# Fractal dual substitution tilings

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30th Summer Conference on Topology and its Applications, June 2015

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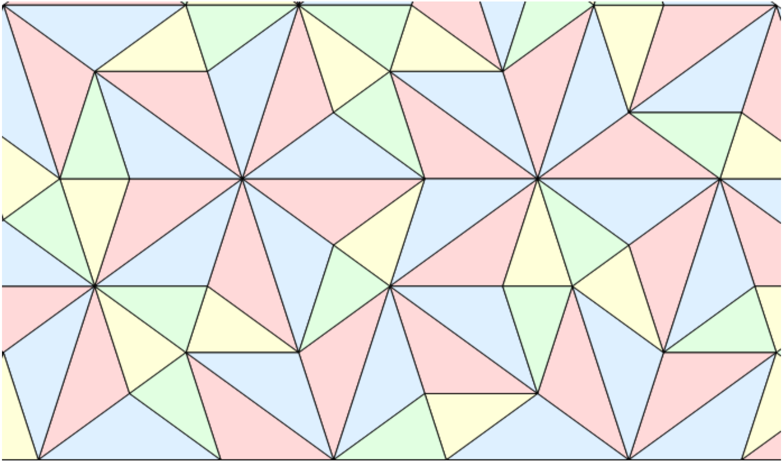
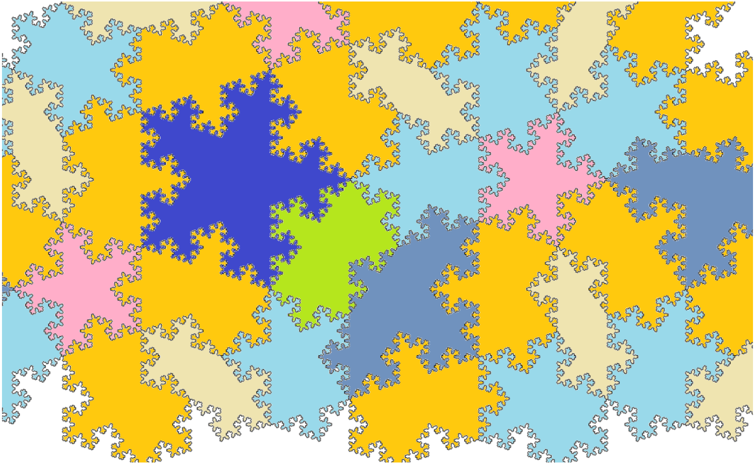




# Overview

- Construction of substitution tilings
  - Famous example: the Penrose tiling
- Construction of fractal dual tilings
  - Combinatorial and geometric graphs
  - Recurrent pairs and edge substitutions
  - Iteration leads to fractal realization of the tiling
- Results and applications

# Substitution tilings

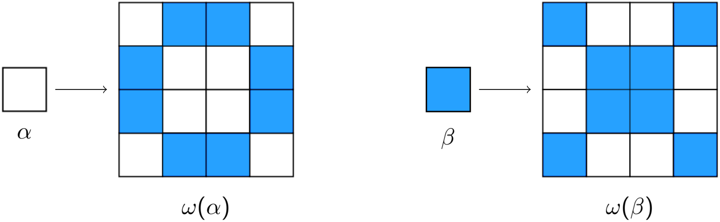
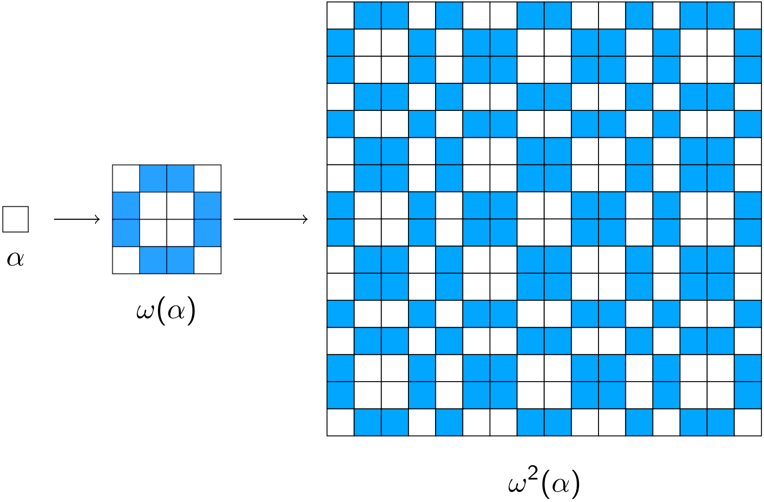


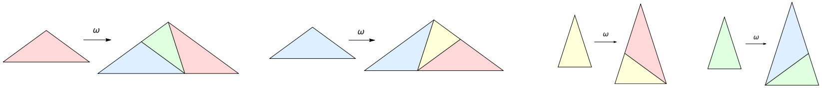

Figure: The two-dimensional Thue-Morse substitution rule.

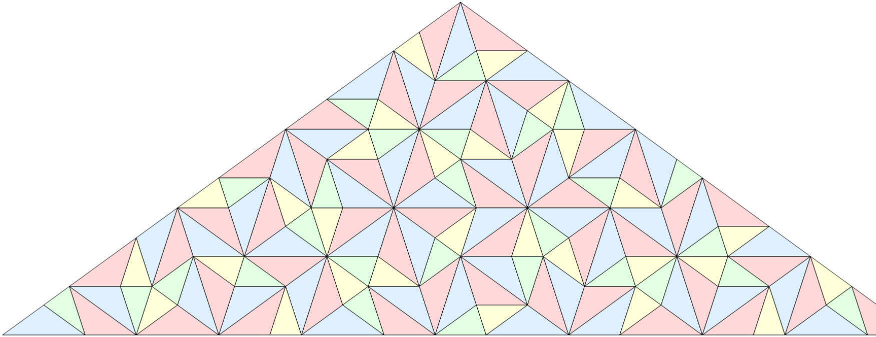
# Iterating the substitution to produce a tiling of $\mathbb{R}^2$



# The Penrose substitution

## Using four types of triangles

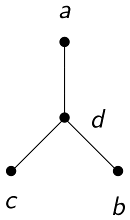


# Combinatorial graphs

## and their topological realizations

- A combinatorial graph $K$ has a finite set of vertices $V(K)$ and an edge set $E(K)$ consisting of two-element subsets of $V(K)$.
  - Ex. $V(K) = \{a, b, c, d\}; E(K) = \{\{a, d\}, \{b, d\}, \{c, d\}\}$
- The topological realisation of $K$ is given by identifying each edge with $[0, 1]$ and gluing together according to $K$.
  - Ex.



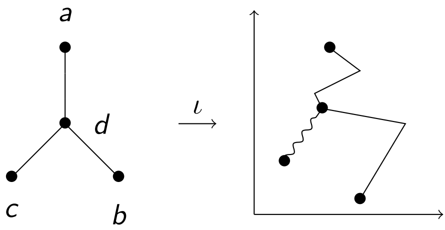

- Our method involves embedding graphs into $\mathbb{R}^2$ in multiple ways.

# Geometric graphs

## Embeddings of combinatorial graphs

- An embedding of $K$ is a continuous injective map $\iota_G$ from the topological realisation of $K$ into $\mathbb{R}^2$.
- A geometric graph $G$ is the image of some embedding $\iota_G$.

Example

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- Two geometric graphs are equivalent if they are embeddings of the same combinatorial graph. Write $G \sim H$.

# Graphs embedded into prototiles

- Let $\mathcal{P}$ be a finite set of prototiles.

  Example. $P =$

$\alpha$ $\beta$

- A geometric graph on $\mathcal{P}$ is a set $\{G_p\}_{p\in\mathcal{P}}$ of geometric graphs with $G_p \subset \operatorname{supp} p$.

  Example. $G =$

$\alpha$ $\beta$

- **Idea:** When you substitute the tiles, you can bring the graphs along to form a new geometric graph on $\mathcal{P}$.
  - From that graph you can try to select a subgraph $G_1$ that is equivalent to $G$.
  - Then you can substitute again, using $G_1$ instead of $G$, and select a subgraph $G_2$ that is equivalent to $G_1$ and $G$.
  - Repeating ad infinitum gives a fractal dual.

# Recurrent pairs

Given substitution $\omega$ with scaling factor $\lambda$, let $\mathcal{R}^N := \lambda^{-N}\omega^N$. We say $(G, S)$ is a recurrent pair if there exists an $N$ for which

- $S \subset \mathcal{R}^N(G)$, and
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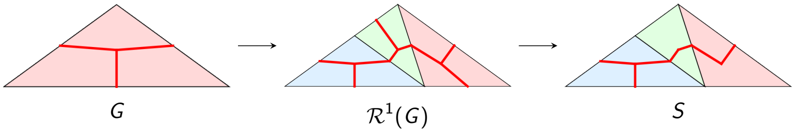
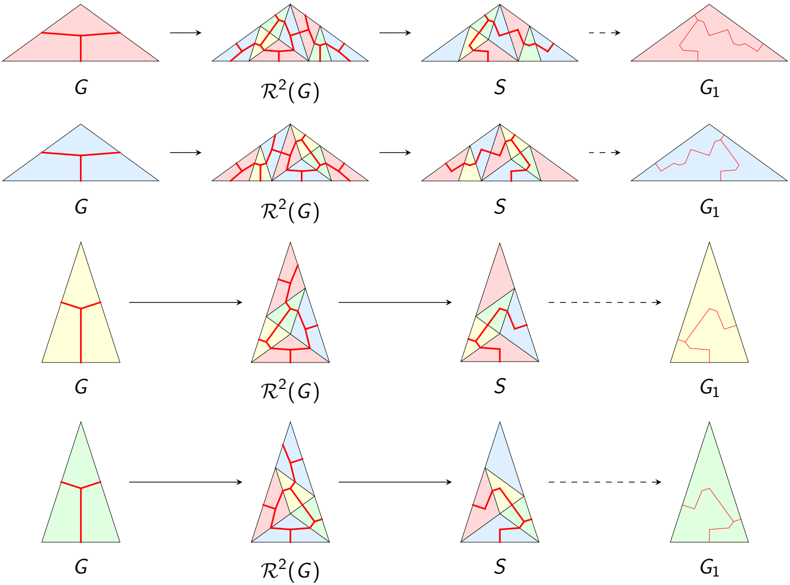
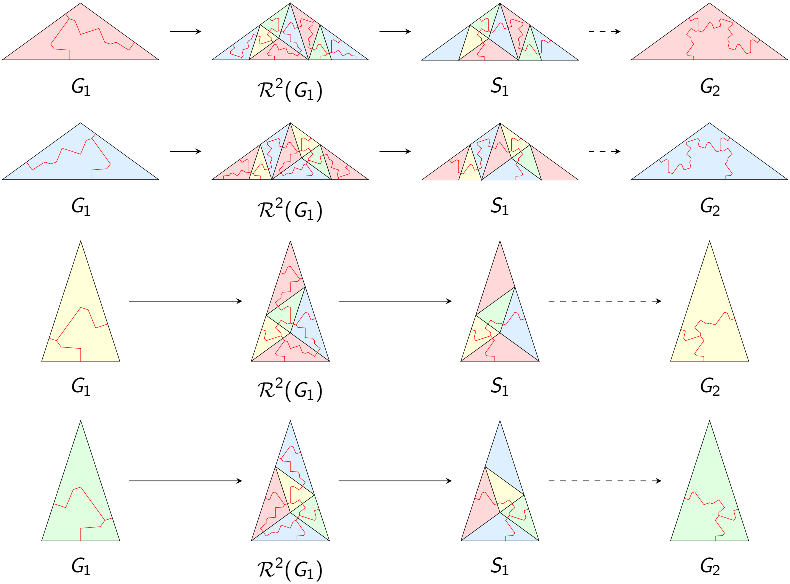



- A recurrent pair $(G, S)$ defines a (graph) iterated function system
- The attractor (or fractal) of that IFS is a geometric graph that forms our new tile boundaries.

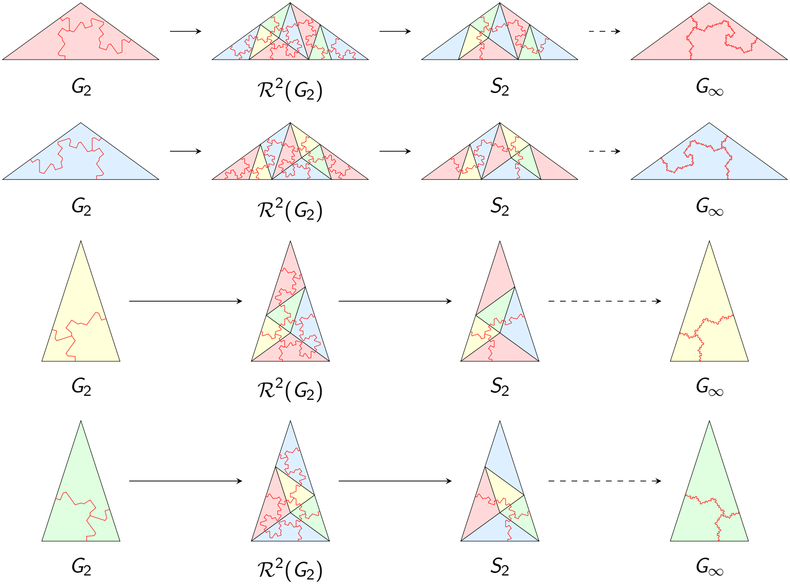
# Example: a $(G, S)$ recurrent pair for the Penrose tiling



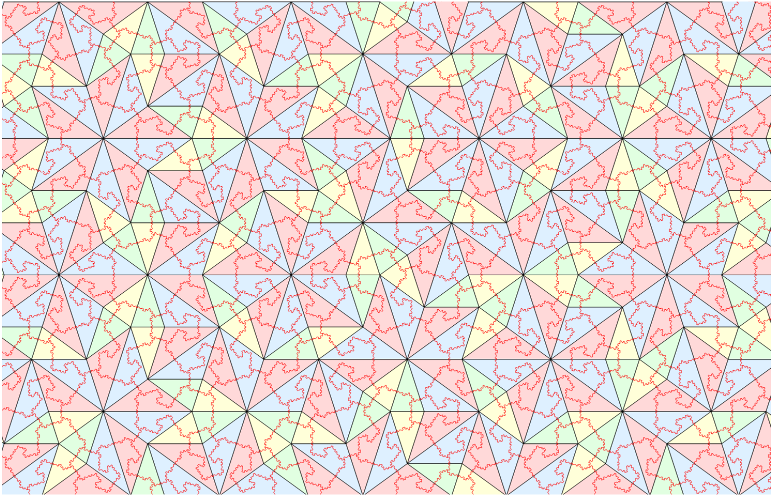
# Example: the first iteration for the Penrose tiling



# Example: the second iteration for the Penrose tiling



# Example: a fractal Penrose tiling



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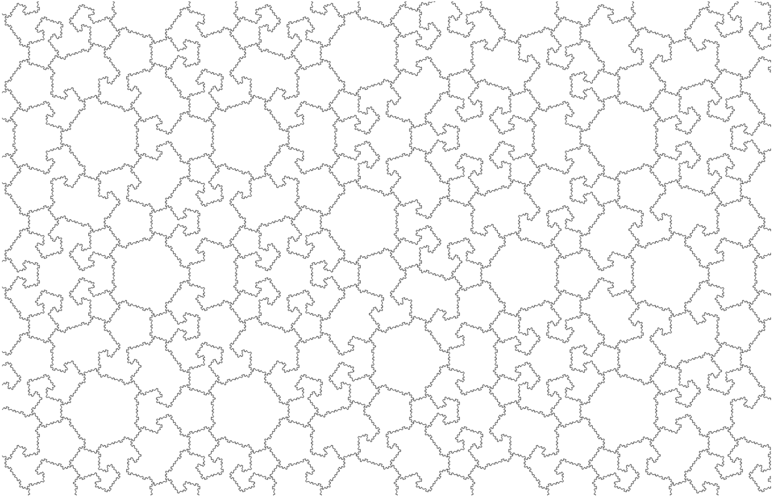

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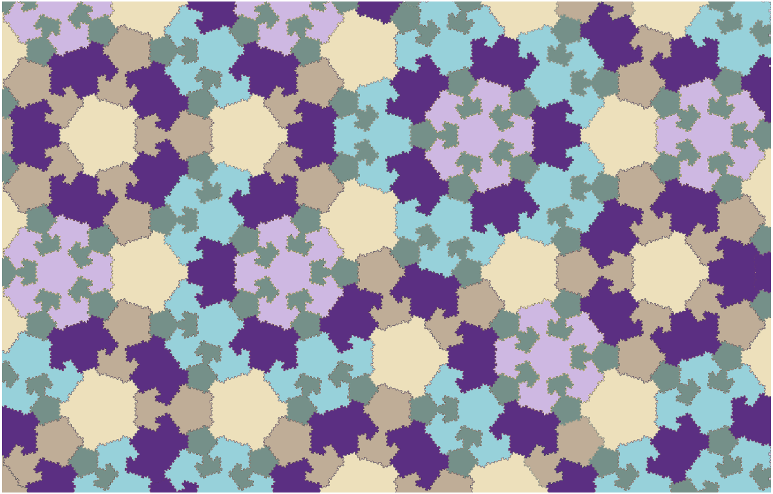
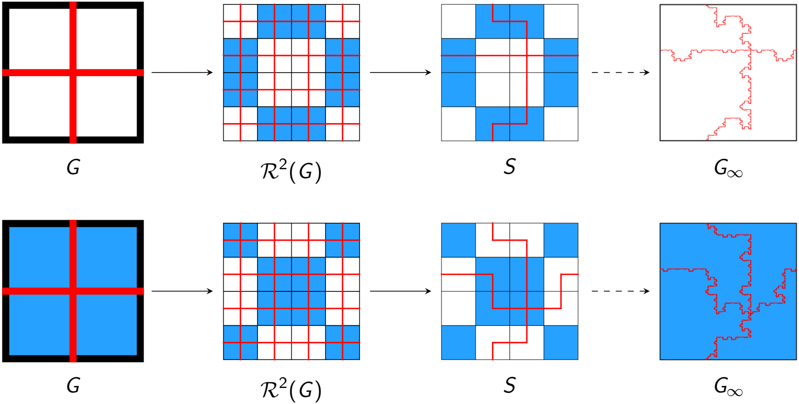


# Example: 2-dimensional Thue-Morse Tiling



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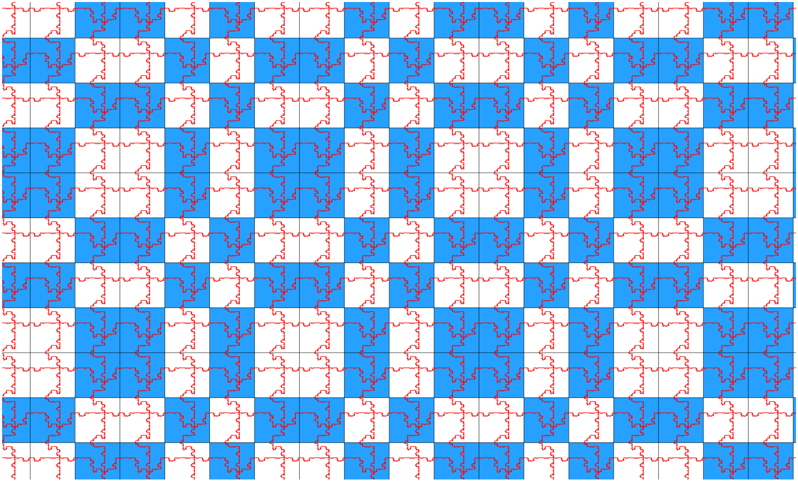
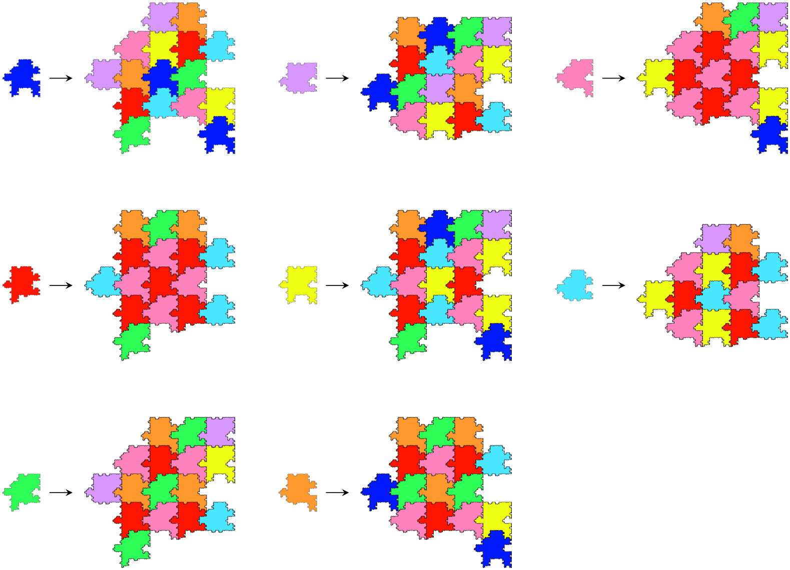


# Example: Fractal Thue-Morse Tiling Substitution



# Results

THEOREM. (F.-Webster-Whittaker '14) If $\psi^{(\infty)}$ is injective, then $\psi^{(\infty)}(\partial T)$ is the boundary of a tiling, denoted $T_\infty$. Moreover, $T_\infty$ is a substitution tiling, is mutually locally derivable from $T$, has FLC, and forces the border.

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# Application: Connes' noncommutative geometry program

THEOREM. (Mampusti-Whittaker '15) Under mild conditions, recurrent pairs give rise to fractal trees that define a geodesic distance on tiles in a tiling. This defines a class of noncommutative spectral triples on the $C^*$-algebra associated with the tiling that respects the hierarchy of the substitution system.

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- Every fractal dual tiling defines a noncommutative Riemannian geometry on the $C^*$-algebra of the original tiling space.

# References


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N. P. Frank, S. B. G. Webster, and M.F. Whittaker, *Fractal dual substitution tilings*, preprint, 2014.

M. Mampusti and M. F. Whittaker, *Fractal spectral triples on Kellendonk's $C^*$-algebra of a substitution tiling*, preprint, 2015.