

# Small Dynamical Heights for Quadratic Polynomials and Rational Functions

Robert L. Benedetto<sup>1</sup>, Ruqian Chen<sup>1</sup>, Trevor Hyde<sup>2</sup>, Yordanka Kovacheva<sup>3</sup>,  
and Colin White<sup>1</sup>

<sup>1</sup>Department of Mathematics and Statistics, Amherst College, Amherst, Massachusetts, USA

<sup>2</sup>Department of Mathematics, University of Michigan, Ann Arbor, Michigan, USA

<sup>3</sup>Department of Mathematics, University of Chicago, Chicago, Illinois, USA

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Let  $\phi \in \mathbb{Q}(z)$  be a polynomial or rational function of degree 2. A special case of Morton and Silverman's dynamical uniform boundedness conjecture states that the number of rational preperiodic points of  $\phi$  is bounded above by an absolute constant. A related conjecture of Silverman states that the canonical height  $\hat{h}_\phi(x)$  of a nonpreperiodic rational point  $x$  is bounded below by a uniform multiple of the height of  $\phi$  itself. We provide support for these conjectures by computing the set of preperiodic and small-height rational points for a set of degree-2 maps far beyond the range of previous searches.

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## 1. INTRODUCTION

In this paper, we consider the dynamics of a rational function  $\phi(z) \in \mathbb{Q}(z)$  acting on  $\mathbb{P}^1(\mathbb{Q})$ . The degree of  $\phi = f/g$  is  $\deg \phi := \max\{\deg f, \deg g\}$ , where  $f, g \in \mathbb{Q}[z]$  have no common factors. Define  $\phi^0(z) = z$ , and for every  $n \geq 1$ , let  $\phi^n(z) = \phi \circ \phi^{n-1}(z)$ ; that is,  $\phi^n$  is the  $n$ th iterate of  $\phi$  under composition. In this context, the automorphism group  $\text{PGL}(2, \mathbb{Q})$  of  $\mathbb{P}^1(\mathbb{Q})$  acts on  $\mathbb{Q}(z)$  by conjugation.

The *forward orbit* of a point  $x \in \mathbb{P}^1(\mathbb{Q})$  is the set of iterates

$$\mathcal{O}(x) = \mathcal{O}_\phi(x) := \{\phi^n(x) : n \geq 0\}.$$

The point  $x$  is said to be *periodic* under  $\phi$  if there is an integer  $n \geq 1$  such that  $\phi^n(x) = x$ . In that case, we say that  $x$  is  $n$ -periodic, we call the orbit  $\mathcal{O}(x)$  an  $n$ -cycle, and we call  $n$  the *period* of  $x$ , or of the cycle. The smallest period  $n \geq 1$  of a periodic point  $x$  is called the *minimal* period of  $x$ , or of the cycle. More generally,  $x$  is *preperiodic* under  $\phi$  if there are integers  $n > m \geq 0$  such that  $\phi^n(x) = \phi^m(x)$ . Equivalently,  $\phi^m(x)$  is periodic for some  $m \geq 0$ ; also equivalently, the forward orbit  $\mathcal{O}(x)$  is finite. We denote the set of preperiodic points of  $\phi$  in  $\mathbb{P}^1(\mathbb{Q})$  by  $\text{Preper}(\phi, \mathbb{Q})$ .

Using the theory of arithmetic heights, it was proved in [Northcott 50] that if  $\deg \phi \geq 2$ , then  $\phi$  has only finitely many

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Address correspondence to Robert L. Benedetto, Department of Mathematics and Statistics, Amherst College, Amherst, MA 01002, USA.  
Email: rbenedetto@amherst.edu

preperiodic points in  $\mathbb{P}^1(\mathbb{Q})$ . (In fact, Northcott proved a far more general finiteness result, for morphisms of  $\mathbb{P}^N$  over an arbitrary number field.) Then a dynamical uniform boundedness conjecture was proposed in [Morton and Silverman 94, Morton and Silverman 95]; for  $\phi \in \mathbb{Q}(z)$  acting on  $\mathbb{P}^1(\mathbb{Q})$ , it says the following.

**Conjecture 1.1.** [Morton and Silverman 94] *For every  $d \geq 2$ , there is a constant  $M = M(d)$  such that for every  $\phi \in \mathbb{Q}(z)$  of degree  $d$ ,*

$$\# \text{Preper}(\phi, \mathbb{Q}) \leq M.$$

Only partial results toward Conjecture 1.1 have been proven, including nonuniform bounds of various strengths, as well as conditions under which certain preperiodic orbit structures are possible or impossible. See, for example, [Benedetto 07, Call and Goldstine 97, Flynn et al. 97, Manes 07, Morton 92, Morton 98, Morton and Silverman 94, Morton and Silverman 95, Narkiewicz 89, Pezda 94, Poonen 98, Zieve 96], as well as [Silverman 07, Section 4.2].

A sharper version of the conjecture for the special case of quadratic polynomials over  $\mathbb{Q}$  was later stated in [Poonen 98].

**Conjecture 1.2.** [Poonen 98] *Let  $\phi \in \mathbb{Q}[z]$  be a polynomial of degree 2. Then  $\# \text{Preper}(\phi, \mathbb{Q}) \leq 9$ .*

If true, Conjecture 1.2 is sharp; for example,  $z^2 - 29/16$  and  $z^2 - 21/16$  each have exactly nine rational preperiodic points, including the point at  $\infty$ . However, even though it is the simplest case of Conjecture 1.1, a proof of Conjecture 1.2 seems to be very far off at this time.

As little as we know about the uniform boundedness conjecture for quadratic polynomials, we know even less about the conjecture for rational functions. The first systematic attack on preperiodic points of quadratic rational functions was made in [Manes 07], including a conjecture that  $\# \text{Preper}(\phi, \mathbb{Q}) \leq 12$  when  $\phi(z) \in \mathbb{Q}(z)$  has  $\deg \phi = 2$ . In this paper, we give examples with 14 rational preperiodic points, showing that Manes’s conjecture is false.

On the one hand, we found a single map with a rational 7-cycle, along with the immediate preimages of all seven points; see equation (1–1). On the other hand, we found many maps with a rational point  $x$  whose sixth iterate  $\phi^6(x)$  is 2-periodic; the immediate preimages of all those preperiodic points again give a total of 14 points. We also found a single map with a rational point  $x$  for which  $\phi^5(x)$  is 3-periodic, again giving a total of 14 points. See Table 1 for examples. It would appear that there are only finitely many maps with a 7-cycle or with a 3-periodic cycle with a tail of length 5, while there seem to be

infinitely many with a 2-periodic cycle having a tail of length 6; for the moment, however, these finiteness questions remain open.

Meanwhile, in their study of the space of degree-two rational functions with a rational point of period 6, Blanc et al. announced an infinite family of quadratic maps with 14  $\mathbb{Q}$ -rational preperiodic points; see [Blanc et al. 13, Lemma 4.7]. This family uses two separate orbits: a 6-cycle and a fixed point, together with the preimages of all seven periodic points.

Besides its preperiodic orbits, every rational function  $\phi \in \mathbb{Q}(z)$  of degree  $d \geq 2$  has an associated *canonical height*. The canonical height is a function  $\hat{h}_\phi : \mathbb{P}^1(\overline{\mathbb{Q}}) \rightarrow [0, \infty)$  satisfying the functional equation  $\hat{h}_\phi(\phi(z)) = d \cdot \hat{h}_\phi(z)$ , and it has the property that  $\hat{h}_\phi(x) = 0$  if and only if  $x$  is a preperiodic point of  $\phi$ ; see Section 2.

For a nonpreperiodic point  $y$ , on the other hand,  $\hat{h}_\phi(y)$  measures how fast the standard Weil height  $h(\phi^n(y))$  of the iterates of  $y$  increases with  $n$ . By analogy with Lang’s height lower bound conjecture for elliptic curves, Silverman has asked how small  $\hat{h}_\phi(y)$  can be for nonpreperiodic points  $y$ . More precisely, considering  $\phi$  as a point in the appropriate moduli space, and defining  $h(\phi)$  to be the Weil height of that point, he stated the following conjecture; see [Silverman 07, Conjecture 4.98] for a more general version.

**Conjecture 1.3.** [Silverman 07] *Let  $d \geq 2$ . Then there is a positive constant  $M' = M'(d) > 0$  such that for every  $\phi \in \mathbb{Q}(z)$  of degree  $d$  and every point  $x \in \mathbb{P}^1(\overline{\mathbb{Q}})$  that is not preperiodic for  $\phi$ , we have  $\hat{h}_\phi(x) \geq M'h(\phi)$ .*

Conjecture 1.3 essentially says that the height of a nonpreperiodic rational point must start to grow rapidly within a bounded number of iterations. Some theoretical evidence for Conjecture 1.3 appears in [Baker 06, Ingram 09], and computational evidence for polynomials of degree  $d = 2, 3$  appears in [Benedetto et al. 09, Doyle et al. 13, Gillette 04]. The smallest known value of  $\hat{h}_\phi(x)/h(\phi)$  when  $\phi$  is a polynomial of degree 2 occurs for  $x = 7/12$  and  $\phi(z) = z^2 - 181/144$ . The first few iterates of this pair  $(x, \phi)$ , first discovered in [Gillette 04], are

$$\begin{aligned} \frac{7}{12} &\mapsto -\frac{11}{12} \mapsto \frac{5}{12} \mapsto -\frac{13}{12} \mapsto -\frac{1}{12} \mapsto -\frac{5}{4} \mapsto \frac{11}{36} \\ &\mapsto -\frac{377}{324} \mapsto \frac{2445}{26244} \mapsto \dots \end{aligned}$$

The small canonical height ratio

$$\frac{\hat{h}_\phi(7/12)}{h(\phi)} \approx 0.0066$$

makes precise the observation that although the numerators and denominators of the iterates eventually explode in size, it takes several iterations for the explosion to get underway.

$\phi$	Orbit	Tail	Period	Total Length
$\frac{330z^2 - 187z - 143}{330z^2 + 1217z + 429}$	$\infty, 1, 0, -\frac{1}{3}, -\frac{11}{15}, -\frac{3}{5}, -\frac{55}{114}, -\frac{13}{44}, -\frac{3}{5}$	5	3	8
$\frac{21z^2 - 84z + 63}{21z^2 - 16z - 21}$	$\infty, 1, 0, -3, \frac{7}{3}, -\frac{1}{3}, -7, -\frac{3}{2}, -7$	6	2	8
$\frac{52z^2 - 30z - 22}{52z^2 + 245z + 88}$	$\infty, 1, 0, -\frac{1}{4}, -\frac{3}{8}, -1, -\frac{4}{7}, -\frac{9}{26}, -\frac{4}{7}$	6	2	8
$\frac{120z^2 - 98z - 22}{120z^2 + 749z + 132}$	$\infty, 1, 0, -\frac{1}{6}, -\frac{2}{9}, -\frac{1}{5}, -\frac{12}{65}, -\frac{1}{12}, -\frac{12}{65}$	6	2	8
$\frac{30z^2 - 10z - 20}{30z^2 + 7z - 30}$	$\infty, 1, 0, \frac{2}{3}, \frac{10}{9}, \frac{2}{5}, \frac{6}{7}, \frac{10}{3}, \frac{6}{7}$	6	2	8
$\frac{33z^2 - 429z + 396}{33z^2 - 197z + 132}$	$\infty, 1, 0, 3, \frac{11}{3}, 5, 33, \frac{3}{4}, 33$	6	2	8
$\frac{176z^2 + 1397z - 1573}{176z^2 + 500z - 1144}$	$\infty, 1, 0, \frac{11}{8}, -\frac{11}{2}, -\frac{11}{4}, \frac{55}{16}, 2, \frac{55}{16}$	6	2	8
$\frac{1350z^2 - 837z - 513}{1350z^2 + 5585z + 1710}$	$\infty, 1, 0, -\frac{3}{10}, -\frac{9}{10}, -\frac{3}{5}, -\frac{72}{175}, -\frac{1}{6}, -\frac{72}{175}$	6	2	8
$\frac{700z^2 - 95z - 605}{700z^2 + 1336z + 880}$	$\infty, 1, 0, -\frac{11}{16}, -\frac{5}{7}, -\frac{7}{11}, -\frac{5}{6}, -\frac{11}{70}, -\frac{5}{6}$	6	2	8
$\frac{784z^2 - 416z - 368}{784z^2 + 3885z + 644}$	$\infty, 1, 0, -\frac{4}{7}, -\frac{2}{21}, -\frac{8}{7}, -\frac{20}{49}, \frac{1}{12}, -\frac{20}{49}$	6	2	8
$\frac{1428z^2 - 1668z + 240}{1428z^2 - 1723z + 900}$	$\infty, 1, 0, \frac{4}{15}, -\frac{4}{21}, \frac{10}{21}, -\frac{4}{7}, \frac{12}{17}, -\frac{4}{7}$	6	2	8
$\frac{308z^2 + 19292z - 19600}{308z^2 + 1937z + 7700}$	$\infty, 1, 0, -\frac{28}{11}, -14, -\frac{28}{5}, -\frac{308}{17}, -\frac{40}{11}, -\frac{308}{17}$	6	2	8
$\frac{9009z^2 - 17094z + 8085}{9009z^2 - 18454z - 10395}$	$\infty, 1, 0, -\frac{7}{9}, \frac{77}{27}, \frac{35}{11}, \frac{63}{31}, -\frac{77}{78}, \frac{63}{31}$	6	2	8
$\frac{5712z^2 - 5937z + 225}{5712z^2 - 137612z + 5400}$	$\infty, 1, 0, \frac{1}{24}, \frac{1}{26}, \frac{3}{68}, \frac{117}{2992}, \frac{2}{35}, \frac{117}{2992}$	6	2	8
$\frac{51480z^2 + 910z - 52390}{51480z^2 + 275477z - 120900}$	$\infty, 1, 0, \frac{13}{30}, -\frac{26}{5}, -\frac{91}{11}, \frac{780}{253}, \frac{13}{36}, \frac{780}{253}$	6	2	8
$\frac{24255z^2 - 277830z + 253575}{24255z^2 + 314788z + 65205}$	$\infty, 1, 0, \frac{35}{9}, -\frac{5}{18}, -\frac{49}{3}, \frac{105}{13}, -\frac{15}{154}, \frac{105}{13}$	6	2	8

TABLE 1.  $(\infty, \phi) \in \mathcal{PM}_2(\mathbb{Q})$  in the search region with preperiodic orbit of length (at least) 8.

In this paper, we investigate quadratic polynomials and rational functions with coefficients in  $\mathbb{Q}$ , looking for rational points that either are preperiodic or have small canonical height. More precisely, we search for pairs  $(x, \phi)$  for which either  $x$  is preperiodic under  $\phi$  or the ratio  $\hat{h}_\phi(x)/h(\phi)$  is positive but especially small.

Past computational investigations of this type (such as those in [Doyle et al. 13, Gillette 04] for quadratic polynomials and [Benedetto et al. 09] for cubic polynomials) have begun with the map  $\phi$  and then computed the full set  $\text{Preper}(\phi, \mathbb{Q})$  of rational preperiodic points, or the full set of rational points

of small canonical height. This is a slow process, because for any given  $\phi$ , the region in  $\mathbb{P}^1(\mathbb{Q})$  that must be exhaustively searched is usually quite large, and the overwhelming majority of points in the region turn out to be false alarms. However, since we are looking for only a single point  $x$  with an interesting forward orbit, we can start with the point  $x$  and then search for maps  $\phi$  that give interesting orbits for  $x$ . This strategy ends up testing far fewer pairs  $(x, \phi)$  that never really had a chance of being preperiodic or having small height ratio, and hence we can push our computations much further.

Our computations provide further evidence for Conjectures 1.1, 1.2, and 1.3. First, in spite of our very large search region, we found no quadratic polynomials with any  $\mathbb{Q}$ -rational preperiodic structures not already observed and classified in [Poonen 98], providing support for Conjecture 1.1, or more precisely, for Conjecture 1.2. Second, we also found a number of new pairs  $(x, \phi)$  with small canonical height ratio for  $\phi$  a quadratic polynomial, but  $(7/2, z^2 - 181/144)$  remains the record-holder. This supports Conjecture 1.3, since even our much larger search region turned up no points breaking the previously existing record. In particular, we are led to propose the following.

**Conjecture 1.4.** *Let*

$$C_{\text{poly},2} := \frac{\hat{h}_{z^2-181/144}(7/12)}{h(181/144)} = \frac{0.03433\dots}{\log 181} \approx 0.00660.$$

For every polynomial  $\phi \in \mathbb{Q}[z]$  with  $\deg \phi = 2$ , let  $h(\phi) = h(c)$ , where  $c \in \mathbb{Q}$  is the unique rational number such that  $\phi$  is conjugate to  $z \mapsto z^2 + c$ . Then  $\hat{h}_\phi(x) \geq C_{\text{poly},2} \cdot h(c)$  for every  $x \in \mathbb{P}^1(\mathbb{Q})$ .

Third, regarding quadratic rational functions, which had not been previously studied at this level of generality, we found several new preperiodic orbit structures. Most notably, we found many degree-two maps with 14  $\mathbb{Q}$ -rational preperiodic points, including one with a  $\mathbb{Q}$ -rational periodic cycle of period 7:

$$\phi(z) = \frac{4655z^2 - 4826z + 171}{4655z^2 - 8071z + 798}, \tag{1-1}$$

with periodic cycle

$$\infty \mapsto 1 \mapsto 0 \mapsto \frac{3}{14} \mapsto \frac{19}{21} \mapsto \frac{1}{7} \mapsto \frac{57}{35} \mapsto \infty.$$

Each of the seven rational points above also has exactly one rational preimage outside the cycle. Those points are

$$\frac{2}{19}, \frac{57}{295}, \frac{9}{245}, \frac{563}{665}, -\frac{29}{5}, \frac{3}{190}, \frac{27}{10},$$

respectively, and  $\text{Preper}(\phi, \mathbb{Q})$  consists of precisely these four-teen points.

Fourth, we found pairs  $(x, \phi) \in \mathbb{P}^1(\mathbb{Q}) \times \mathbb{Q}(z)$  with  $\deg(\phi) = 2$  and with canonical height ratio  $\hat{h}_\phi(x)/h(x)$  much smaller than 0.0066, the record for quadratic polynomials. But again, the pairs  $(x, \phi)$  of especially small height ratio, as well as the first maps with 14 rational preperiodic points, turned up early in our large search. This support for Conjectures 1.1 and 1.3 inspired the following more specific statements for quadratic rational maps.

**Conjecture 1.5.** *Let*

$$\psi(z) = (10z^2 - 7z - 3)/(10z^2 + 37z + 9),$$

and let

$$C_{\text{rat},2} := \frac{\hat{h}_\psi(\infty)}{h(\psi)} \approx 0.000466.$$

For a rational function  $\phi \in \mathbb{Q}(z)$  with  $\deg \phi = 2$ , and any  $x \in \mathbb{P}^1(\mathbb{Q})$ , the following hold.

1.  $\# \text{Preper}(\phi, \mathbb{Q}) \leq 14$ .
2. If  $x$  is preperiodic, then  $\#\mathcal{O}_\phi(x) \leq 8$ .
3. If  $x$  is periodic, then  $\#\mathcal{O}_\phi(x) \leq 7$ .
4. If  $x$  is not preperiodic, then  $\hat{h}_\phi(x) \geq C_{\text{rat},2} \cdot h(\phi)$ .

The outline of the paper is as follows. In Section 2, we review some background: heights, canonical heights, multipliers of periodic points, valuations, and good reduction. In Section 3, we recall and then sharpen some known facts about the dynamics of quadratic polynomials over  $\mathbb{Q}$ , and we describe our search algorithm. We then summarize and discuss our data from that search. Finally, in Section 4, we do a similar analysis for quadratic rational functions over  $\mathbb{Q}$ . We also state and prove the following result suggested by our data; see Theorem 4.5 for a more precise version.

**Theorem 1.6.** *Let  $X_{5,2}$  be the parameter space of all pairs  $(x, \phi)$  with  $x \in \mathbb{P}^1$  and  $\phi$  a rational function of degree 2, up to coordinate change, for which the forward orbit of  $x$  consists of five strictly preperiodic points followed by a periodic cycle of period 2. Then  $X_{5,2}$  is birational over  $\mathbb{Q}$  to an elliptic surface of positive rank over  $\mathbb{Q}(t)$  with infinitely many  $\mathbb{Q}$ -rational points.*

**2. BACKGROUND**

The standard (Weil) height function on  $\mathbb{Q}$  is the function  $h : \mathbb{P}^1(\mathbb{Q}) \rightarrow \mathbb{R}$  given by

$$h(x) := \log \max\{|m|, |n|\},$$

if we write  $x = m/n$  in lowest terms and write  $\infty$  as  $1/0$ . It has a well-known extension to a function  $h : \mathbb{P}^1(\overline{\mathbb{Q}}) \rightarrow \mathbb{R}$ , although that extension will not be of much concern to us here. The height function satisfies two important properties. First, for every  $\phi(z) \in \mathbb{Q}(z)$ , there is a constant  $C = C(\phi)$  such that

$$|h(\phi(x)) - d \cdot h(x)| \leq C \quad \text{for all } x \in \mathbb{P}^1(\overline{\mathbb{Q}}),$$

where  $d = \deg \phi$ . Second, for every bound  $B \in \mathbb{R}$ ,

$$\{x \in \mathbb{P}^1(\mathbb{Q}) : h(x) \leq B\} \quad \text{is a finite set.} \tag{2-1}$$

For a fixed  $\phi \in \mathbb{Q}(z)$  of degree  $d \geq 2$ , the *canonical height* function  $\hat{h}_\phi : \mathbb{P}^1(\overline{\mathbb{Q}}) \rightarrow \mathbb{R}$  for  $\phi$  is given by

$$\hat{h}_\phi(x) := \lim_{n \rightarrow \infty} d^{-n} h(\phi^n(x)),$$

and it satisfies the functional equation

$$\hat{h}_\phi(\phi(x)) = d \cdot \hat{h}_\phi(x) \quad \text{for all } x \in \mathbb{P}^1(\overline{\mathbb{Q}}). \quad (2-2)$$

In addition, there is a constant  $C' = C'(\phi)$  such that

$$|\hat{h}_\phi(x) - h(x)| \leq C' \quad \text{for all } x \in \mathbb{P}^1(\overline{\mathbb{Q}}). \quad (2-3)$$

Northcott's theorem [Northcott 50] that

$$\# \text{Preper}(\phi, \mathbb{Q}) < \infty$$

is immediate from properties (2-1), (2-2), and (2-3), since they imply that for every  $x \in \mathbb{P}^1(\mathbb{Q})$ ,  $\hat{h}(x) = 0$  if and only if  $x$  is preperiodic under  $\phi$ . (In fact, Northcott proved this equivalence for every  $x \in \mathbb{P}^1(\overline{\mathbb{Q}})$ ). Meanwhile, our canonical height computations will require the following result.

**Lemma 2.1.** *Let  $\phi = f/g \in \mathbb{Q}(z)$ , where  $f, g$  are relatively prime polynomials in  $\mathbb{Z}[z]$  with  $d := \max\{\deg f, \deg g\} \geq 2$ . Let  $R = \text{Res}(f, g) \in \mathbb{Z}$  be the resultant of  $f$  and  $g$ , and let*

$$D := \min_{t \in \mathbb{R} \cup \{\infty\}} \frac{\max\{|f(t)|, |g(t)|\}}{\max\{|t|^d, 1\}}.$$

Then  $D > 0$ , and for all  $x \in \mathbb{P}^1(\mathbb{Q})$  and all integers  $i \geq 0$ ,

$$\hat{h}_\phi(x) \geq d^{-i} \left[ h(\phi^i(x)) - \frac{1}{d-1} \log \left( \frac{|R|}{D} \right) \right].$$

*Proof.* The function

$$F(t) = \frac{\max\{|f(t)|, |g(t)|\}}{\max\{|t|^d, 1\}}$$

is defined at  $\infty$  because  $\max\{\deg f, \deg g\} = d$ . Moreover,  $F$  is real-valued, positive, and continuous on the compact set  $\mathbb{R} \cup \{\infty\}$ , and hence the minimum  $D$  is indeed both attained and positive. As shown in [Silverman and Tate 92, proof of Lemma III.3'(b)],

$$h(\phi(x)) \geq dh(x) + \log \left( \frac{F(x)}{|R|} \right)$$

for all  $x \in \mathbb{P}^1(\mathbb{Q})$ . Bounding  $F(x)$  by  $D$ , and applying  $\phi$  repeatedly to  $\phi^i(x)$ , we have

$$\begin{aligned} h(\phi^{i+m}(x)) &\geq d^m h(\phi^i(x)) + (1 + d + d^2 + \dots + d^{m-1}) \log \left( \frac{D}{|R|} \right) \end{aligned}$$

for all  $x \in \mathbb{Q}$  and all  $i, m \geq 0$ . Dividing by  $d^{i+m}$  and taking the limit as  $m \rightarrow \infty$  gives the desired inequality.  $\square$

For more background on heights and canonical heights, see [Hindry and Silverman 00, Section B.2], [Lang 83, Chapter 3], or [Silverman 07, Chapter 3].

The following notion of good reduction of a dynamical system, which we state here only over  $\mathbb{Q}$ , was first proposed in [Morton and Silverman 94].

**Definition 2.2.** Let  $\phi(z) = f(z)/g(z) \in \mathbb{Q}(z)$  be a rational function, where  $f, g \in \mathbb{Z}[z]$  have no common factors in  $\mathbb{Z}[z]$ . Let  $d = \deg \phi = \max\{\deg f, \deg g\}$ , and let  $p$  be a prime number. Let  $\bar{f}, \bar{g} \in \mathbb{F}_p[z]$  be the reductions of  $f$  and  $g$  modulo  $p$ . If  $\deg(\bar{f}/\bar{g}) = d$ , we say that  $\phi$  has *good reduction* at  $p$ ; otherwise, we say that  $\phi$  has *bad reduction* at  $p$ .

Of course,  $\deg(\bar{f}/\bar{g}) \leq d$  always; but the degree can drop if either  $\max\{\deg \bar{f}, \deg \bar{g}\} < d$  or  $\bar{f}$  and  $\bar{g}$  are no longer relatively prime. If  $\phi = a_d z^d + \dots + a_0$  is a polynomial, then  $\phi$  has good reduction at  $p$  if and only if  $v_p(a_i) \geq 0$  for all  $i$ , and  $v_p(a_d) = 0$ ; see [Morton and Silverman 95, Example 4.2]. Here,  $v_p(x)$  denotes the  $p$ -adic valuation of  $x \in \mathbb{Q}^\times$ , given by  $v_p(p^r a/b) = r$ , where  $a, b, r \in \mathbb{Z}$  and  $p \nmid ab$ .

Finally, if  $\phi \in \mathbb{Q}(z)$  is a rational function and  $x \in \overline{\mathbb{Q}}$  is a periodic point of  $\phi$  of minimal period  $n$ , the *multiplier* of  $x$  is  $(\phi^n)'(x)$ . The multiplier is invariant under coordinate change, and therefore, one can compute the multiplier of a periodic point at  $x = \infty$  by changing coordinates to move it elsewhere. We will need multipliers to discuss rational functions in Section 4.

### 3. QUADRATIC POLYNOMIALS

It is well known that (except in characteristic 2), every quadratic polynomial is conjugate over the base field to a unique polynomial of the form  $\phi_c(z) := z^2 + c$ . (Scaling guarantees that the polynomial is monic, and an appropriate translation to complete the square eliminates the linear term.) Thus, the moduli space of quadratic polynomials up to conjugacy is  $\mathbb{A}^1$ , where the parameter  $c \in \mathbb{A}^1$  corresponds to  $\phi_c$ . For the purposes of Conjecture 1.3, then, the height of  $\phi_c$  itself is

$$h(\phi_c) := h(c).$$

In addition, let us denote the canonical height associated with  $\phi_c$  simply by  $\hat{h}_c$ .

However, we are really interested in *pairs*  $(x, c)$  for which the point  $x \in \mathbb{P}^1$  either is preperiodic with a long forward orbit under  $\phi_c$  or else has very small canonical height under  $\phi_c$ , as compared with  $h(c)$ . The appropriate moduli space of such pairs is therefore  $\mathbb{P}^1 \times \mathbb{A}^1$ . If  $x = \infty$ , then  $x$  is simply a fixed point with canonical height 0. Thus, we may restrict ourselves to the subspace  $\mathbb{A}^2 \cong \mathbb{A}^1 \times \mathbb{A}^1 \subseteq \mathbb{P}^1 \times \mathbb{A}^1$ . When working

over  $\mathbb{Q}$ , we can restrict ourselves even further. The following lemma is well known, but we include the short proof for the convenience of the reader.

**Lemma 3.1.** *Let  $x, c \in \mathbb{Q}$ , and suppose that  $x$  is preperiodic under  $\phi_c(z) = z^2 + c$ . Then writing  $x = m/n$  in lowest terms, we must have  $c = k/n^2$ , where  $k \in \mathbb{Z}$  satisfies*

1.  $k \equiv -m^2 \pmod{n}$ ,
2. the integers  $n$  and  $(k + m^2)/n$  are relatively prime,
3.  $k \leq n^2/4$ ,
- 4.

$$|x| \in \begin{cases} [0, 2] & \text{if } c \geq -2, \\ [\sqrt{-c - B}, B] & \text{if } c < -2, \end{cases}$$

where  $B := (1 + \sqrt{1 - 4c})/2$ .

*Proof.* Write  $c = k/N$ , where  $k, N \in \mathbb{Z}$  are relatively prime, and  $N \geq 1$ . If  $N \neq n^2$ , then there is a prime  $p$  such that  $v_p(N) \neq 2v_p(n)$ . Let  $r := v_p(n)$  and  $s := v_p(N)$ . Then

$$\phi_c(x) = \frac{m^2}{n^2} + \frac{k}{N} = \frac{m^2 N + kn^2}{n^2 N}. \tag{3-1}$$

The denominator of (3-1) has  $v_p(n^2 N) = 2r + s$ . If  $s > 2r$ , then the numerator has  $v_p(m^2 N + kn^2) = 2r$ , and hence  $v_p(\phi_c(x)) = -s$ . On the other hand, if  $s < 2r$ , then the numerator has  $v_p(m^2 N + kn^2) = s$ , and hence  $v_p(\phi_c(x)) = -2r$ . Either way, then,  $v_p(\phi_c(x)) < -r = v_p(x)$ . Thus,  $v_p(\phi_c^i(x))$  will strictly decrease with  $i$ , contradicting the hypothesis that  $x$  is preperiodic. Therefore,  $c = k/n^2$ , with  $k \in \mathbb{Z}$  relatively prime to  $n$ .

Applying the previous paragraph to an arbitrary preperiodic point, not just  $x$ , we see that all preperiodic points of  $\phi$  in  $\mathbb{Q}$  have denominator exactly  $n$  when written in lowest terms. In particular,  $\phi_c(x) = (k + m^2)/n^2$  is preperiodic. Hence,  $n \mid (k + m^2)$ , and  $(k + m^2)/n$  must be relatively prime to  $n$ , giving us statements (a) and (b).

Next, if  $c > 1/4$ , then for every  $y \in \mathbb{R}$ , we have

$$\phi(y) - y = y^2 - y + c > y^2 - y + \frac{1}{4} = \left(y - \frac{1}{2}\right)^2 \geq 0.$$

Thus,  $\{\phi_c^n(x)\}_{n \geq 0}$  is a strictly increasing sequence, contradicting the hypothesis that  $x$  is preperiodic. Hence,  $c \leq 1/4$ , giving us statement (c).

Finally, if  $c \geq -2$ , then for every  $y \in \mathbb{R}$  with  $|y| > 2$ ,

$$\begin{aligned} \phi(y) - |y| &= |y|^2 - |y| + c \geq |y|^2 - |y| - 2 \\ &= (|y| - 2)(|y| + 1) > 0. \end{aligned}$$

Just as in the previous paragraph, then,  $\{\phi_c^n(x)\}_{n \geq 0}$  is strictly increasing if  $|x| > 2$ , contradicting the preperiodicity hy-

pothesis. Similarly, if  $c < -2$ , then noting that  $A = (1 - \sqrt{1 - 4c})/2 < -1$  and  $B = (1 + \sqrt{1 - 4c})/2 > 2$  are the two (real) roots of  $\phi(z) - z = 0$ , we have, for every  $y \in \mathbb{R}$  with  $|y| > B$ ,

$$\phi(y) - |y| = |y|^2 - |y| + c = (|y| - A)(|y| - B) > 0.$$

Meanwhile,

$$\phi_c^{-1}([-B, B]) = [-B, -\sqrt{-c - B}] \cup [\sqrt{-c - B}, B],$$

and hence for  $x \in \mathbb{Q}$  to be preperiodic, we must have  $|x| \in [\sqrt{-c - B}, B]$ .  $\square$

We can improve the bound of Lemma 3.1(c) with a minor extra assumption, as follows.

**Lemma 3.2.** *With notation as in Lemma 3.1, let  $\mathcal{O}_c(x)$  denote the forward orbit of  $x$  under  $\phi_c$ . If  $\#\mathcal{O}_c(x) \geq 4$ , then  $k < -3n^2/4$ .*

*Proof.* Given Lemma 3.1(c), we need to show that if  $c \in [-3/4, 1/4]$ , then the preperiodic point  $x$  has  $\#\mathcal{O}_c(x) \leq 3$ .

First, using Lemma 3.1, it is easy to see that

$$\text{Preper}(\phi_{1/4}, \mathbb{Q}) = \left\{ \pm \frac{1}{2} \right\},$$

and

$$\text{Preper}(\phi_{-3/4}, \mathbb{Q}) = \left\{ \pm \frac{1}{2}, \pm \frac{3}{2} \right\}.$$

In both cases, it follows immediately that each preperiodic point has forward orbit of length at most 2.

Second, for every  $c \in (-3/4, 1/4)$ , the dynamics of  $\phi_c$  on  $\mathbb{R}$  has two fixed points, at  $a < b \in \mathbb{R}$ , and every point in  $\mathbb{R}$  is attracted to  $\infty$  under iteration, attracted to  $a$  under iteration, or equal to  $\pm b$ ; and  $\phi_c(\pm b) = b$ . This well-known fact is easy to check; see, for example, Section VIII.1, and especially Theorem VIII.1.3, of [Carleson and Gamelin 91]. (Indeed, the  $c$ -interval  $(-3/4, 1/4)$  is precisely the intersection of the main cardioid of the Mandelbrot set with the real line.) Thus, the only rational preperiodic orbits of  $\phi$  must end in fixed points. However, in part 6 of Theorem 3 of [Poonen 98], it is proved that for  $x, c \in \mathbb{Q}$ , if  $x$  is preperiodic to a fixed point, then  $\#\mathcal{O}(x) \leq 3$ , as desired.  $\square$

Lemmas 3.1 and 3.2 are only about the case in which the point  $x \in \mathbb{Q}$  is preperiodic for  $\phi_c$ . The following result, using similar ideas to those in Lemma 3.1, is also relevant to finding nonpreperiodic points of small canonical height.

**Lemma 3.3.** *Let  $x, c \in \mathbb{Q}$ , let  $\phi_c(z) = z^2 + c$ , and let  $p$  be a prime number. Set  $s := v_p(c)$ , and suppose  $v_p(\phi_c^i(x)) <$*

$\min\{0, s/2\}$  for some  $i \geq 0$ . Then

$$\hat{h}_c(x) \geq \begin{cases} 2^{-i-1}(2-s) \log p & \text{if } s \leq -2 \text{ is even,} \\ 2^{-1} \log 2 & \text{if } p = 2 \text{ and } s = -2, \\ \log 2 & \text{if } p = 2 \text{ and } s = -4, \\ 2^{-1} \log p & \text{otherwise.} \end{cases}$$

*Proof.* Let  $r = v_p(x)$ . If  $s \geq 0$ , suppose first that  $r \geq 0$  as well. Then  $v_p(\phi^j(x)) \geq 0$  for all  $j \geq 0$ , contradicting our hypotheses. Thus, we must have  $r \leq -1$ . Applying  $\phi$ , it is immediate that  $v_p(\phi^j(x)) = 2^j r$  for all  $j \geq 1$ . That is, the denominator of  $\phi^j(x)$  features  $p$  raised to the power  $2^j |r|$ , and hence

$$\hat{h}_c(x) = \lim_{j \rightarrow \infty} 2^{-j} h(\phi^j(x)) \geq |r| \log p \geq \log p.$$

If  $s \leq -1$  is odd, then regardless of the value of  $r = v_p(x)$ , we must have  $v_p(\phi(x)) \leq s$ , and hence  $v_p(\phi^j(x)) \leq 2^{j-1} s$  for all  $j \geq 1$ . Thus,

$$\hat{h}_c(x) = \lim_{j \rightarrow \infty} 2^{-j} h(\phi^j(x)) \geq 2^{-1} \log p.$$

If  $s \leq -2$  is even, then the hypothesis that  $v_p(\phi^i(x)) < s/2$  gives  $v_p(\phi^i(x)) \leq (s-2)/2$ , and hence  $v_p(\phi^j(x)) \leq 2^{j-i-1}(s-2)$  for all  $j \geq i$ . Thus,

$$\hat{h}_c(x) = \lim_{j \rightarrow \infty} 2^{-j} h(\phi^j(x)) \geq 2^{-i-1}(2-s) \log p.$$

If  $p = 2$  and  $s = -2$ , write  $c = a/4$ , where  $a \in \mathbb{Q}$  with  $v_2(a) = 0$ . If  $r \leq -2$ , then  $v_2(\phi^j(x)) = 2^j r$  for all  $j \geq 0$ , and hence  $\hat{h}_c(x) \geq -r \log 2 \geq 2 \log 2$ . If  $r \geq 0$ , then  $v_2(\phi(x)) = -2$ , so that  $\hat{h}_c(\phi(x)) \geq 2 \log 2$ , and hence  $\hat{h}_c(x) \geq \log 2$ . Lastly, if  $r = -1$ , suppose first that  $a \equiv 1 \pmod{4}$ . Writing  $x = b/2$  with  $v_2(b) = 0$ , we have  $v_2(b^2 + a) = 1$ , and therefore  $v_2(\phi(x)) = -1$  as well. Thus,  $v_2(\phi^j(x)) = -1$  for all  $j \geq 0$ , contradicting the hypotheses. Hence, we must instead have  $a \equiv 3 \pmod{4}$ . Then  $v_2(b^2 + 1) \geq 2$ , so that  $v_2(\phi(x)) \geq 0$ , implying that  $\hat{h}_c(\phi(x)) \geq \log 2$ , and therefore that  $\hat{h}_c(x) \geq 2^{-1} \log 2$ .

It remains to consider  $p = 2$  and  $s = -4$ . If  $r \leq -3$ , then  $v_2(\phi^j(x)) = 2^j r$  for all  $j \geq 0$ , and hence  $\hat{h}_c(x) \geq -r \log 2 \geq 3 \log 2$ . If  $r \geq -1$ , then  $v_2(\phi(x)) = -4$ , so that  $\hat{h}_c(\phi(x)) \geq 4 \log 2$ , and hence  $\hat{h}_c(x) \geq 2 \log 2$ .

Lastly, if  $p = 2$ ,  $s = -4$ , and  $r = -2$ , write  $c = a/16$  and  $x = b/4$  with  $v_2(a) = v_2(b) = 0$ . We consider three cases. First, if  $a \equiv 3 \pmod{8}$ , then  $v_2(b^2 + a) = 2$ , and hence  $v_2(\phi(x)) = -2$ ; thus,  $v_2(\phi^j(x)) = -2$  for all  $j \geq 0$ , contradicting the hypotheses. Second, if  $a \equiv 7 \pmod{8}$ , then  $v_2(b^2 + a) \geq 3$ , and hence  $v_2(\phi(x)) \geq -1$ . By the previous paragraph, then,  $\hat{h}_c(\phi(x)) \geq 2 \log 2$ , and therefore  $\hat{h}_c(x) \geq \log 2$ . Third, if  $a \equiv 1 \pmod{4}$ , then  $v_2(b^2 + a) = 1$ , and hence  $v_2(\phi(x)) = -3$ . Again by the previous paragraph,  $\hat{h}_c(\phi(x)) \geq 3 \log 2$ , and therefore  $\hat{h}_c(x) \geq (3/2) \log 2$ .  $\square$

In Algorithm 1, we will check whether the denominator of  $\phi^3(x)$  is too large, that is, whether it is more than the square root of the denominator of  $c$ . If that happens, then some prime  $p$  appears in the denominator of  $\phi^3(x)$  to too large a power. By Lemma 3.3 with  $i = 3$ , then, we would have

$$\begin{aligned} \hat{h}_c(x) &\geq \min\{2^{-4} \cdot 4, 2^{-1}\} \log p = \frac{\log p}{4} \geq \frac{\log 3}{4} \\ &= 0.275 \dots, \end{aligned}$$

if  $p \geq 3$ , or

$$\hat{h}_c(x) \geq \min\{2^{-4} \cdot 8, 2^{-1}\} \log 2 = \frac{\log 2}{2} = 0.347 \dots,$$

if  $p = 2$ . Meanwhile, the parameters  $c$  in the search range we used had denominator at most  $60060^2$  and absolute value  $|c| \leq 10$ , so that  $h(c) \leq 2 \log(60060) + \log(10)$ . Thus, the associated height ratio would be at least

$$\frac{\hat{h}_c(x)}{h(c)} \geq \frac{\log 3}{4(2 \log(60060) + \log 10)} = 0.0113 \dots,$$

and usually much larger. In short, if the third iterate  $\phi^3(x)$  has the wrong denominator, the pair  $(x, c)$  will not be of interest to us.

Similarly, even though Lemmas 3.1 and 3.2 do not expressly disallow points of small but positive canonical height when  $c \geq -3/4$ , they strongly suggest that such points will be unlikely to show up with denominators in our range of  $n \leq 60060$ . We confirmed the absence of such points for smaller denominators in some preliminary searches. Thus, although we risked missing a very small handful of points of small height and larger denominators, we decided to exclude the parameters  $c \geq -3/4$  from our search.

Our entire search, then, was over pairs  $(x, c)$  with  $x = m/n$  and  $c = k/n^2$  fitting the restrictions of Lemmas 3.1 and 3.2. Meanwhile, since  $\phi_c(-x) = \phi_c(x)$ , we may assume that the starting point  $x$  is positive. Thus, we were led to our search algorithm, Algorithm 1, which is guaranteed to find all previously unknown rational preperiodic pairs  $(x, c)$  in its search range, as well as nearly all pairs in the same range with especially small canonical height ratio  $\hat{h}_c(x)/h(c)$ .

In [Poonen 98], all quadratic polynomial preperiodic orbits in  $\mathbb{Q}$  with  $\#\mathcal{O}_c(x) \leq 4$  are fully classified, which is why we discarded them in step 5 of the algorithm. Poonen also showed that for  $x, c \in \mathbb{Q}$ , if  $x$  is preperiodic for  $\phi_c$ , then there are only two ways in which the forward orbit  $\mathcal{O}_c(x)$  can have more than four points. Specifically, either  $c = -29/16$  and  $x = \pm 3/4$ , or else the periodic cycle of the orbit has period at least 6; the latter should be impossible, according to Conjecture 1.2.

If such a point did exist, Algorithm 1 would simply declare it to be a point of extremely small canonical height. We would

$c$	$x$	$\hat{h}_c(x)$	$\hat{h}_c(x)/h(c)$	$\phi_c(x), \phi_c^2(x), \dots$
$-\frac{181}{144}$	$\frac{7}{12}$	0.03433	0.00660	$-\frac{11}{12}, -\frac{5}{12}, -\frac{13}{12}, -\frac{1}{12}, -\frac{5}{4}, \frac{11}{36}, -\frac{377}{324}, \dots$
$-\frac{1153}{576}$	$\frac{11}{24}$	0.06505	0.00923	$-\frac{43}{24}, \frac{29}{24}, -\frac{13}{24}, -\frac{41}{24}, \frac{11}{12}, -\frac{223}{192}, -\frac{8021}{12288}, \dots$
$-\frac{517}{144}$	$\frac{17}{12}$	0.06885	0.01102	$-\frac{19}{12}, -\frac{13}{12}, -\frac{29}{12}, \frac{9}{4}, \frac{53}{36}, -\frac{461}{324}, -\frac{41093}{26244}, \dots$
$-\frac{36989}{19600}$	$\frac{153}{140}$	0.12319	0.01171	$-\frac{97}{140}, -\frac{197}{140}, \frac{13}{140}, -\frac{263}{140}, \frac{1609}{980}, -\frac{38821}{48020}, \dots$
$-\frac{31949}{19600}$	$\frac{27}{140}$	0.12319	0.01187	$-\frac{223}{140}, \frac{127}{140}, -\frac{113}{140}, -\frac{137}{140}, -\frac{659}{980}, -\frac{56561}{48020}, \dots$
$-\frac{5149}{3600}$	$\frac{23}{60}$	0.10274	0.01202	$-\frac{77}{60}, \frac{13}{60}, -\frac{83}{60}, \frac{29}{60}, -\frac{359}{300}, \frac{13}{7500}, -\frac{6704413}{4687500}, \dots$
$-\frac{205}{144}$	$\frac{1}{12}$	0.06866	0.01290	$-\frac{17}{12}, \frac{7}{12}, -\frac{13}{12}, -\frac{1}{4}, -\frac{49}{36}, \frac{139}{324}, -\frac{32531}{26244}, \dots$
$-\frac{181}{144}$	$\frac{11}{12}$	0.06866	0.01321	$-\frac{5}{12}, -\frac{13}{12}, -\frac{1}{12}, -\frac{5}{4}, \frac{11}{36}, -\frac{377}{324}, -\frac{2545}{26244}, \dots$
$-\frac{16381}{7056}$	$\frac{97}{84}$	0.13059	0.01346	$-\frac{83}{84}, -\frac{113}{84}, \frac{43}{84}, -\frac{173}{84}, \frac{1129}{588}, -\frac{39331}{28812}, \dots$
$-\frac{931161001}{476985600}$	$\frac{30379}{21840}$	0.28548	0.01382	$-\frac{379}{21840}, -\frac{42629}{21840}, \frac{40571}{21840}, \frac{32731}{21840}, \frac{27809}{94640}, -\frac{76737829}{41127840}, \dots$
$-\frac{10381}{3600}$	$\frac{121}{60}$	0.12758	0.01380	$\frac{71}{60}, -\frac{89}{60}, -\frac{41}{60}, -\frac{29}{12}, \frac{887}{300}, \frac{43947}{7500}, -\frac{147354737}{4687500}, \dots$
$-\frac{9901}{3600}$	$\frac{131}{60}$	0.12912	0.01403	$\frac{121}{60}, \frac{79}{60}, -\frac{61}{60}, -\frac{103}{60}, \frac{59}{300}, -\frac{6779}{2500}, -\frac{64722611}{14062500}, \dots$
$-\frac{293749}{176400}$	$\frac{433}{420}$	0.17685	0.01405	$-\frac{253}{420}, -\frac{547}{420}, \frac{13}{420}, -\frac{233}{140}, \frac{6959}{6300}, -\frac{630923}{1417500}, \dots$
$-\frac{271909}{176400}$	$\frac{43}{420}$	0.17685	0.01413	$-\frac{643}{420}, \frac{337}{420}, -\frac{377}{420}, -\frac{103}{140}, -\frac{6301}{6300}, -\frac{767033}{1417500}, \dots$
$-\frac{1513}{576}$	$\frac{31}{24}$	0.10590	0.01446	$-\frac{23}{24}, -\frac{41}{24}, \frac{7}{24}, -\frac{61}{24}, \frac{23}{6}, \frac{2317}{192}, -\frac{1757219}{12288}, \dots$
$-\frac{373}{144}$	$\frac{23}{12}$	0.08640	0.01459	$\frac{13}{12}, -\frac{17}{12}, -\frac{7}{12}, -\frac{9}{4}, \frac{89}{36}, \frac{1141}{324}, -\frac{257491}{26244}, \dots$
$-\frac{1013082841}{476985600}$	$\frac{10541}{21840}$	0.30762	0.01483	$-\frac{41299}{21840}, \frac{31709}{21840}, -\frac{349}{21840}, -\frac{46381}{21840}, \frac{677449}{283920}, -\frac{5994294739}{1679386800}, \dots$
$-\frac{160021}{63054}$	$\frac{181}{252}$	0.17952	0.01498	$-\frac{505}{252}, \frac{377}{252}, -\frac{71}{252}, -\frac{205}{84}, \frac{7793}{2268}, -\frac{1706041}{183708}, \dots$

TABLE 2.  $(x, c) \in \mathbb{Q}^2$  in the search region with canonical height ratio  $\hat{h}_c(x)/h(c) < 0.015$ .

have checked all such points by hand for preperiodicity, but our search found none.

Our choice of  $n_{\max} = 60060 = 2^2 \cdot 3 \cdot 5 \cdot 7 \cdot 11 \cdot 13$  was motivated firstly to ensure that the computation finished in reasonable time, but also because the results of [Morton and Silverman 94] show that preperiodic orbits in  $\mathbb{P}^1(\mathbb{Q})$  cannot be very long if there are small primes of good reduction. Moreover, the results of [Benedetto 07] show that there cannot be too many rational preperiodic points unless

there is a large number of bad primes. The bad primes of  $z^2 + c$  are precisely those dividing the denominator of  $c$ , and hence we wanted our search to run at least to  $n = 60060$ .

In the case of  $p = 2$ , we needed the 2 in the denominator of  $x$  to appear to at least the power 2, for the following reason. Although every map  $\phi_{q/4}(z) = z^2 + q/4$  with  $q \in \mathbb{Q}$  and  $q \equiv 1 \pmod{4}$  has bad reduction as written, the conjugate  $\phi_{q/4}(z + 1/2) - 1/2 = z^2 + z + (q - 1)/4$  would have good reduction at 2; by the results of [Zieve 96] (summarized in [Silverman 07,



**Algorithm 1** Quadratic polynomial height search.

Fix integers  $n_{\max}$  and  $N_{\max}$  as bounds. (We used  $n_{\max} = 60060$  and  $N_{\max} = 10$ .)

1. Let  $n$  run through the integers from 1 to  $n_{\max}$ .
2. For each such  $n$ , let  $m$  run through the integers from 1 to  $n$  that are relatively prime to  $n$ .
3. For each such  $n$  and  $m$ , let  $k$  run through all the integers congruent to  $m^2$  modulo  $n$  and lying between  $-3n^2/4$  and  $-n^2N_{\max}$ . Let  $c := k/n^2$  and  $\phi_c(z) := z^2 + c$ .
4. Let  $x$  run through all rational numbers of the form  $x := (m + nj)/n$  with  $j \in \mathbb{Z}$  and  $x$  in the interval  $[0, 2]$  or  $[\sqrt{-c - B}, B]$  from Lemma 3.1(d).
5. Compute  $\phi^i(x)$  for  $i = 1, 2, 3, 4$ . Stop iterating and discard  $x$  if  $\phi^i(x)$  coincides with an earlier iterate for some  $i \leq 4$  or has denominator larger than  $n$  for some  $i \leq 3$ .
6. Otherwise, compute

$$2^{-12}h(\phi^{12}(x)).$$

If this value is less than 0.02 times  $h(c)$ , record it as our approximation for the canonical height  $\hat{h}_c(x)$ , and record  $\hat{h}_c(x)/h(c)$  as the associated height ratio.

Theorem 2.28]), the forward orbit of a preperiodic point  $x \in \mathbb{Q}$  could then have length at most 4. (On the other hand, if  $q \equiv 3 \pmod{4}$ , then all points of  $\mathbb{Q}$  would have canonical height at least  $\log 2$ , as shown in the proof of Lemma 3.3.)

The results of Algorithm 1 are summarized in Table 2, which lists all pairs  $(x, c)$  that we found with height ratio at most 0.015. For the sake of accuracy, we manually computed the canonical heights and height ratios in the table to many more than the 12 iterates originally computed by the algorithm.

Note that although there continue to be pairs with canonical height ratio about 0.013 for  $c$  up to fairly large height (e.g., two on the list with  $n = 21840 = 2^4 \cdot 3 \cdot 5 \cdot 7 \cdot 13$ ), none comes close to the ratio of less than 0.007 attained by the already known pair  $(7/12, -181/144)$ . Given the large size of this search (far larger than any prior search for such pairs), we view this as strong computational support for Conjecture 1.3, at least for quadratic polynomials over  $\mathbb{Q}$ . In addition, as noted above, we found no preperiodic orbits outside of those classified in [Poonen 98], thus providing further computational evidence for Conjecture 1.2, and hence for Conjecture 1.1 in the case of quadratic polynomials over  $\mathbb{Q}$ .

Our data suggests that just as a small prime of good reduction limits the length of a  $\mathbb{Q}$ -rational preperiodic orbit, such a prime also appears to make it difficult for the canonical height

$\hat{h}_c(x)$  to be particularly small. Indeed, even though our search included pairs  $(x, c)$  for which the denominator  $n$  of  $x$  was not divisible by 4, the reader may have noticed that almost all of the pairs in Table 2 have  $n$  divisible by 12, and usually by 60. Motivated by this observation, we did another search allowing height ratios up to 0.03 when  $4 \nmid n$ . Table 3 lists all pairs  $(x, c)$  from that search with height ratio  $\hat{h}_c(x)/h(c)$  less than 0.025. There were only four, only one of which even came close to the cutoff of 0.015 used for Table 2. In addition, all four had  $n$  divisible by 15. The full data from both searches may be found at <http://www3.amherst.edu/~rlbenedetto/quadpolydata/>.

**4. QUADRATIC RATIONAL FUNCTIONS**

The space  $\mathcal{M}_2$  of conjugacy classes of degree-2 self-morphisms of  $\mathbb{P}^1$  is known, from [Milnor 93] and [Silverman 98], to be isomorphic to  $\mathbb{A}^2$ . In particular, those authors proved the following result, which appears as [Silverman 07, Theorem 4.56].

**Lemma 4.1.** *Let  $\phi \in \mathbb{Q}(z)$  be a rational function of degree 2. Then  $\phi$  has three fixed points in  $\mathbb{P}^1(\overline{\mathbb{Q}})$ , counting multiplicities, and the multipliers of the fixed points are precisely the roots of a cubic polynomial*

$$T^3 + \sigma_1(\phi)T^2 + \sigma_2(\phi)T + (\sigma_1(\phi) - 2),$$

where  $\sigma_1(\phi), \sigma_2(\phi) \in \mathbb{Q}$  are certain explicit rational functions of the coefficients of  $\phi$ .

Moreover, the function  $\mathcal{M}_2 \rightarrow \mathbb{A}^2$  given by  $\phi \mapsto (\sigma_1(\phi), \sigma_2(\phi))$  is an isomorphism of algebraic varieties defined over  $\mathbb{Q}$ .

The precise formulas for  $\sigma_1$  and  $\sigma_2$  in terms of the coefficients of  $\phi$  are not important here, but the interested reader can find them in [Silverman 07, p. 189]. Instead, what is important is that Lemma 4.1 allows us to define a height function on  $\mathcal{M}_2$ , given by

$$h(\phi) = h(\sigma_1(\phi), \sigma_2(\phi)) := \max\{|a|, |b|, |c|\}, \tag{4-1}$$

where  $\sigma_1(\phi) = a/c$  and  $\sigma_2(\phi) = b/c$ , with  $a, b, c \in \mathbb{Z}$  and  $\gcd(a, b, c) = 1$ .

However, we are interested instead in the moduli space not of morphisms  $\phi$ , but rather of pairs  $(x, \phi)$  consisting of a point  $x$  and a morphism  $\phi$ . More precisely, given a field  $K$ , we define

$$\mathcal{PM}_2(K) := \{(x, \phi) \in \mathbb{P}^1(K) \times K(z) : \deg \phi = 2\} / \sim,$$

where the equivalence relation  $\sim$  is given by

$$(x, \phi) \sim (\eta(x), \eta \circ \phi \circ \eta^{-1}) \quad \text{for every } \eta \in \text{PGL}(2, K).$$

Of course, to verify that this quotient is well behaved from the perspective of algebraic geometry, the machinery of geometric

$c$	$x$	$\hat{h}_c(x)$	$\hat{h}_c(x)/h(c)$	$\phi_c(x), \phi_c^2(x), \dots$
$-\frac{9142351}{5832225}$	$\frac{1394}{2415}$	0.27774	0.01733	$-\frac{2981}{2415}, -\frac{106}{2415}, -\frac{3781}{2415}, \frac{2134}{2415}, -\frac{43699}{55545}, -\frac{139366718}{146916525}, \dots$
$-\frac{28236091}{12744900}$	$\frac{5981}{3570}$	0.35856	0.02090	$\frac{2111}{3570}, -\frac{6661}{3570}, \frac{4519}{3570}, -\frac{2189}{3570}, -\frac{260493}{141610}, \frac{21085814279}{18048052890}, \dots$
$-\frac{322457899}{143280900}$	$\frac{24347}{11970}$	0.48733	0.02487	$\frac{22583}{11970}, \frac{15667}{11970}, -\frac{919}{1710}, -\frac{164371}{83790}, \frac{124637651}{78008490}, \dots$
$-\frac{136643866}{79655625}$	$\frac{3854}{8925}$	0.46675	0.02492	$-\frac{13646}{8925}, \frac{5554}{8925}, -\frac{11854}{8925}, \frac{62}{1275}, -\frac{9097034}{5310375}, \frac{3820075881834}{3133342515625}, \dots$

TABLE 3.  $(x, c) \in \mathbb{Q}^2$  in the search region with  $v_2(x) \geq -1$  and canonical height ratio  $\hat{h}_c(x)/h(c) < 0.025$ .

invariant theory is required. However, given the parameterization we are about to construct, and given that we are really interested in  $\mathcal{PM}_2$  merely as a set, we can safely sidestep these issues.

Our parameterization, based on an idea suggested to us by Elkies, begins with an informal dimension count. The set  $\mathcal{PM}_2$  should be a space of dimension 3, because there are five dimensions of choices for  $\phi$  (the six coefficients of the rational function, minus one for multiplying both top and bottom by an element of  $K$ ) and one for  $x$ , but then the quotient by the action of  $\text{PGL}(2, K)$  subtracts three dimensions.

Since we will be interested in both infinite orbits of small canonical height and preperiodic points with long forward orbits, we will restrict our attention to pairs  $(x, \phi)$  for which the first six iterates  $\{\phi^i(x) : 0 \leq i \leq 5\}$  are all distinct. (Incidentally, the moduli spaces corresponding to pairs  $(x, \phi)$  where  $x$  is periodic of period 5 or less are all birational over  $\mathbb{Q}$  to  $\mathbb{P}^2$ , as shown in [Blanc et al. 13, Theorem 1(1)].) Call the subset of  $\mathcal{PM}_2$  consisting of (equivalence classes of) such pairs  $\mathcal{PM}'_2$ . Given  $(x, \phi) \in \mathcal{PM}'_2$ , and writing  $x_i := \phi^i(x)$ , then, there is a unique  $\eta \in \text{PGL}(2, K)$  such that  $\eta(x_0) = \infty, \eta(x_1) = 1$ , and  $\eta(x_2) = 0$ . Hence, the equivalence class of  $(x, \phi)$  under  $\sim$  contains a unique element whose orbit begins in the form

$$\infty \mapsto 1 \mapsto 0 \mapsto x_3 \mapsto x_4 \mapsto x_5. \tag{4-2}$$

Thus, we can view  $\mathcal{PM}'_2$  as coinciding, as a set, with the Zariski open subset of  $\mathbb{A}^3(K)$  consisting of triples  $(x_3, x_4, x_5)$  for which the map  $\phi$  giving the partial orbit (4-2) actually has degree 2. (In particular, the three coordinates must be distinct, with none equal to 0 or 1, and  $\phi$  must not degenerate to a lower-degree map.) The restrictions  $\phi(\infty) = 1$  and  $\phi(1) = 0$  dictate that the associated map  $\phi$  is of the form

$$\phi(z) = \frac{(a_1z + a_0)(z - 1)}{a_1z^2 + b_1z + b_0}. \tag{4-3}$$

Moreover, it is straightforward to check that (4-2) stipulates

$$\begin{aligned} a_1 &= x_4(x_3^2x_4 - x_3^2x_5 - x_3^2 + 2x_3x_5 - x_4x_5), & (4-4) \\ a_0 &= -x_3b_0 = x_3^2x_4(x_4 - 1)(x_5 - x_4 + x_4x_5 - x_3x_5), \\ b_1 &= x_3^2x_4^2x_5 - x_3^3x_4^2 + 2x_3^3x_4 - x_3^2x_4^2 + x_4^3x_5 \\ &\quad - x_3^2x_4x_5 - x_3x_4^2x_5 - x_3^3 + x_3x_4^2 - x_4^3 \\ &\quad + x_3^2x_5 + x_3^2 - x_3x_4 + x_4^2 - x_3x_5. \end{aligned}$$

Of course, the Zariski closed subset of  $\mathbb{A}^3$  on which  $\phi$  does not have degree 2 is precisely the zero locus of the resultant of the two polynomials  $(a_1z + a_0)(z - 1)$  and  $a_1z^2 + b_1z + b_0$ .

**Definition 4.2.** For every pair of integers  $m \geq 0$  and  $n \geq 1$  with  $m + n \geq 6$ , let  $X_{m,n}$  denote the space of triples  $(x_3, x_4, x_5) \in \mathcal{PM}'_2 \subseteq \mathbb{A}^3$  for which the map  $\phi$  given by equations (4-3) and (4-4) satisfies  $\phi^{m+n}(\infty) = \phi^m(\infty)$ , but  $\phi^i(\infty) \neq \phi^j(\infty)$  for every other  $i \neq j$  between 0 and  $m + n$ .

As a set, then,  $X_{m,n}$  is the moduli space of pairs  $(x, \phi)$  up to coordinate change for which  $\phi^m(x)$  is periodic of minimal period  $n$ , but  $\phi^i(x)$  is not periodic for  $0 \leq i \leq m - 1$ . (Presumably, it is in fact birational to the actual geometric moduli space of such pairs up to coordinate change, but as we noted before, we will not need to know that here.)

Our preliminary searches showed an abundance of orbits with  $\phi^6(\infty) = \phi^4(\infty)$ . This observation led us to the following result.

**Lemma 4.3.**  $X_{4,2}$  is birational to  $\mathbb{P}^2$ .

*Proof.* We have that  $X_{4,2}$  is the locus of triples for which  $\phi(x_5) = x_4$ . Solving this equation using the formula for  $\phi$  from equations (4-3) and (4-4) gives

$$\begin{aligned} &x_4(x_4 - 1)(x_3 - x_5)(x_4 - x_5) \\ &\quad \times (x_4 - x_4x_5 - x_3^2x_5 + 2x_3x_5 - x_3) = 0. \end{aligned}$$

Dividing out by the first four factors, which correspond to parameter choices outside  $\mathcal{PM}'_2$ , we get

$$x_5 = \frac{x_4 - x_3}{x_3^2 - 2x_3 + x_4}. \tag{4-5}$$

Thus,  $X_{4,2}$  is parameterized by  $(x_3, x_4)$ , with  $x_5$  given by equation (4-5).  $\square$

**Remark 4.4.** With  $x_5$  given by equation (4-5), it is easy to check with computational software that the resultant of the numerator and denominator of the corresponding map  $\phi$  from equations (4-3) and (4-4) is

$$x_3^2 x_4^2 (x_3 - x_4)(x_3 - 1)^8 (x_4 - 1)^2 (x_3^2 - 2x_3 + x_4) \times (x_3 x_4 - x_3 + x_4)^5. \tag{4-6}$$

It is fairly clear why most of the terms in expression (4-6) appear if we recall that none of  $x_3, x_4, x_5$  can equal 0, 1, or  $\infty$ . The term  $x_3 x_4 - x_3 + x_4$ , meanwhile, appears because it is zero precisely when the degree-1 map such that  $\infty \mapsto 1 \mapsto 0 \mapsto x_3$  already maps  $x_3$  to  $x_4$ , and hence  $\phi$  degenerates.

Bearing in mind everything so far in this section, we are led to our Algorithm 2.<sup>1</sup>

The most noticeable feature of the data was the huge number of points on  $X_{5,2}$ . We therefore analyzed  $X_{5,2}$ , leading us to the following result, which is the precise form of the theorem stated in the introduction.

**Theorem 4.5.** *The space  $X_{5,2}$  is birational to the elliptic surface*

$$E : y^2 = 4x^3 + (4t^4 + 4t^3 + 1)x^2 - 2t^3(t + 1)^2(2t^2 + 2t + 1)x + t^6(t + 1)^4. \tag{4-7}$$

Moreover,  $E$  has positive rank over  $\mathbb{Q}(t)$ , including the non-torsion point  $P$  given by  $(x, y) = (0, t^3(t + 1)^2)$ .

*Proof.* There is a morphism  $X_{5,2} \rightarrow X_{4,2}$  taking a pair  $(x, \phi)$  to the pair  $(\phi(x), \phi)$ . Parameterizing  $X_{4,2}$  by  $(x_3, x_4)$  via equation (4-5) as in the proof of Lemma 4.3, and letting  $\psi$  denote the morphism giving the orbit  $\infty, 1, 0, x_3, x_4, x_5, x_4$ , we can parameterize  $X_{5,2}$  by triples  $(x_3, x_4, w)$  for which  $\psi(w) = \infty$ . That is,  $X_{5,2}$  is the subvariety of  $\mathbb{A}^3$  defined by  $g(x_3, x_4, w) = 0$ , where

$$g(x_3, x_4, w) = w(1 - x_4)x_3^3 + (1 - w)x_3x_4^2 + w(x_4w + x_4 - 2)x_3^2 - (2w^2 - 2w + 1)x_3x_4 + wx_3 + (w^2 - w)x_4^2$$

is the denominator of  $\psi(w)$ .

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**Algorithm 2** Quadratic rational function height search.

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Fix a height bound  $B > 0$  and a threshold  $r > 0$  for small height ratios. (We used  $B = \log(100)$  and  $r = 0.002$ .)

1. Let  $(x_3, x_4, x_5)$  run through all triples in  $\mathbb{Q}^3$  with  $h(x_3), h(x_4), h(x_5) \leq B$  and with  $0, 1, x_3, x_4, x_5$  all distinct.
2. For each triple, discard it if the resulting map  $\phi$  of equations (4-3) and (4-4) degenerates to degree less than 2. In light of Lemma 4.3, also discard the triple if  $\phi(x_5) = x_4$ .
3. Compute  $\phi^i(\infty) = \phi^{i-5}(x_5)$  for  $i = 6, \dots, 10$ . For each such  $i$ , stop iterating and record  $(\infty, \phi)$  as preperiodic if  $\phi^i(\infty)$  coincides with an earlier iterate.
4. Let  $C = \log(|R|/D) > 0$  be the constant of Lemma 2.1 for  $\phi$ , and let  $h(\phi) \geq 0$  be the height of  $\phi$  from Lemma 4.1 and equation (4-1).
5. Discard the triple  $(x_3, x_4, x_5)$  if

$$h(\phi^8(\infty)) \geq 2^8 r \cdot h(\phi) + C$$

or

$$h(\phi^{10}(\infty)) \geq 2^{10} r \cdot h(\phi) + C,$$

since either one forces the height ratio  $\hat{h}_\phi(\infty)/h(\phi)$  to be larger than the threshold  $r$ .

6. Otherwise, compute

$$2^{-15} h(\phi^{15}(\infty)).$$

If this value is less than  $r \cdot h(\phi) + 2^{-15} C$ , record it as the approximation for the canonical height  $\hat{h}_\phi(\infty)$ , and record  $\hat{h}_\phi(x)/h(\phi)$  as the associated height ratio.

---

This variety is singular along the line  $x_3 = w = 1$ . Blowing up along this line via  $(x_3 - 1)s = w - 1$  gives the surface

$$s(s - 1)x_3^2 x_4 + sx_3^2 - (2s^2 - 2s + 1)x_3 x_4 - (s - 1)x_3 + s(s - 1)x_4^2 = 0,$$

which is cubic in  $(x_3, x_4)$ . Standard manipulations, along with the substitution  $s = t + 1$ , give equation (4-7), with the birationality given by

$$t = \frac{w - x_3}{x_3 - 1}, \quad x = \frac{(w - x_3)^2(w - 1)}{x_3(x_3 - 1)^3},$$

$$y = \frac{(w - x_3)^2(w - 1)}{x_3^2(x_3 - 1)^5} [(w - x_3)(w - 1)(2x_4 + x_3^2) - x_3((w - x_3)^2 + (w - 1)^2)]$$

---

<sup>1</sup> See <http://www3.amherst.edu/~rlbenedetto/quadratdata/> for the full data we found with this algorithm.

and

$$w = \frac{t}{x}(t(t+1)^2 - x), \quad x_3 = \frac{t^2(t+1)}{x},$$

$$x_4 = \frac{t}{2x^2}(y + (2t^2 + 2t + 1)x - t^3(t+1)^2).$$

The point  $P$  in the statement of the theorem obviously lies on the  $\mathbb{Q}(t)$ -curve  $E$  of equation (4–7). Finally, specializing  $E$  at  $t = 1$  gives an isomorphic copy of curve 142a1 in Cremona’s tables, which has trivial torsion and rank 1 over  $\mathbb{Q}$ . The point  $P$  specializes to the generator, and hence  $P$  must have infinite order in  $E(\mathbb{Q}(t))$ .  $\square$

Obviously, the point  $P$  itself lies in the degeneracy locus of the birationality of Theorem 4.5, given the formula  $x_3 = t^2(t+1)/x$ . Meanwhile, it is easy to compute the first few multiples of  $P$  on  $E$ :

$$P = (0, t^3(t+1)^2), \quad [2]P y = (t(t+1)^2, t(t+1)^4),$$

$$[3]P = (-t^2(t+1)^2, t^2(t+1)^3),$$

$$[4]P = (t^2(t+1), -t^2(t+1)(t^2+t-1)),$$

$$[5]P = (-t(t+1), -t(t+1)(t^3+3t^2+2t-1)),$$

$$[6]P = (t^3(t+1)^2(t+2), -t^3(t+1)^3(2t^3+6t^2+4t-1)),$$

and it turns out that all of the points  $[n]P$ , for  $n \in \{0, \pm 1, \pm 2, \pm 3, \pm 4, \pm 5\}$ , also lie in the degeneracy locus. However, all but finitely many  $[n]P$  must lie off it, and in particular, both  $[6]P$  and  $[-6]P$  are nondegenerate. For example,  $[6]P$  corresponds to

$$x_3 = \frac{1}{t(t+1)(t+2)}, \quad x_4 = -\frac{(t^2+t-1)}{t^2(t+2)^2}, \quad x_5 = \frac{t+1}{t+2},$$

with pole

$$w = -\frac{(t+1)(t^2+t-1)}{t(t+2)}.$$

Meanwhile, given the results of [Blanc et al. 13] on the infinitude of conjugacy classes admitting a rational 6-periodic point, there are also certainly infinitely many rational functions with a length-7 preperiodic orbit. Indeed, given any noncritical point  $x$  in the 6-cycle (which consists mostly, or probably entirely, of noncritical points), the other preimage  $y$  of  $\phi(x)$  is necessarily rational and strictly preperiodic with orbit length 7. Thus,  $(y, \phi)$  gives a point in  $X_{1,6}(\mathbb{Q})$ .

We also found numerous rational points (though not nearly as many as for  $X_{5,2}$ ) on the other moduli surfaces corresponding to strictly preperiodic orbits of length 7:  $X_{6,1}$ ,  $X_{4,3}$ ,  $X_{3,4}$ , and  $X_{2,5}$ . It would be interesting to find descriptions of these other moduli surfaces, some of which probably have infinitely many rational points. By contrast, the only rational points we found on  $X_{0,7}$ , the moduli space of maps with 7-cycles, corresponded to conjugates of the map  $\phi$  of equation (1–1).

As for preperiodic orbits of length 8, certainly  $X_{1,7}(\mathbb{Q})$  is nonempty. After all, each of the seven strictly preperiodic points of the map  $\phi$  of equation (1–1) has forward orbit of length 8. The resulting elements of  $X_{1,7}$  did not show up directly in our search, however, because the coordinate changes required (to move any such preperiodic point  $x$  to  $\infty$ ,  $\phi(x)$  to 1, and  $\phi^2(x)$  to 0) result in triples  $(x_3, x_4, x_5)$  outside our search region.

We found several other triples  $(x_3, x_4, x_5)$  corresponding to rational forward orbits of length 8; they are listed in Table 1. We found 26 points in  $X_{6,2}$ , two in  $X_{5,3}$ , and none in any other surface  $X_{m,n}$  with  $m+n=8$ . (It should be expected that 2-cycles are the easiest to realize, since a degree-two map has three fixed points, six 3-periodic points, and many more of every higher period, but only two 2-periodic points. Thus, the 2-periodic points are roots of a quadratic polynomial, while all the other periodic points are roots of higher-degree polynomials.)

Note, however, that for  $m \geq 1$ , points in the moduli space  $X_{m,n}$  generally come in sets of two. After all, if  $(x, \phi)$  is a point in the moduli space, then unless  $x$  is a critical point,  $\phi(x)$  has two rational preimages:  $x$  and some other point  $y$ . Thus,  $(y, \phi)$  is also a point in the same moduli space. The two points we found in  $X_{5,3}$  were related to each other in this way, as were 11 sets of two points we found in  $X_{6,2}$ . In Table 1, therefore, we have listed only one point in  $X_{5,3}$ , and fifteen in  $X_{6,2}$ . The four unpaired points we found in  $X_{6,2}$  appear last in the table; the points they would be paired with lie outside our search region.

With both preimages of each point in the strict forward orbit of  $\infty$  taken into account, each of the maps listed in Table 1 comes with 14 rational preperiodic points, assuming that none of the points involved are critical images. And a simple but tedious computation shows that this assumption is accurate. (In fact, none of the maps in Table 1 has any  $\mathbb{Q}$ -rational critical points at all.) Moreover, a longer computation, using the algorithm in [Hutz 14] for computing  $\text{Preper}(\phi, \mathbb{Q})$ , shows that in each case,  $\text{Preper}(\phi, \mathbb{Q})$  is precisely this set of 14 points. The foregoing evidence from our data led us to propose parts (a)–(c) of Conjecture 1.5.

The abundance of points in  $X_{6,2}$  suggests that  $X_{6,2}$  may be infinite, although we have not yet found a proof or disproof of this statement. Meanwhile, the surfaces  $X_{m,n}$  for other  $m, n$  with  $m+n=8$  all appear to have at most finitely many rational points. However, here the limitations of our search region, even though it is quite large, should inspire some caution. In particular, as noted above, we know that  $X_{1,7}$  must have some  $\mathbb{Q}$ -rational points, but they all lie outside our region. That is, all such points  $(x_3, x_4, x_5)$  have  $\max\{h(x_3), h(x_4), h(x_5)\} > \log 100$ .

$\phi$	$\hat{h}_\phi(\infty)$	$\hat{h}_\phi(\infty)/h(\phi)$	$\phi^3(x), \phi^4(x), \dots$
$\frac{10z^2 - 7z - 3}{10z^2 + 37z + 9}$	0.00360	0.000466	$-\frac{1}{3}, -\frac{1}{5}, -\frac{3}{5}, -\frac{1}{2}, -\frac{3}{7}, -\frac{15}{41}, -\frac{9}{32}, -\frac{41}{105}, \dots$
$\frac{48165z^2 - 54663z + 6498}{48165z^2 - 49361z + 1482}$	0.01425	0.000747	$\frac{57}{13}, \frac{38}{39}, \frac{76}{65}, \frac{57}{65}, \frac{12}{13}, \frac{2109}{2197}, \frac{12084}{11525}, \dots$
$\frac{91z^2 + 399z - 490}{91z^2 - 16z - 350}$	0.01221	0.000867	$\frac{7}{5}, -\frac{14}{11}, \frac{14}{3}, \frac{28}{13}, 21, \frac{28}{23}, -\frac{329}{591}, \frac{975016}{446071}, \dots$
$\frac{1701z^2 - 427z - 1274}{1701z^2 - 3222z + 546}$	0.01553	0.000919	$-\frac{7}{3}, \frac{14}{27}, \frac{14}{9}, -\frac{56}{9}, \frac{7}{9}, \frac{106}{171}, \frac{4424}{3987}, -\frac{6245167}{16848423}, \dots$
$\frac{7z^2 - 6z - 1}{7z^2 + 20z - 3}$	0.00721	0.000931	$\frac{1}{3}, -\frac{1}{2}, -\frac{1}{3}, -\frac{1}{5}, -\frac{1}{14}, \frac{5}{41}, \frac{57}{16}, \frac{3403}{8043}, \dots$
$\frac{60z^2 - 24z - 36}{60z^2 - 143z + 6}$	0.01128	0.000935	$-6, \frac{3}{4}, \frac{3}{10}, \frac{6}{5}, -\frac{3}{11}, -\frac{48}{95}, -\frac{78}{853}, -\frac{96159}{56528}, \dots$
$\frac{42z^2 - 67z + 25}{42z^2 - 75z + 30}$	0.00829	0.000958	$\frac{5}{6}, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{13}{21}, \frac{8}{7}, -\frac{23}{6}, \frac{2697}{2804}, \dots$
$\frac{845z^2 - 20z - 825}{845z^2 + 3302z + 2145}$	0.01243	0.001028	$-\frac{5}{13}, -\frac{9}{13}, -\frac{20}{13}, -\frac{285}{221}, -\frac{345}{403}, \frac{510}{169}, \frac{1395}{4057}, -\frac{829730}{3831763}, \dots$
$\frac{8190z^2 - 10983z + 2793}{8190z^2 - 23941z + 7182}$	0.02187	0.001040	$\frac{7}{18}, \frac{7}{26}, \frac{21}{65}, \frac{28}{85}, \frac{9}{26}, \frac{119}{537}, \frac{50022}{149725}, \dots$
$\frac{1625z^2 - 1417z - 208}{1625z^2 + 11075z + 260}$	0.01780	0.001101	$-\frac{4}{5}, -\frac{13}{50}, \frac{7}{65}, \frac{1}{25}, -\frac{13}{35}, \frac{71}{475}, -\frac{1183}{40085}, \frac{113139661}{44931025}, \dots$
$\frac{14z^2 - 9z - 5}{14z^2 + 76z + 20}$	0.01192	0.001125	$-\frac{1}{4}, -1, -\frac{3}{7}, -\frac{1}{7}, -\frac{4}{11}, -\frac{3}{140}, -\frac{611}{2339}, -\frac{308865}{201037}, \dots$
$\frac{945z^2 - 980z + 35}{945z^2 - 3372z + 315}$	0.02104	0.001145	$\frac{1}{9}, \frac{35}{27}, -\frac{1}{7}, \frac{5}{21}, \frac{1}{3}, \frac{35}{132}, \frac{26287}{85071}, \frac{86934355}{311687886}, \dots$
$\frac{90z^2 - 20826z + 20736}{90z^2 - 533z + 1080}$	0.02642	0.001161	$\frac{96}{5}, -\frac{72}{5}, \frac{99}{8}, -27, 8, -\frac{1251}{23}, \frac{131004}{27343}, -\frac{532888651446}{4100060723}, \dots$
$\frac{1375z^2 - 847z - 528}{1375z^2 + 4025z + 1320}$	0.01778	0.001183	$-\frac{2}{5}, -\frac{11}{25}, -\frac{3}{5}, -\frac{99}{125}, -1, -\frac{121}{95}, -\frac{19017}{10775}, -\frac{36280937}{10371365}, \dots$
$\frac{1400z^2 - 2005z + 605}{1400z^2 - 3148z + 1100}$	0.01621	0.001186	$\frac{11}{20}, \frac{5}{14}, \frac{7}{16}, \frac{5}{11}, \frac{23}{56}, \frac{110}{269}, \frac{2809}{6844}, \frac{23315575}{56982922}, \dots$
$\frac{290z^2 - 966z + 676}{290z^2 - 1077z + 754}$	0.02290	0.001198	$\frac{26}{29}, 2, \frac{2}{5}, \frac{10}{11}, \frac{18}{7}, -\frac{122}{109}, \frac{190862}{209065}, \frac{26084280098}{9085163135}, \dots$

TABLE 4.  $(\infty, \phi) \in \mathcal{PM}'_2(\mathbb{Q})$  in the search region with positive canonical height ratio less than 0.0012.

We close with Table 4, which gives the pairs  $(x, \phi)$  in our search region with the smallest positive height ratios  $\hat{h}_\phi(x)/h(\phi)$  that we found, specifically, with ratio less than 0.0012. (Recall that  $h(\phi)$  is the height of  $\phi$  itself as given by Lemma 4.1 and equation (4-1). As in Tables 2 and 3, we manually computed the canonical heights and height ratios in the table to many more than the 15 iterates originally computed by the algorithm.) In addition, as with the preperiodic points in Table 1, pairs  $(x, \phi)$  with small height ratio come in sets of two; Table 4 lists only one of the two pairs in this situation.

Incidentally, the fifth map listed in Table 4 is simply a coordinate change of the first. More precisely, if we call the first pair  $(\infty, \psi)$ , the fifth is the equivalence class of  $(\psi(\infty), \psi)$ , and the canonical height ratio of the latter pair is exactly double that of  $(\infty, \psi)$ . This pair  $(\infty, \psi)$  provides further evidence for Conjecture 1.3. First discovered by Elkies in a similar but smaller-scale search (personal communication), it showed up very early in our search, because the point  $(\infty, \psi) \in \mathcal{PM}'_2$  is represented by the small-height triple  $(x_3, x_4, x_5) = (-1/3, -1/5, -3/5)$ . It is telling that the second-place candidate had height ratio one and a half times as large, even though our search extended

to points in  $\mathcal{PM}'_2$  of far larger height. This observation led us to pose part (d) of Conjecture 1.5. After all, if there were pairs  $(x, \phi) \in \mathcal{PM}'_2$  with even smaller positive canonical height ratio, presumably at least one of them would have shown up.

**Remark 4.6.** The decision to check the height of  $\phi^i(\infty)$  for iterates  $i = 8, 10, 15$  in Algorithm 2 was somewhat arbitrary, but the goal was to minimize the algorithm's run time. In practice, most points tested tend to blow up in height rapidly under iteration, and therefore we did a preliminary check only a few iterations after  $x_5$  and another two iterations later, to avoid computing ten iterates of  $x_5$  for every candidate. On the other hand, the computation time required to test all the iterates  $i = 6, 7, \dots$  for each candidate also slowed the program down. Thus, we ultimately settled on testing only at  $i = 8, 10, 15$  by trial and error on relatively small search regions. Similarly, the threshold of 0.002 as an upper bound for height ratios of interest was chosen by trial and error; small preliminary searches suggested that there were plenty of interesting points with ratio below 0.002, but simply too many larger than that.

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## REFERENCES

- [Baker 06] M. Baker. "A Lower Bound for Average Values of Dynamical Green's Functions." *Math. Res. Lett.* 13 (2006), 245–257.
- [Benedetto 07] R. Benedetto. "Preperiodic Points of Polynomials over Global Fields." *J. Reine Angew. Math.* 608 (2007), 123–153.
- [Benedetto et al. 09] R. Benedetto, B. Dickman, S. Joseph, B. Krause, D. Rubin, and X. Zhou. "Computing Points of Small Height for Cubic Polynomials." *Involve* 2 (2009), 37–64.
- [Blanc et al. 13] J. Blanc, J. K. Canci, and N. D. Elkies. "Moduli Spaces of Quadratic Rational Maps with a Marked Periodic Point of Small Order." arXiv:1305.1054, 2013.
- [Call and Goldstone 97] G. Call and S. Goldstone. "Canonical Heights on Projective Space." *J. Number Theory* 63 (1997), 211–243.
- [Carleson and Gamelin 91] L. Carleson and T. Gamelin. *Complex Dynamics*. Springer, 1991.
- [Doyle et al. 13] J. R. Doyle, X. Faber, and D. Krumm. "Preperiodic Points for Quadratic Polynomials over Quadratic Fields." arXiv:1309.6401, 2013.
- [Flynn et al. 97] E. V. Flynn, B. Poonen, and E. Schaefer. "Cycles of Quadratic Polynomials and Rational Points on a Genus-2 Curve." *Duke Math. J.* 90 (1997), 435–463.
- [Gillette 04] A. Gillette. "Searching for Small Canonical Heights: Finding a Minimum Speed Limit on the Road to Infinity." BA thesis, Amherst College, 2004.
- [Gouvêa 97] F. Gouvêa. *p-adic Numbers: An Introduction*, 2nd ed. Springer, 1997.
- [Hindry and Silverman 00] M. Hindry and J. Silverman. *Diophantine Geometry: An Introduction*. Springer, 2000.
- [Hutz 14] B. Hutz. "Determination of All Rational Preperiodic Points for Morphisms of  $\mathbb{P}^N$ ." To appear in *Math. Comp.*, 2014.
- [Ingram 09] P. Ingram. "Lower Bounds on the Canonical Height Associated to the Morphism  $\phi(z) = z^d + c$ ." *Monatsh. Math.* 157 (2009), 69–89.
- [Lang 83] S. Lang. *Fundamentals of Diophantine Geometry*. Springer, 1983.
- [Manes 07] M. Manes. "Arithmetic Dynamics of Rational Maps." PhD thesis, Brown University, 2007.
- [Milnor 93] J. Milnor. "Geometry and Dynamics of Quadratic Rational Maps," with an appendix by the author and Lei Tan. *Experiment. Math.* 2 (1993), 37–83.
- [Morton 92] P. Morton. "Arithmetic Properties of Periodic Points of Quadratic Maps." *Acta Arith.* 62 (1992), 343–372.
- [Morton 98] P. Morton, "Arithmetic Properties of Periodic Points of Quadratic Maps, II." *Acta Arith.* 87 (1998), 89–102.
- [Morton and Silverman 94] P. Morton and J. Silverman. "Rational Periodic Points of Rational Functions." *Inter. Math. Res. Notices* 2 (1994), 97–110.
- [Morton and Silverman 95] P. Morton and J. Silverman. "Periodic Points, Multiplicities, and Dynamical Units." *J. Reine Angew. Math.* 461 (1995), 81–122.
- [Narkiewicz 89] W. Narkiewicz. "Polynomial Cycles in Algebraic Number Fields." *Coll. Math.* 58 (1989), 151–155.
- [Northcott 50] D. Northcott. "Periodic Points of an Algebraic Variety." *Ann. Math.* 51 (1950), 167–177.

- [Pezda 94] T. Pezda. “Polynomial Cycles in Certain Local Domains.” *Acta Arith.* 66 (1994), 11–22.
- [Poonen 98] B. Poonen. “The Classification of Rational Preperiodic Points of Quadratic Polynomials over  $\mathbb{Q}$ : A Refined Conjecture.” *Math. Z.* 228 (1998), 11–29.
- [Silverman 98] J. Silverman. “The Space of Rational Maps on  $\mathbb{P}^1$ .” *Duke Math. J.* 94 (1998), 41–77.
- [Silverman 07] J. Silverman. *The Arithmetic of Dynamical Systems*, Springer, 2007.
- [Silverman and Tate 92] J. Silverman and J. Tate. *Rational Points on Elliptic Curves*. Springer, 1992.
- [Zieve 96] M. Zieve. “Cycles of Polynomial Mappings.” PhD thesis, University of California, Berkeley, 1996.