What is Aperiodic Order? Some examples.

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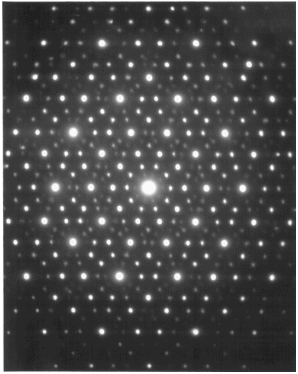
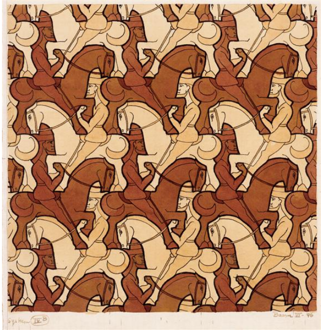
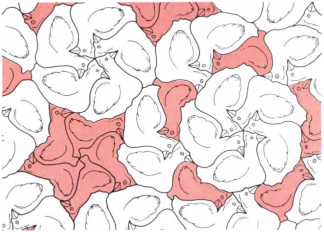






Penrose's nonperiodic chickens

What I hope to accomplish

The field of Aperiodic Order is highly multidisciplinary, with many sides to every coin. I'll talk about

- The kinds of objects we tend to consider
- The spaces they tend to occupy, and
- How to construct some of your own examples using "substitution".
- I hope this provides an overall perspective and base of examples for the rest of the talks here.

In order to understand aperiodic things, we should understand periodic things first.

Periodic sequences

For $j \in \mathbb{Z}$, let

$$\mathbf{x}(j) = \begin{cases} a & j \text{ even} \\ b & j \text{ odd} \end{cases}$$

We use the decimal point to designate the $j = 0$ entry.

$$\mathbf{x} = \cdots ababab.ababab \cdots$$

Applying the “shift operator” to move one unit to the left:

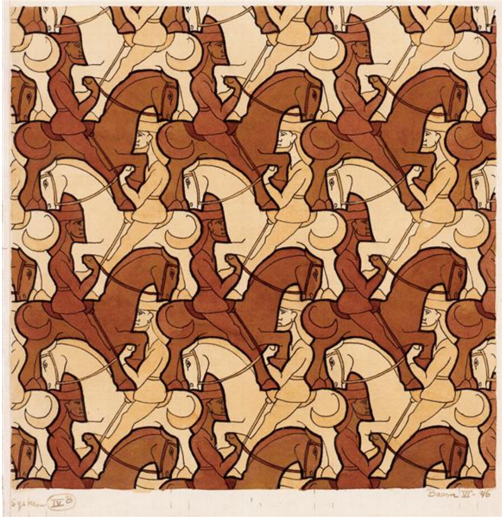
$$\mathbf{x} - 1 = \cdots abababa.babab \cdots$$

Applying the shift operator again:

$$\mathbf{x} - 2 = \cdots abababab.abab \cdots = \mathbf{x}$$

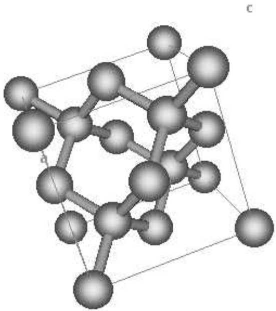
$\Rightarrow$ Period 2

Periodic tilings

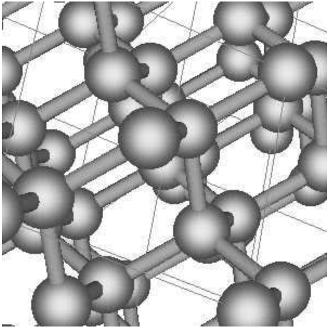


There is a lattice of translations under which the horsemen are invariant.

Crystalline Structure



The unit cell of a diamond. Each atom forms a perfect tetrahedron with four of its neighbors.



The diamond's atomic structure is composed of translations of the unit cell along coordinate axes.

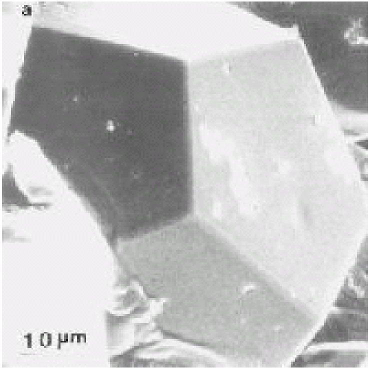
p://chemistry.umeche.maine.edu/~fgamar/Spring2001/diamond.html

The Crystallographic Restriction

"Rotational symmetry of order *greater than six*, and also *five-fold* rotational symmetry, are impossible for a periodic pattern in the plane or in three-dimensional space."

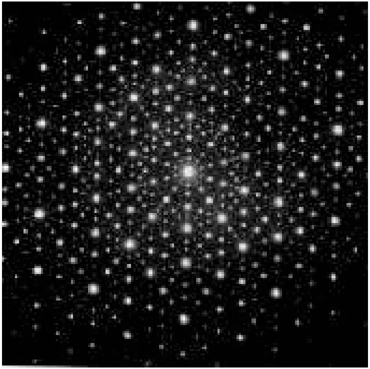
Quasicrystals and Geometry, page 7
Marjorie Senechal

Discovery of Quasicrystals



Single grain of icosahedral Al–Pd–Mn phase

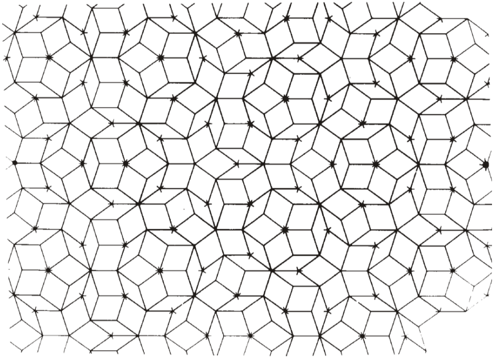
source: A. P. Tsai

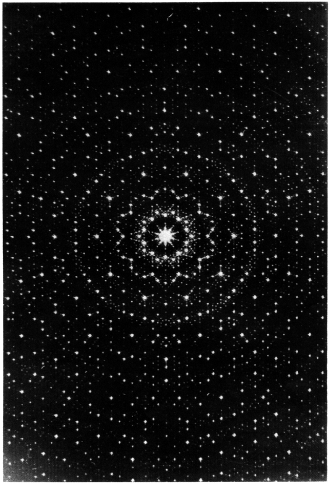


Diffraction image of Al6Mn

www.ph.melb.edu.au/diffraction/image/fivefold.html

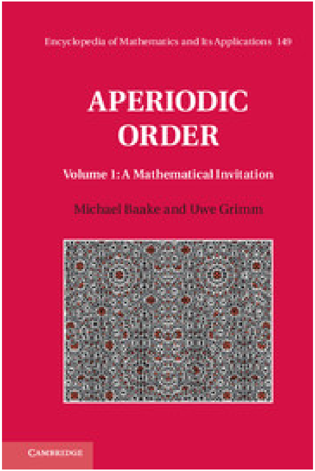
Alan Mackay’s 1982 optical diffraction of Penrose tilings





Defining Aperiodic Order

"Quasicrystals are non-periodic solids that were discovered in 1982 by Dan Shechtman, Nobel Prize Laureate in Chemistry in 2011. The underlying mathematics, known as the theory of Aperiodic Order, ..."



Whirlwind review of symbolic dynamical systems

- $\mathcal{A}$ = some finite set= *alphabet*.
- *sequence* $\mathbf{x} : \mathbb{Z} \to \mathcal{A}$ (I'm doing two-sided today)
- Master space: the set of all sequences is denoted $\mathcal{A}^{\mathbb{Z}}$.
- We study "subshifts" $\mathbb{X} \subset \mathcal{A}^{\mathbb{Z}}$ satisfying:
  1. $\mathbb{X}$ is invariant under the action of the shift, and
  2. $\mathbb{X}$ is closed in an appropriate metric topology

Three subshifts to ponder

Example

$$\text{If } \mathbf{x}_1 = \ldots 0101.0101\ldots \text{ and } \mathbf{x}_2 = \ldots 1010.1010\ldots,$$

then $\mathbb{X} = \{\mathbf{x}_1, \mathbf{x}_2\}$ is a subshift.

Example

The space $\mathbb{X}$ of all sequences of 0s and 1s with no consecutive ones is a subshift.

Example

Let $\mathbf{x}$ be a sequence with a 1 at the origin and 0s elsewhere, and let $\mathbb{X}$ be the smallest subshift containing $\mathbf{x}$.

- Metric (a “big ball” metric)

$$N(\mathbf{x},\mathbf{y}) = \min\{n \geq 0 \text{ such that } \mathbf{x}(j) \neq \mathbf{y}(j) \text{ for some } |j| = n\}$$

$$d(\mathbf{x},\mathbf{y}) = \exp(-N(\mathbf{x},\mathbf{y}))$$

Idea: **x** and **y** are close if they agree on a large ball centered at the origin.

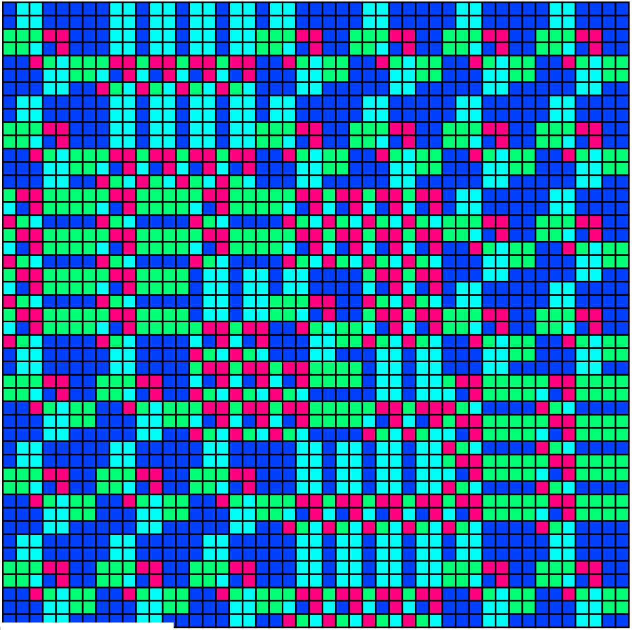
Sequences in $\mathbb{Z}^d$

- (Multidimensional) sequence $\mathbf{x}\colon \mathbb{Z}^d \to \mathcal{A}$, where $\mathbf{x}(i_1, i_2, \ldots, i_d) \in \mathcal{A}$.
  - Imagine coloring all the dots in $\mathbb{Z}^d$ with one of $|\mathcal{A}|$ colors.
  - Or, use colored d-cubes centered at the dots.
- Master space: $\mathcal{A}^{\mathbb{Z}^d}$ is the set of all sequences in $\mathbb{Z}^d$ (the "full shift")
- Shift $\mathbf{x}$ by $\vec{j}$ to get $\mathbf{x} - \vec{j} \in \mathcal{A}^{\mathbb{Z}^d}$

$$(\mathbf{x} - \vec{j})(\vec{k}) = \mathbf{x}(\vec{k} + \vec{j})$$

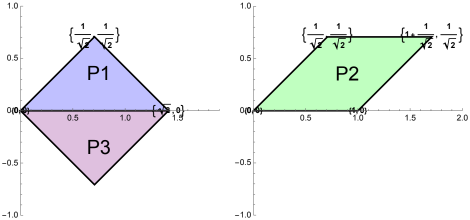
- $(\mathcal{A}^{\mathbb{Z}^d}, \mathbb{Z}^d)$ is the *full shift*; a closed translation-invariant subset $\mathbb{X}$ is a *subshift*

Often we visualize $\mathbb{Z}^2$ sequences as tilings...



Tilings from an ‘alphabet’ of prototiles

- A *prototile* p is a nice compact subset of $\mathbb{R}^d$, possibly labelled. The subset is called the “support” of p.
- A finite set $\mathcal{P}$ of prototiles will act as our alphabet.

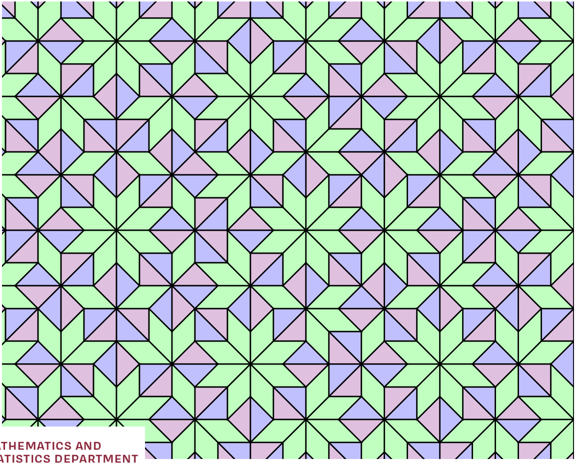


- *Tiles* (or $\mathcal{P}$-tiles) are translated copies of prototiles $p \in \mathcal{P}$, we may write $t = p - \vec{v}$, where $\vec{v} \in \mathbb{R}^d$.
- To make a tiling, we ‘cover’ and ‘pack’ $\mathbb{R}^d$ with $\mathcal{P}$-tiles.

Definition

A $\mathcal{P}$-*tiling* or just *tiling* is a countably infinite set of $\mathcal{P}$-tiles $\mathcal{T} = \{t_i \mid i \in \mathbb{Z}\}$ such that

1. If $i \neq j$, t_i and t_j intersect at most on their boundaries
2. Supports of the t_is cover all of $\mathbb{R}^d$



The action of translation

- We have

$$\mathcal{T} = \{t_i \,|\, i \in \mathbb{Z}\}$$

- $\mathcal{T} - \vec{v}$ is the new tiling

$$\mathcal{T} - \vec{v} = \{t_i - \vec{v} \,|\, t_i \in \mathcal{T}\}.$$

- Note the origin in $\mathcal{T} - \vec{v}$ corresponds to $\vec{v}$ in $\mathcal{T}$
- $\mathcal{T}$ is *nonperiodic* if there is no $\vec{v} \neq 0$ for which $\mathcal{T} - \vec{v} = \mathcal{T}$
  - If $d > 1$, $\mathcal{T}$ can be periodic but not *fully periodic*, if the directions of periodicity do not form a basis for $\mathbb{R}^d$.

Tiling spaces

- Master space: $\mathbb{X}_{\mathcal{P}}$ = the space of all $\mathcal{P}$-tilings
  - Note: Elements of $\mathcal{A}^{\mathbb{Z}}$ are infinite sequences, likewise elements of $\mathbb{X}_{\mathcal{P}}$ are infinite tilings of $\mathbb{R}^d$.
- A tiling space $\mathbb{X}$ is a closed, translation-invariant subset of $\mathbb{X}_{\mathcal{P}}$.
- We write $(\mathbb{X}, \mathbb{R}^d)$ for the dynamical system under the action of translation
- In Aperiodic Order you'll typically see
  - The "hull" $\mathbb{X}_{\mathcal{T}}$ of a tiling $\mathcal{T}$, or
  - The set of all tilings made of specified patches.

Dynamical Systems for Aperiodic Order TLDR:

- We define a space $\mathbb{X}$ that is invariant under translation.
  - We requre the elements of $\mathbb{X}$ to be locally identical to one another, like crystals formed on different days.
- We have a "big ball" metric that says how similar two elements of the space are, at least near the origin.
- The action of translation and the big ball metric work together allow us to pull disparate regions of an object 'into view' at the origin.
- Periodic objects are invariant under nontrivial translations, i.e. there's some $\vec{v} \neq 0$ such that

$$\mathcal{T} - \vec{v} = \mathcal{T}.$$

Symbolic substitutions

Definition

A *substitution* is a map $S : \mathcal{A} \to \mathcal{A}^+$, where $\mathcal{A}^+$ is the set of non-empty words on $\mathcal{A}$.

Example

(A constant-length substitution.) Let $S(a) = abb$ and $S(b) = aaa$.

Definition

If $w = a_1 ... a_k \in \mathcal{A}^+$, then $S(w) = S(a_1) ... S(a_k)$. In particular An *n-superword* is a word of the form $S^n(a)$ for some $a \in \mathcal{A}$.

Example

(A constant-length substitution, superwords.)

$$a \to abb \to abb\,aaa\,aaa \to abb\,aaa\,aaa\;\; abb\,abb\,abb\;\; abb\,abb\,abb$$

Example

(Non-constant length) Let $S(a) = abbb$ and $S(b) = a$.

$$b \to a \to abbb \to abbb\,a\,a\,a \to abbb\,a\,a\,a\,abbb\,abbb\,abbb \;\to \cdots ,$$

The subshift associated to the substitution S

Definition

$\mathbf{x} \in \mathcal{A}^{\mathbb{Z}}$ *is said to be* admitted *by S if every subword of $\mathbf{x}$ is a subword of $S^n(a)$ for some n and a. The* substitution subshift $\mathbb{X}_S \subset \mathcal{A}^{\mathbb{Z}}$ *is defined to be*

$$\mathbb{X}_S = \{\mathbf{x} \in \mathcal{A}^{\mathbb{Z}} \text{ such that } \mathbf{x} \text{ is admitted by } S\}.$$

- We are using the set of superwords $S^n(a)$ as a sort of "language" for $\mathbb{X}_S$.
- It is clear that $\mathbb{X}_S$ is a shift-invariant subset of $\mathcal{A}^{\mathbb{Z}}$

Example

For our constant length example the superwords are:
$a, b, abb, aaa, abbaaaaaa, abbabbabb, abbaaaaaaabbabbabbabbabb,$
$abbaaaaaaabbaaaaaaabbaaaaaa, \ldots$ If $\mathbf{x}$ is admitted by S, then we expect it to look locally like these words.

Conclusion: The dynamical system $(\mathbb{X}_S, \mathbb{Z})$ is ready for study.

Properties commonly found in substitution subshifts

- Under mild conditions they display aperiodic order in this sense:
  - If a word appears in $\mathbf{x} \in \mathbb{X}$, then it appears elsewhere in $\mathbf{x}$ and in all other elements of $\mathbb{X}$ with “bounded gaps”
- Most substitutions do not have any periodic elements in their subshifts.

1D self-similar tilings from substitutions

Example

$$S(a) = abbb \text{ and } S(b) = a$$

- Define interval prototiles $t_a = [0, |t_a|]$ and $t_b = [0, |t_b|]$

$$\mathcal{P} = \{t_a, t_b\}.$$

- We define a tile substitution $\mathcal{S}$:
  - $\mathcal{S}(t_a)$ is the tile t_a followed by 3 copies of t_b.
  - $\mathcal{S}(t_b)$ is just t_a.
- The lengths of the supertiles are $|\mathcal{S}(t_a)| = |t_a| + 3|t_b|$ and $|\mathcal{S}(t_b)| = |t_a|$.
- The ideal situation would be if there was an *inflation factor* $\lambda > 1$ such that $|\mathcal{S}(t_a)| = \lambda|t_a|$ and $|\mathcal{S}(t_b)| = \lambda|t_b|$.

Solving for 'natural' tile lengths

If we know

$$|\mathcal{S}(t_a)| = |t_a| + 3|t_b| \qquad |\mathcal{S}(t_b)| = |t_a|$$

and we want

$$|\mathcal{S}(t_a)| = \lambda|t_a| \qquad |\mathcal{S}(t_b)| = \lambda|t_b|$$

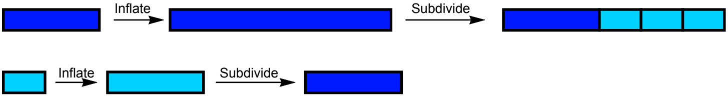
we quickly see that λ must satisfy $3 = \lambda^2 - \lambda$. So we can let

$$\lambda = \frac{1+\sqrt{13}}{2}, \quad |t_a| = \lambda, \quad |t_b| = 1.$$

(In general this process boils down to a quick eigenvalue/eigenvector computation on an easily obtained "substitution matrix".)

Inflate-and-subdivide rule

The symbolic substitution becomes a tiling *inflate-and-subdivide* rule:



Extend $\mathcal{S}$ to be a map on $\mathbb{X}_{\mathcal{P}}$ as follows

- Let $\mathcal{T} \in \mathbb{X}_{\mathcal{P}}$ be a tiling and let $t \in \mathcal{T}$ be any tile
- $\mathcal{S}(t) =$ patch given by the substitution of the prototile of t, translated so that it occupies the set $\lambda \operatorname{supp}(t)$
- Apply $\mathcal{S}$ to all tiles in $\mathcal{T}$ simultaneously to get $\mathcal{S}(\mathcal{T})$

$$\mathcal{S}(\mathcal{T}) = \bigcup_{t \in \mathcal{T}} \mathcal{S}(t)$$

If $\mathcal{S}(\mathcal{T}) = \mathcal{T}$, then $\mathcal{T}$ is called a *self-similar tiling*.

Self-similar tiling for our example

Part of a self-similar tiling for our example:

Suppose the origin is at the far left, so $\lambda(\mathcal{T})$ looks like:

When you subdivide this to compute $\mathcal{S}(\mathcal{T})$, you once again get

$\mathcal{S}(\mathcal{T}) = \mathcal{T}$, so $\mathcal{T}$ is self-similar.

The "hull" of $\mathcal{T}$

$\mathbb{X}_{\mathcal{T}} =$ the smallest closed, translation-invariant subset of $\mathbb{X}_{\mathcal{P}}$ containing $\mathcal{T}$.

A direct product substitution

$$S(a) = abb, S(b) = aa$$

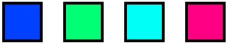
$$\mathcal{P} = \mathcal{A} \times \mathcal{A} = \{(a,a),(a,b),(b,a),(b,b)\}$$

$$\mathcal{S}((a,a)) = \begin{matrix} (a,b) & (b,b) & (b,b) \\ (a,b) & (b,b) & (b,b) \\ (a,a) & (b,a) & (b,a) \end{matrix}, \qquad \mathcal{S}((a,b)) = \begin{matrix} (a,a) & (b,a) & (b,a) \\ (a,a) & (b,a) & (b,a) \end{matrix},$$

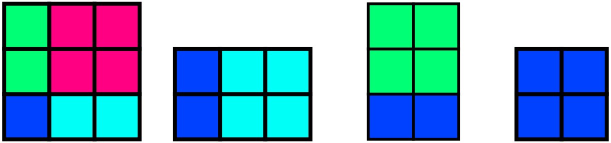
$$\mathcal{S}((b,a)) = \begin{matrix} (a,b) & (a,b) \\ (a,b) & (a,b) \\ (a,a) & (a,a) \end{matrix}, \qquad \mathcal{S}((b,b)) = \begin{matrix} (a,a) & (a,a) \\ (a,a) & (a,a) \end{matrix}$$

Substitute the first coordinate horizontally and the second coordinate vertically.

Direct product tiling

$\mathcal{P} = \{(a,a),(a,b),(b,a),(b,b)\} =$ 

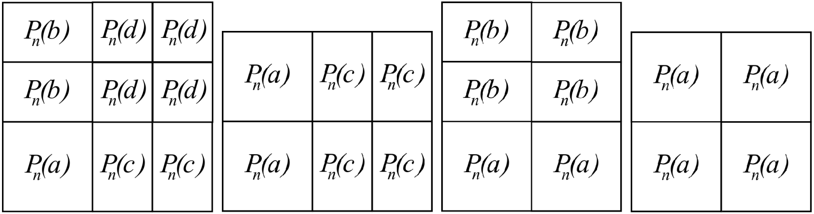
and the 1-supertiles look like



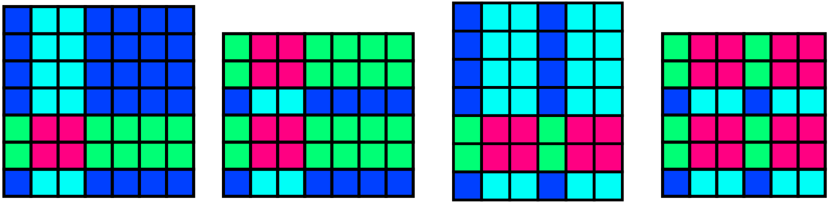
1-D subs for reference:

$$S(a) = abb, S(b) = aa$$

Template for DP concatenation

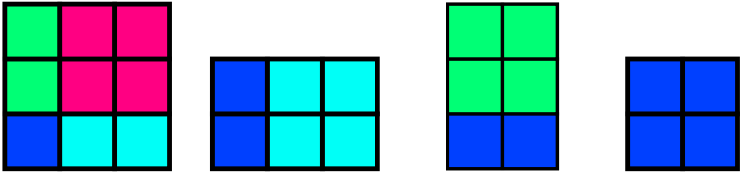


The 2-supertiles come out to be

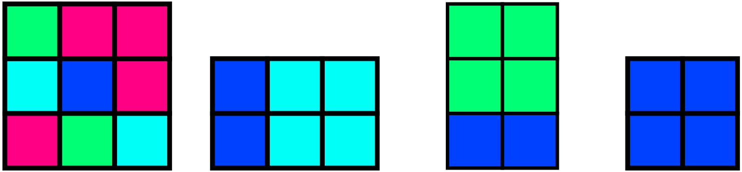


DP Variations: breaking the direct product structure

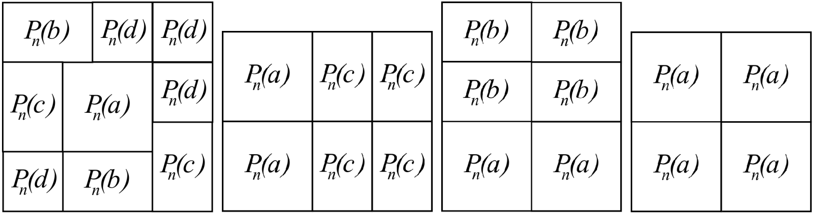
Start with the direct product:



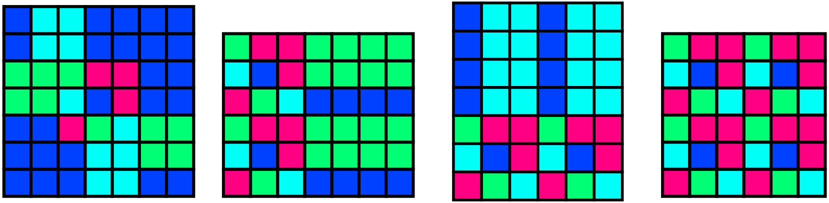
The tile on the left has been carefully rearranged:



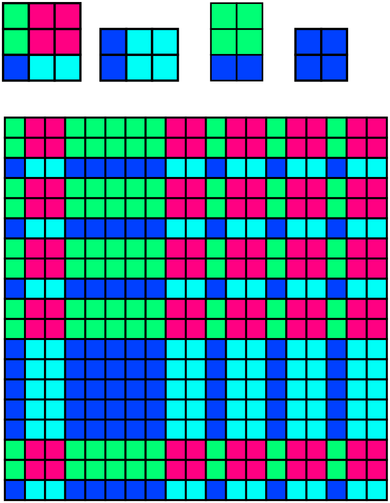
Template for DPV concatenation

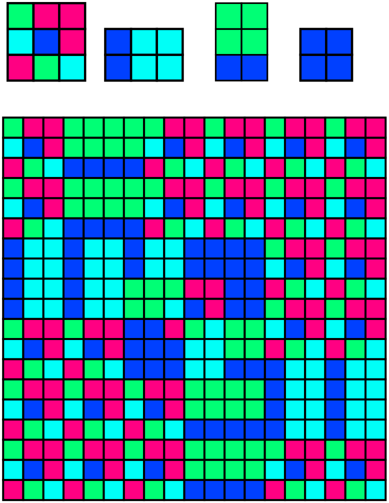


The 2-supertiles come out to be



A comparison of the DP and DPV





Tiling space construction

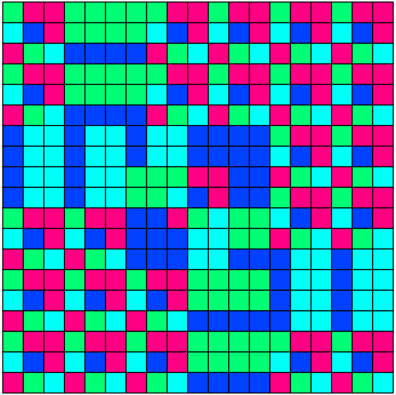
Consider the set of all possible supertiles

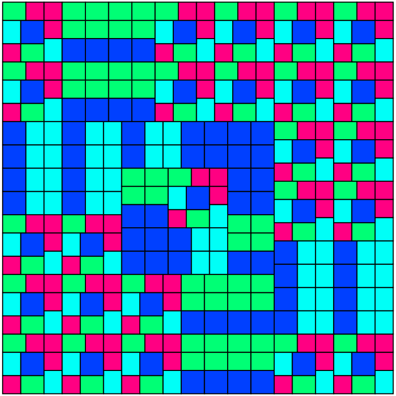
$$\mathcal{R}_{\mathcal{S}} = \{S^n(a) \,|\, a \in \mathcal{A} \text{ and } n \in \mathbb{N}\}$$

We say that $\mathcal{T} \in \mathbb{X}_{\mathcal{P}}$ is *admitted* by $\mathcal{S}$ if every patch in $\mathcal{T}$ is translation-equivalent to a subpatch of an element of $\mathcal{S}$.

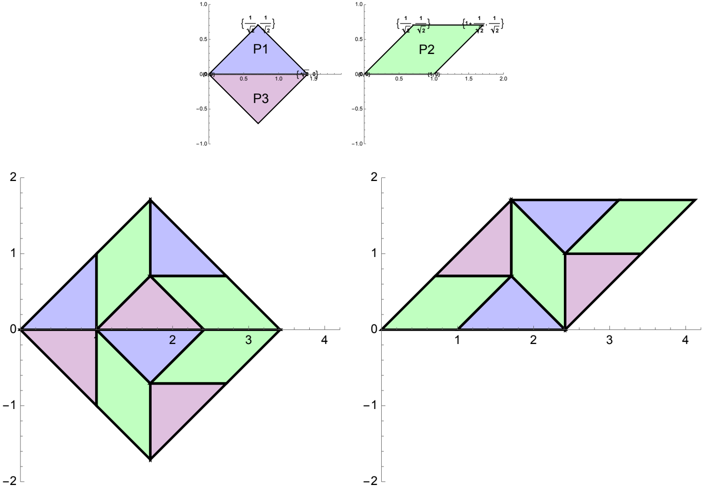
The tiling space $\mathbb{X}_{\mathcal{S}}$ is the set of all admitted tilings.

‘Natural’ tile shapes ⇒ infinite local complexity





A famous inflate-and-subdivide rule.



('Octagonal' or 'Ammann-Beenker' tiling; expansion factor is $1 + \sqrt{2}$.)

Self-similar tilings

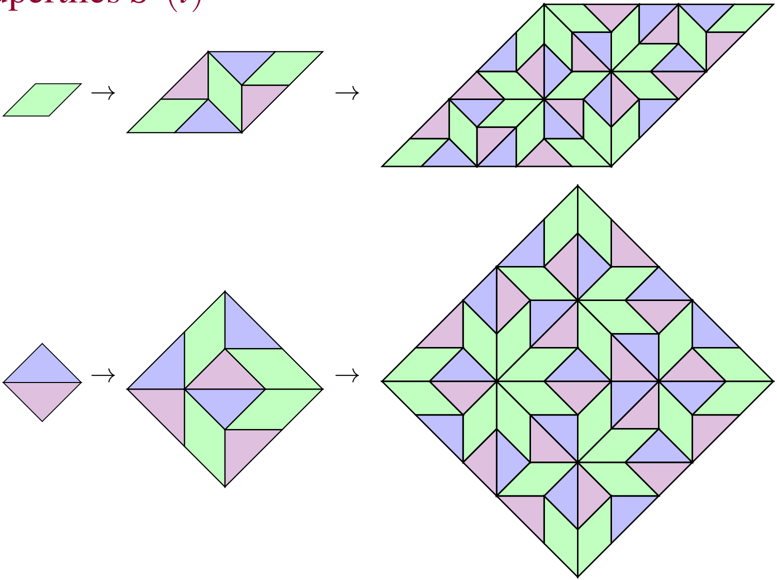
Definition

Let $\mathcal{P}$ be a prototile set in $\mathbb{R}^d$ and let ϕ be an expanding linear similarity of $\mathbb{R}^d$. A function $S : \mathcal{P} \to \mathcal{P}^*$ is called an *inflate-and-subdivide rule with inflation map* ϕ if for every $p \in \mathcal{P}$,

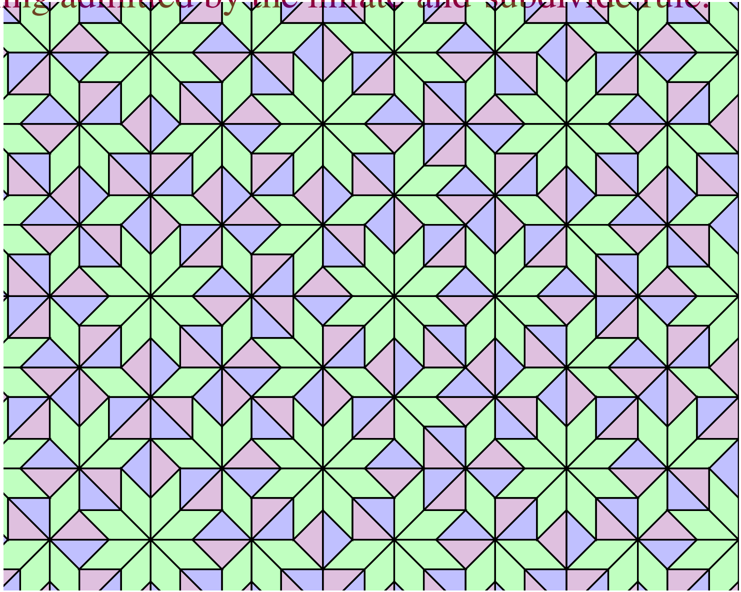
$$\phi(\text{supp}(p)) = \text{supp}(\mathcal{S}(p)).$$

Expanded support of p = Union of supports of its subtiles

Supertiles $S^n(t)$



A tiling admitted by the inflate-and-subdivide rule.



Tiling self-similarity

We can extend S to tiles, patches, and tilings:

- If $t = p - \vec{v}$ for $p \in \mathcal{P}$ and $\vec{v} \in \mathbb{R}^d$ we define

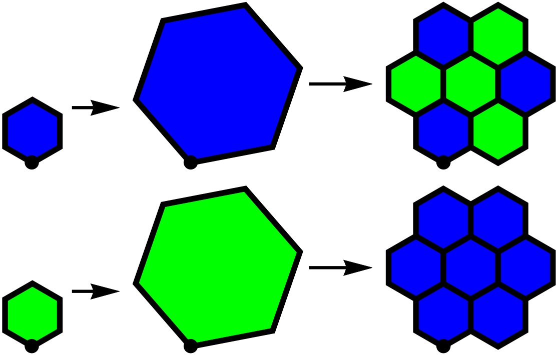
$$S(t) := S(p) - \phi(\vec{v})$$

- $\mathcal{T}$ tiling:

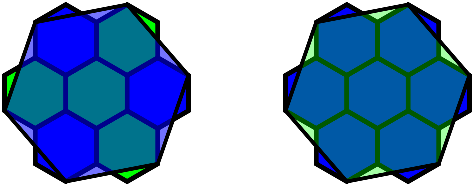
$$S(\mathcal{T}) = \bigcup_{t \in \mathcal{T}} S(t)$$

A tiling $\mathcal{T}$ is said to be **self-similar** if $\mathcal{S}(\mathcal{T}) = \mathcal{T}$.

$$\phi = \begin{pmatrix} 5/2 & \sqrt{3}/2 \\ -\sqrt{3}/2 & 5/2 \end{pmatrix}$$



An inflate-and-replace rule for a hexagonal pseudo-self-similar tiling.



The 'local derivation rule' taking $\phi(\mathcal{P})$-tiles back to $\mathcal{P}$-patches.

Pseudo-self-similar tilings

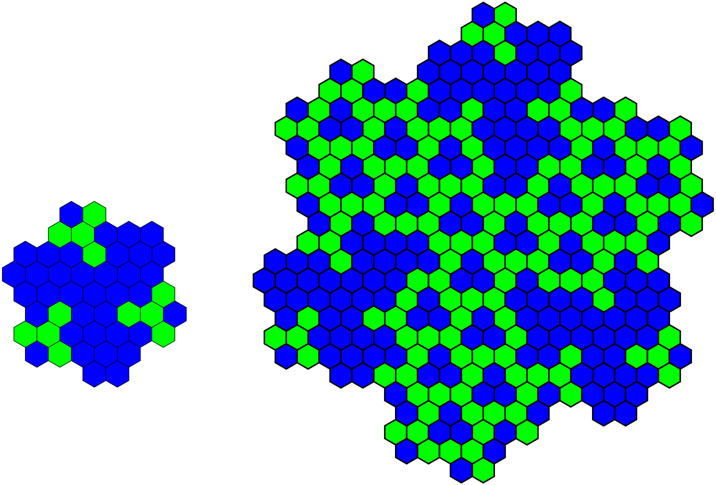
Definition

Let $\mathcal{P}$ be a finite prototile set in $\mathbb{R}^d$ and let $\phi : \mathbb{R}^d \to \mathbb{R}^d$ be an expanding linear similarity. We say a tiling $\mathcal{T} \in \mathbb{X}_{\mathcal{P}}$ is *pseudo-self-similar with expansion* ϕ if $\mathcal{T}$ is locally derivable from $\phi(\mathcal{T})$.

Note: $\phi(\mathcal{T}) = \{\phi(t) \,|\, t \in \mathcal{T}\}$ is a tiling made of inflated $\mathcal{P}$-tiles.

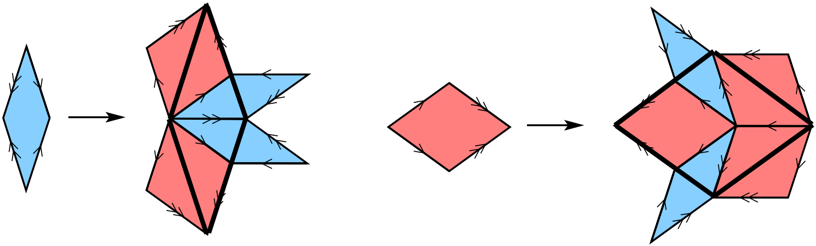
The local derivation rules tell you how to *replace* the *inflated* tiles with tiles from the original scale..

Use your supertiles to define a language for a tiling space



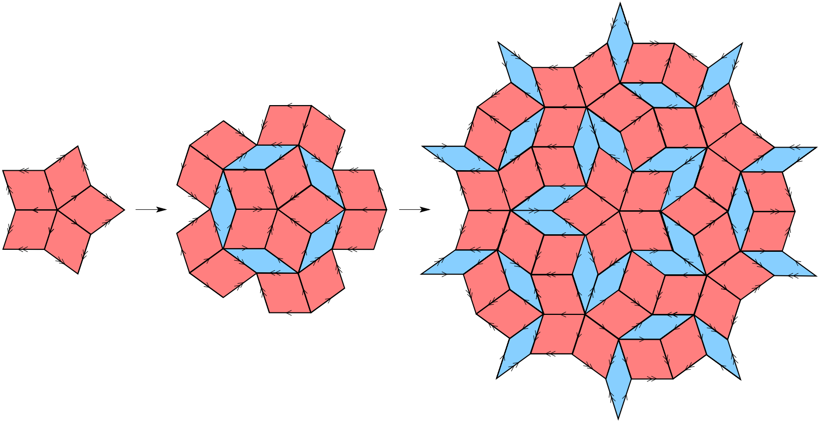
2- and 3-supertiles for the blue prototile.

A particularly famous pseudo-self-similar tiling



The Penrose rhombuses!

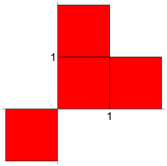
A particularly famous pseudo-self-similar tiling

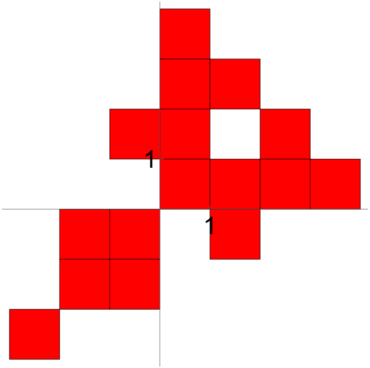


Example of a digit system

$$Q = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}, \quad \mathcal{D} = \{(0,0),(1,0),(0,1),(-1,-1)\}.$$

$$\begin{aligned} Q\mathcal{D} + \mathcal{D} = \{&(0,0),(1,0),(0,1),(-1,-1),\\ &(2,0),(3,0),(2,1),(1,-1)\\ &(0,2),(1,2),(0,3),(-1,1)\\ &(-2,-2),(-1,-2),(-2,-1),(-3,-3)\} \end{aligned}$$





Digit systems: the underlying 'shape'

- expansive endomorphism Q of $\mathbb{Z}^m$
- complete set $\mathcal{D}$ of coset representatives of $\mathbb{Z}^m/Q\mathbb{Z}^m$ called a ***digit set***
- $\mathbb{Z}^m = Q\mathbb{Z}^m + \mathcal{D}$
- Iterate: $\mathcal{D}^{(k)} = Q\mathcal{D}^{(k-1)} + \mathcal{D}$

Digit substitutions

- Let $(Q, \mathcal{D})$ be a digit system and let $\mathcal{A}$ be the ***alphabet***.
- Let $\mathcal{A}^{\mathcal{D}}$ be the set of finite words over $\mathcal{A}$ whose support is $\mathcal{D}$.
- A ***digit substitution*** is a map $\mathcal{S} : \mathcal{A} \to \mathcal{A}^{\mathcal{D}}$.
- The ***n-supertiles*** are defined recursively as $S^{n+1}(\mathsf{a}) = \bigcup_{\vec{d} \in \mathcal{D}} S^n(\mathcal{S}_{\vec{d}}(\mathsf{a})) + Q^n(\vec{d}\,)$.

$$Q = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}, \quad \mathcal{D} = \{(0,0),(1,0),(0,1),(-1,-1)\}$$

$$\mathcal{A} = \{\mathsf{a}, \mathsf{b}\}.$$

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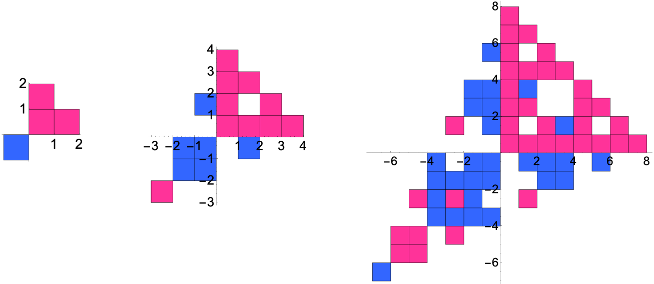
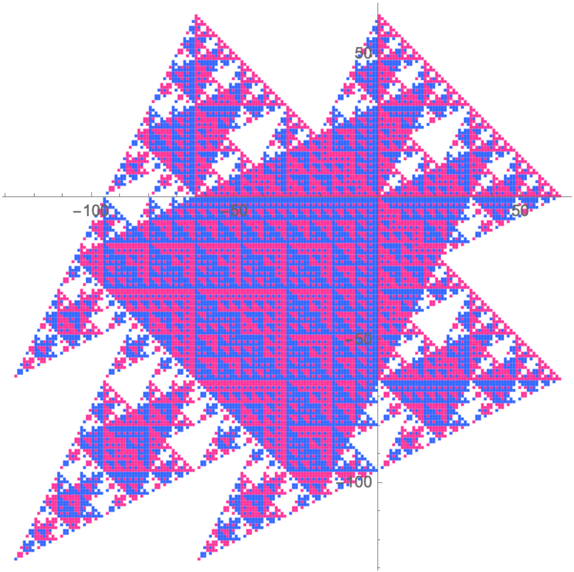
- Let $\mathcal{S}(\mathsf{a})$ take the four digits to a, a, a and b respectively
- Let $\mathcal{S}(\mathsf{b})$ take the digits to b, b, b, and a, the opposite of $\mathcal{S}(\mathsf{a})$. (This type of substitution is called *bijective*).

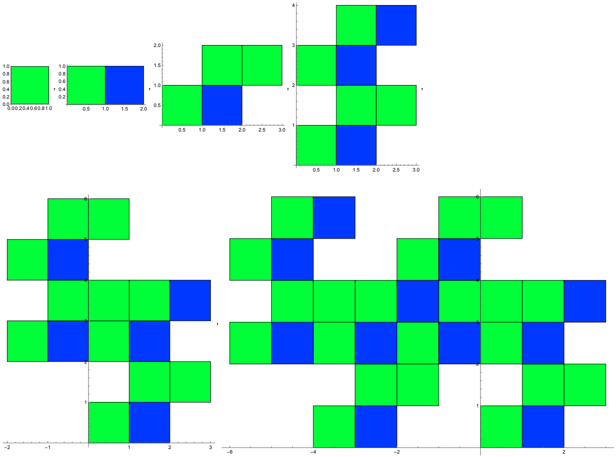




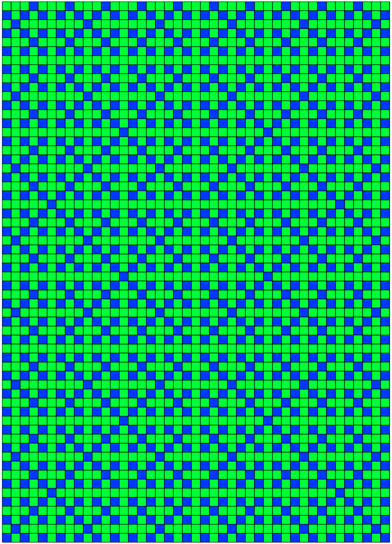
A 2×2 block substituted many times contains a large contiguous region.

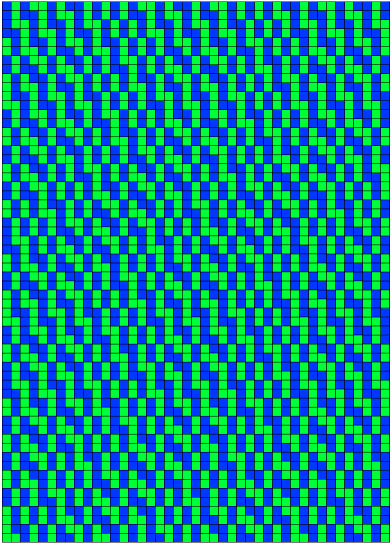
Substitutions based on the "twindragon" fractal

$$Q = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}, \quad \mathcal{D} = \{(0,0),(1,0)\}, \quad S(a) = ab, S(b) = aa.$$



Comparing the PDdragon and TMdragon





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