, we fix a positive integer m. For n a nonnegative integer, we define

$$I_m(n) = \int_0^{\varpi_m} \varphi_m(x)^n \, dx.$$

Each integral is finite and positive. When $m = 2$, $\{I_2(n)\}$ reduces to the sequence of definite integrals used in the proof of the classic Wallis product. We first establish a recursive relation among the elements of $\{I_m(n)\}_{n\geq 1}$.

Lemma 3.1. *For all nonnegative integers n,*

$$\frac{I_m(n+m)}{I_m(n)} = \frac{2(n+1)}{2(n+1)+m}.$$

Proof. Our strategy is to transform $I_m(n+m)$ with integration by parts. Let

$$u = \varphi_m(x)^{n+1}, \quad \text{and}$$
$$dv = \varphi_m(x)^{m-1}\,dx.$$

Since $\varphi_m(x)^{m-1} = -\frac{2}{m}\varphi_m''(x)$ by Proposition 2.1, we have

$$du = (n+1)\varphi_m(x)^n\varphi_m'(x)\,dx, \quad \text{and}$$
$$v = -\frac{2}{m}\varphi_m'(x).$$

Thus,

$$\begin{aligned} I_m(n+m) &= \int_0^{\varpi_m} \varphi_m(x)^{n+m}\,dx \\ &= \frac{2(n+1)}{m}\int_0^{\varpi_m} \varphi_m(x)^n\varphi_m'(x)^2\,dx \qquad (\text{since } \varphi_m(0)=\varphi_m(\varpi_m)=0) \\ &= \frac{2(n+1)}{m}\int_0^{\varpi_m} \varphi_m(x)^n(1-\varphi_m(x)^m)\,dx \quad (\text{by Prop. 2.1(1)}) \\ &= \frac{2(n+1)}{m}\left(I_m(n) - I_m(n+m)\right). \end{aligned}$$

Rearranging leads to

$$\frac{I_m(n+m)}{I_m(n)} = \frac{2(n+1)}{2(n+1)+m}. \qquad \blacksquare$$

To derive the generalized Wallis product, we consider the following limit,

$$\lim_{k\to\infty} \frac{I_m(mk-1)}{I_m(mk)}. \tag{3.1}$$

First we compute (3.1) analytically, then express the limit as an infinite product using the recurrence of Lemma 3.1.

For $x \in [0, \varpi_m]$ and $n \geq 0$, we have

$$0 \leq \varphi_m(x) \leq 1 \Longrightarrow \varphi_m(x)^{n+1} \leq \varphi_m(x)^n \Longrightarrow I_m(n+1) \leq I_m(n).$$

Therefore, $\{I_m(n)\}_{n\geq 1}$ is a decreasing sequence. It follows that

$$0 < I_m(n) \leq I_m(n-1) \leq I_m(n-m).$$

Dividing through by $I_m(n)$, we obtain

$$1 \leq \frac{I_m(n-1)}{I_m(n)} \leq \frac{I_m(n-m)}{I_m(n)} = \frac{2(n-m+1)+m}{2(n-m+1)},$$

where the last equality is an application of Lemma 3.1. As $n \to \infty$, we conclude that $I_m(n-1)/I_m(n) \to 1$. Then, for the subsequence $n = mk$, it follows that

$$\lim_{k\to\infty} \frac{I_m(mk-1)}{I_m(mk)} = 1. \tag{3.2}$$

Next, we express the limit (3.1) as an infinite product.

Lemma 3.2. *For all positive integers k,*

1. $I_m(mk) = \frac{2\varpi_m}{m+2} \prod_{n=1}^{k-1} \frac{2(mn+1)}{2(mn+1)+m}$, *and*
2. $I_m(mk-1) = \frac{4}{m} \prod_{n=1}^{k-1} \frac{2n}{2n+1}$.

Proof. We proceed inductively. Clearly, $I_m(0) = \varpi_m$, so

$$I_m(m) = \frac{2}{m+2} I_m(0) = \frac{2}{m+2}\varpi_m$$

by Lemma 3.1. To compute $I_m(m-1)$, we use $\varphi_m(x)^{m-1} = -\frac{2}{m}\varphi_m''(x)$ and $\varphi_m'(0) = -\varphi_m'(\varpi_m) = 1$ to conclude that

$$I_m(m-1) = \int_0^{\varpi_m} \varphi_m(x)^{m-1}\,dx = -\frac{2}{m}\int_0^{\varpi_m} \varphi_m''(x)\,dx = \frac{4}{m}.$$

These computations establish the base case. The inductive step is a consequence of Lemma 3.1:

$$I_m(m(k+1)) = \frac{2(mk+1)}{2(mk+1)+m} I_m(mk), \quad \text{and}$$

$$I_m(m(k+1)-1) = \frac{2mk}{2mk+m} I_m(mk-1) = \frac{2k}{2k+1} I_m(mk-1). \qquad \blacksquare$$

Theorem 3.3 (Wallis Product on Clovers). *For any positive integer m, we have*

$$\varpi_m = \frac{2(m+2)}{m} \prod_{n=1}^{\infty} \frac{n\,(2(mn+1)+m)}{(mn+1)(2n+1)}.$$

Proof. From Lemma 3.2, we have

$$\frac{I_m(mk-1)}{I_m(mk)} = \frac{2(m+2)}{m\varpi_m} \prod_{n=1}^{k-1} \frac{n\,(2(mn+1)+m)}{(mn+1)(2n+1)}.$$

Taking the limit as $k \to \infty$, we have $\lim_{k\to\infty} \frac{I_m(mk-1)}{I_m(mk)} = 1$ by (3.2). Therefore,

$$1 = \lim_{k\to\infty} \frac{I_m(mk-1)}{I_m(mk)} = \frac{2(m+2)}{m\varpi_m} \prod_{n=1}^{\infty} \frac{n\,(2(mn+1)+m)}{(mn+1)(2n+1)}.$$

Multiplying the final identity by ϖ_m results in the Wallis product. $\blacksquare$

4. EULER AND THE BETA FUNCTION.

We conclude this article by suggesting how Euler may have derived Theorem 3.3. On page 33 of *De fractionibus continuis Wallisii* [**7**], Euler recalls an eighteenth century "identity:"

$$\int_0^1 \frac{x^{M-1}\,dx}{(1-x^N)^{1-\frac{K}{N}}} = \frac{M+K}{M}\cdot\frac{M+K+N}{M+N}\cdot\frac{M+K+2N}{M+2N}\cdots\int_0^1 \frac{x^{\infty}\,dx}{(1-x^N)^{1-\frac{K}{N}}}.$$

To a modern reader, the infinite exponent in the integrand may cause concern. However, following Euler's lead, we consider the ratio of the two cases $(M, N, K) = (1, m, m/2)$ and $(m, m, m/2)$,

$$\frac{\int_0^1 \frac{1}{\sqrt{1-x^m}}\,dx}{\int_0^1 \frac{x^{m-1}}{\sqrt{1-x^m}}\,dx} = (m+2)\cdot\frac{2(3m+2)}{3(m+1)}\cdot\frac{3(5m+2)}{5(2m+1)}\cdot\frac{4(7m+1)}{7(3m+1)}\cdots.$$

The integrals with dubious integrands naïvely cancel. The left-hand side evaluates to $\frac{m}{4}\varpi_m$ after substituting $x = \varphi_m(t)$ into each integral. Scaling by $4/m$ recovers the generalized Wallis product. In [**7**], Euler uses this technique to show that the values of certain continued fractions may be expressed by ratios of integrals. Earlier in the treatise, Euler notes a connection between infinite products and continued fractions. He chooses to focus on values expressed by continued fractions, but was surely aware of the corresponding products.

Euler's product identity is closely related to a product formula [**1**] for the beta function $B(x, y) = \Gamma(x)\Gamma(y)/\Gamma(x+y)$,

$$B(x, y) = \left(\frac{1}{x}+\frac{1}{y}\right)\prod_{n=1}^{\infty}\frac{1+\frac{x+y}{n}}{(1+\frac{x}{n})(1+\frac{y}{n})}.$$

Then, in terms of the beta function, we have $\varpi_m = \frac{2}{m}B\left(\frac{1}{2}, \frac{1}{m}\right)$. The authors of [**6**] derive this expression for ϖ_m via a change of coordinates in the defining integral (1.2), suggesting a succinct proof of the Wallis product on clovers without reference to the clover functions. In [**8**], the authors further explore the connection between lengths of curves defined by polar equations and values of the beta function.

As a final remark, it would be interesting to see which, if any, other proofs of the classic Wallis product may be generalized to the Wallis product on clovers.

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Anagrams of "American Mathematical Monthly"

Loyal mamma, enchant arithmetic.
Teach an immortal, mythical, mean.
I am a healthy, carnal, commitment.
A nice, timely, mammoth charlatan.
Halt, I'm the maniacal commentary.
Mythical, rotten, mammalian ache.
A macho charm, mentally intimate.
The crotchety mammalian animal.
Hmmm: an ethical, amoral tenacity.
Hmm, I am a catty, incoherent llama.

—Submitted by Vadim Ponomarenko