Generalizations of the Rudin-Shapiro sequence

Natalie Priebe Frank

Vassar College

Aperiodic tilings: A meeting and mathematical art exhibition in honour of Uwe Grimm
Open University, Milton Keynes
June 28, 2022

The “Rudin-Shapiro” Sequence: some history

Quick definitions

$$. + + + - + + - + + + + - - - + - \cdots$$

1. The solution $\{r_n\}_{n>0}$ to the recurrence

$$r_0 = 1, \quad r_{2n} = r_n, \quad r_{2n+1} = (-1)^n r_n$$

2. If $u(n)$ is the occurrence number of “11” in the 2-adic representation of n,

$$r_n = (-1)^{u(n)}$$

3. The factor onto ± 1 of the fixed point of

$$\begin{array}{ll} 1 \to 12 & 2 \to 1\bar{2} \\ \bar{1} \to \bar{1}\bar{2} & \bar{2} \to \bar{1}2 \end{array}$$

Fast facts

- 2-Automatic
- Balanced and non-palindromic
- Has an absolutely continuous spectrum of multiplicity 2
- Satisfies the “root-N property”

Foundational literature on the RS sequence

- H. Shapiro,"Extremal Problems for Polynomials and Power Series", 1951
  - In the context of trigonometric polynomials, see p. 39
- M. Golay, "Statistic multislit spectrometry and its application to the panoramic display of infrared spectra", 1951
- W. Rudin, "Some theorems on Fourier coefficients", 1959
  - Establishes the "root-N property" can be satisfied.
- J. Brillhart and L. Carlitz, "Note on the Shapiro polynomials", 1970
  - The formulation using binary expansion is Theorem 4.
- M. Queffélec, **Substitution Dynamical Systems – Spectral Analysis**, 1987, 2010
  - Extensive documentation of literature pre-1987.

Literature on RS generalizations

- M. Queffélec, “Une nouvelle propriété des suites de Rudin-Shapiro,” 1987
  - Generalization to sequences of roots of unity
- J.-P. Allouche and P. Liardet, “Generalized Rudin-Shapiro Sequences”, 1991
  - Generalization through binary expansion
- N. Frank, “Substitution sequences in $\mathbb{Z}^d$ with a non-simple Lebesgue component in the spectrum”, 2003
  - Uses Hadamard matrices to make rectangular substitutions
- L. Chan and U. Grimm, “Spectrum of a Rudin–Shapiro-like sequence”, 2017; Chan, U. Grimm, and I. Short, “Substitution-based structures with absolutely continuous spectrum”, 2018
  - Uses the root-N property
- N. Mañibo and N. Frank, “Spectral Theory of Spin Substitutions”, 202?
  - Generalizes using digit-based substitutions and abelian groups

Shapiro polynomials: Chan-Grimm-Short method

Trigonometric polynomials and $\sqrt{N}$

Polynomials of the form

$$P_N(z) = \sum_{n<N} a_n z^n \text{ where } ||z|| = 1.$$

Define

$$\|P_N\|_\infty = \sup_{|z|=1} \left| \sum_{n<N} a_n z^n \right|$$

For almost every sequence with $a_n = \pm 1$ it is true that

$$\sqrt{N} \le \|P_N\|_\infty \le \sqrt{N \log N}$$

A question of Raphael Salem, circa 1950

As stated by W. Rudin, "Some theorems on Fourier coefficients", 1959:

Does there exist an absolute constant C such that for all N there exists $\epsilon_1, \ldots, \epsilon_N$ such that

$$\left| \sup_{|x|=1} \sum_{n<N} \epsilon_n x^n \right| \leq C\sqrt{N}?$$

This becomes known as the "root-N" property.

Root-N property IFF absolutely continuous diffraction

(You can deduce from Queffelec's Substitution Dynamical Systems–Spectral Analysis)

The original polynomials

Letting $P_0(x) = Q_0(x) = x$, define

$$P_{k+1}(x) = P_k(x) + x^{2^k} Q_k(x)$$
$$Q_{k+1}(x) = P_k(x) - x^{2^k} Q_k(x)$$

- You can show that the coefficients of P_k are the first 2^k elements of the Rudin-Shapiro sequence.
- Using the parallelogram law

$$|\alpha + \beta|^2 + |\alpha - \beta|^2 = 2|\alpha|^2 + 2|\beta|^2,$$

  the proof of the root-N property falls out of the recursion for $N = 2^k$ and can be extended to any N in a straightforward way.

Chan-Grimm-Short generalizations

A first generalization: Let $P_0(x) = Q_0(x) = x$, and $\sigma_k = \pm 1$:

$$P_{k+1}(x) = P_k(x) + (\sigma_k)x^{2^k}Q_k(x)$$
$$Q_{k+1}(x) = P_k(x) - (\sigma_k)x^{2^k}Q_k(x)$$

- With $\sigma_k == 1$ you get the original
- With $\sigma_k == -1$ you get a different sequence
- In every case, the root-N property is satisfied and so the new sequences generated have absolutely continuous diffraction

Chan-Grimm-Short generalizations

- They define a substitution for $\sigma_k == -1$ analogously to the RS substitution
- For a general sequence of σ_ks, the sequnce becomes S-adic using those two substitutions
- Interesting questions about the relationships between the resulting subshifts are also investigated.

Additionally, a generalization using complex coefficients, more general Shapiro polynomials, and Fourier matrices is shown to maintain the root-N property, producing more examples of sequences with ac spectrum.

Generalization to Z^d using Hadamard matrices

Recall: RS as a substitution

Let $\mathcal{A} = \{1, 2, \bar{1}, \bar{2}\}$ and define

$$\begin{array}{ll} 1 \to 12 & 2 \to 1\bar{2} \\ \bar{1} \to \bar{1}\bar{2} & \bar{2} \to \bar{1}2 \end{array}$$

Things to notice:

1. Only 1s are in the first 'column' and only 2's in the second.
2. The substitution of a barred element is the bar of the substituted element:

$$\mathcal{S}(\bar{1}) = \bar{1}\bar{2} = \overline{12} = \overline{\mathcal{S}(1)}$$

$$\mathcal{S}(\bar{2}) = 1\bar{2} = \overline{\bar{1}2} = \overline{\mathcal{S}(2)}$$

3. A Hadamard matrix appears when we look at the substitution on the unbarred elements: $\begin{pmatrix} + & + \\ + & - \end{pmatrix}$

RS generalization to higher dimensions, F. circa 2002

Recipe:

1. Get an $n \times n$ Hadamard matrix H
   - A matrix of ± 1 with orthogonal rows
2. Make a rectangular array in $\mathbb{Z}^d$ with n total entries
   - each $j \in \{1, ...n\}$ is associated with a spot in this array
3. Make the alphabet $\mathcal{A} = \{1, 2, ..., n, \bar{1}, \bar{2}, ..., \bar{n}\}$.
4. Define the kth spot in $\mathcal{S}(j)$ to be k or $\bar{k}$ depending on whether $H_{jk} = +$ or $-$
5. Define $\mathcal{S}(\bar{j})$ to be $\overline{\mathcal{S}(j)}$

Using the recipe

- Let $H = \begin{pmatrix} + + + - \\ + + - + \\ + - + + \\ - + + + \end{pmatrix}$
- Take the array in $\mathbb{Z}^2$ given by

  $1 = (0,0), 2 = (1,0), 3 = (0,1), 4 = (1,1)$:

3	4
1	2

- The alphabet is then $\mathcal{A} = \{1, 2, 3, 4, \bar{1}, \bar{2}, \bar{3}, \bar{4}\}$.
- $1 \rightarrow$

+	-
+	+

$2 \rightarrow$

-	+
+	+

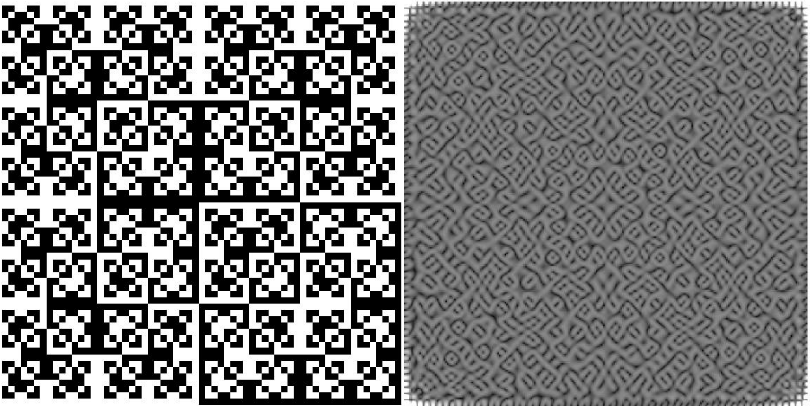
$3 \rightarrow$

+	+
+	-

$4 \rightarrow$

+	+
-	+

The ± 1 factor and its diffraction



Geometric generalization using groups

(Regular) Digit tilings

(Gröchenig/Haas, Lagarias/Wang, Vince, ...)

Ingredients:

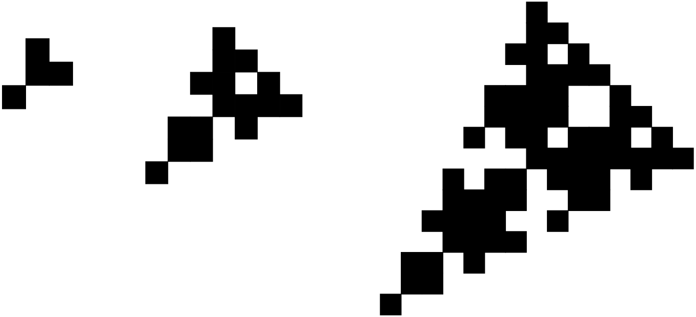
1. a matrix Q that preserves $\mathbb{Z}^d$
2. a full set of coset representatives for $\mathbb{Z}^d/Q\mathbb{Z}^d$ we call $\mathcal{D}$
3. One can think of $\mathcal{D}$ as a 'tile' that tiles $\mathbb{Z}^d$.

We obtain a digit set for Q^n:

$$\mathcal{D}^{(n)} = \left\{ \sum_{k=1}^{n} Q^{k-1} d_k \text{ with } d_1, ..., d_n \in \mathcal{D} \right\}$$

Digit tiling example

$$Q = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}, \quad \mathcal{D} = \{(0,0),(1,0),(0,1),(-1,-1)\}$$



Digit sets for Q, Q^2, and Q^3.

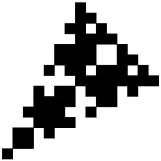
Digit fractile

By iterating and rescaling we get a fractile:

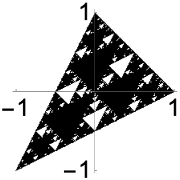
$$\mathfrak{t} = \left\{ \sum_{k=1}^{\infty} Q^{-k} \vec{d}_k \,|\, \vec{d}_k \in \mathcal{D} \right\} = \lim_{k \to \infty} Q^{-k} \mathcal{D}^{(k)}. \qquad (1)$$

How big is this fractile? Is it a rep-tile?

A. Vince has a survey paper condensing results into a 10-point theorem for when the fractile has volume 1.



$\longrightarrow \cdots$



(The tile in our example does have volume 1)

Digit substitutions and their subshifts

Recipe:

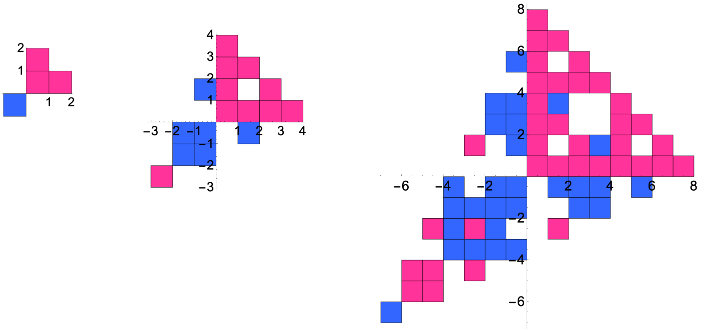
- Get a digit tiling system $(Q, \mathcal{D})$
- Pick a (finite) alphabet $\mathcal{A}$
- Define a substitution rule as $\mathcal{S} : \mathcal{A} \to \mathcal{A}^{\mathcal{D}}$ however you like
- Use the supertiles $\mathcal{S}^{(k)}(a)$ to create a 'language'
- If there are sequences in $\mathbb{Z}^d$ that are allowed by this language, you have a substitution subshift $(\Sigma, \mathbb{Z}^d)$.
  - This will happen if the $\mathcal{D}^{(n)}$s contain arbitrarily large rectangles

Using the recipe

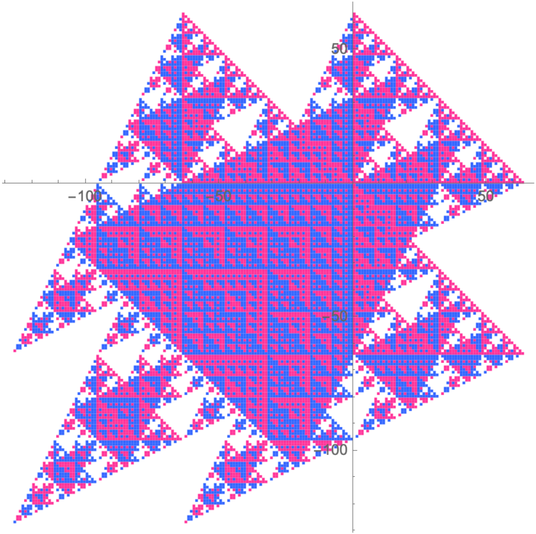
- Using our previous digit system

$$Q = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}, \quad \mathcal{D} = \{(0,0),(1,0),(0,1),(-1,-1)\}$$

- Let $\mathcal{A} = \{\mathsf{a}, \mathsf{b}\} = \{pink, blue\}$
- Let $\mathcal{S}(pink)$ assign the digits to $pink, pink, pink, blue$
- Let $\mathcal{S}(blue)$ be the opposite



Our example makes a subshift



RS generalization: spin digit substutions

RS as a spin substitution

Let $G = C_2 = \{e, g\}$ and consider the alphabet to be digits that have a 'spin' given by elements of G:

$$\mathcal{A} = \{1, 2, \bar{1}, \bar{2}\} = \{e1, e2, g1, g2\},$$

$$\begin{array}{ll} e1 \to e1e2 & e2 \to e1g2 \\ g1 \to g1g2 & g2 \to g1e2 \end{array}$$

The matrix W allocating the spins in this notation is $\begin{pmatrix} e & e \\ e & g \end{pmatrix}$

Spin digit substitutions

Recipe:

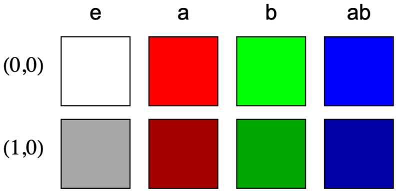
- Get a digit system $(Q, \mathcal{D})$
- Get a finite abelian group G
- Make the alphabet

$$\{g1, g2, ..., g|\mathcal{D}| \text{ such that } g \in G\},$$

which has $|G||\mathcal{D}|$ elements.
- Make a $|\mathcal{D}| \times |\mathcal{D}|$ matrix W with entries from G
- Use the rows of W to distribute the spins for the substitution of the spin-free letters $e1, e2, ..., e|\mathcal{D}|$
- Define $\mathcal{S}(gd) = g\mathcal{S}(d)$ for the rest of the alphabet.

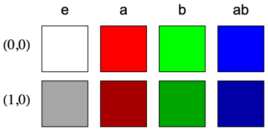
Example: Vierdrachen substitution

- Let $Q = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$ with $\mathcal{D} = \{(0,0),(1,0)\} = \{1,2\}$
    - this digit system makes the 'twindragon' fractile.
- Let $G = C_2 \times C_2 = \{e, a, b, ab\}$
- The alphabet as square tiles.



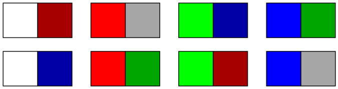
- Let $W = \begin{pmatrix} e & a \\ e & ab \end{pmatrix}$.

Example: Vierdrachen substitution

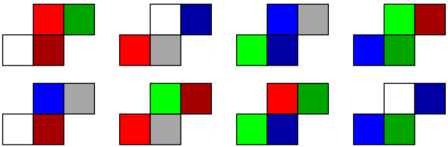


$$W = \begin{pmatrix} e & a \\ e & ab \end{pmatrix}.$$

Level-1 supertiles:

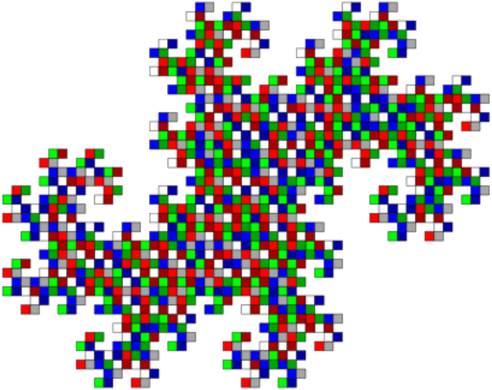


Level-2 supertiles:



Example: Vierdrachen substitution

Level-10 supertile $\mathcal{S}^{10}(\mathsf{d}_0)$ corresponding to d_0:



Fast look at spectral theory

- Given a $\mathbb{Z}^d$ subshift Σ with invariant measure μ, let $H = L^2(\Sigma, \mu)$.
- Each $f \in H$ has a spectral measure associated with it
- That measure can be pure point, singular wrt Lebesgue but continuous, absolutely continuous wrt Lebesgue, or a combination
- These measures, taken together, reveal something about the structure of the subshift

Breaking the spectrum down

- Let $U^g : H \to H$ be given by

$$U^g(f(\mathcal{T})) = f(g\mathcal{T})$$

- Let $\chi : G \to S^1$ be a group character
- let H^χ be the eigenspace of functions $U^g(f) = \chi(g)f$

Proposition (F.–Mañibo, '21)

Let $\mathcal{S} = (Q, \mathcal{D}, G, W)$ be a primitive spin substitution and Σ be the subshift it generates. Suppose further that Σ is fully aperiodic. Then

$$L^2(\Sigma, \mu) = \bigoplus_{\chi \in \widehat{G}} H^\chi$$

Spectral purity

Corollary (F.–Mañibo, '21)

Let $\mathcal{S} = (Q, \mathcal{D}, G, W)$ be a primitive spin substitution and Σ be the subshift it generates. Suppose further that Σ is fully aperiodic. Consider the decompositions

$$H_{\mathsf{pp}} \oplus H_{\mathsf{ac}} \oplus H_{\mathsf{sc}} = L^2(\Sigma, \mu) = \bigoplus_{\chi \in \widehat{G}} H^{\chi}.$$

Each H^{χ} is spectrally pure, i.e., for a fixed χ, $H^{\chi} \subset H_{\alpha}$ where $\alpha \in \{\mathsf{pp}, \mathsf{ac}, \mathsf{sc}\}$.

Characterizing the spectrum

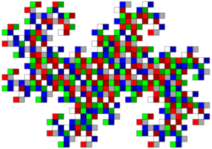
Theorem (F.–Mañibo, '21)

Let $\mathcal{S} = (Q, \mathcal{D}, G, W)$ be a primitive spin substitution and Σ be the subshift it generates. Suppose further that Σ is fully aperiodic. Let $\chi(W) := (\chi(W_{ij}))$

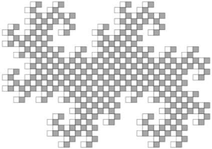
1. χ *trivial* $\implies$ H^χ *pure point*
2. $\frac{1}{\sqrt{|\mathcal{D}|}}\chi(W)$ *unitary* $\implies$ H^χ *is purely absolutely continuous*
3. $\chi(W)$ *rank-1* $\implies$ H^χ *is singular (either pure point or purely singular continuous)*

- Unitarity $\implies$ zero spectral coefficients for $\vec{j} \neq 0$
- Rank-1 $\implies$ χ induces a factor onto a substitution with singular spectrum

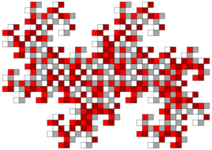
Example: Vierdrachen



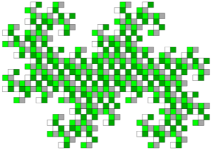
$\mathcal{S}^9(\mathsf{d}_0)$



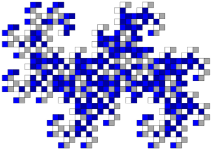
χ_0 (pp)



χ_1 (ac)



χ_2 (sc)



χ_3 (ac)