Introduction to hierarchical tiling dynamical systems

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Lecture 3: Ergodic theory and spectral analysis

Transition (a.k.a. incidence, substitution, abelianization, or subdivision) matrices

Transition matrices

- In an abuse of terminology we use tiling terminology to refer to all the constructions we've discussed.
- Assume the prototile set has been given some arbitrary order $\mathcal{P} = \{p_1, ...p_{|\mathcal{P}|}\}$.
- Then the transition matrix for $\mathcal{S}$ is the $|\mathcal{P}| \times |\mathcal{P}|$ matrix M whose (i,j) entry M_{ij} is the number of tiles of type p_i in $\mathcal{S}(p_j)$.

A transition matrix

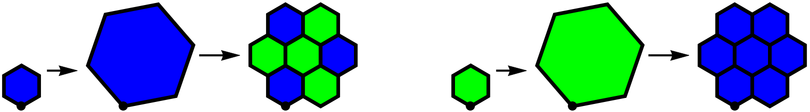
M_{ij} is the number of tiles of type p_i in $\mathcal{S}(p_j)$:

$$a \to abbb \qquad b \to a$$

has transition matrix

$$M = \begin{pmatrix} 1 & 1 \\ 3 & 0 \end{pmatrix}$$

Another transition matrix



$$M = \begin{pmatrix} 3 & 7 \\ 4 & 0 \end{pmatrix}$$

Transition matrices count tiles

$$a \to abbb \qquad b \to a$$

Suppose I want to know how many tiles of each type there are in $\sigma^3(a)$.

In $\sigma(a)$ we get:

$$\begin{pmatrix} 1 & 1 \\ 3 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$$

In $\sigma^2(a)$ we get:

$$\begin{pmatrix} 1 & 1 \\ 3 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 3 \end{pmatrix} = \begin{pmatrix} 4 \\ 3 \end{pmatrix}$$

In $\sigma^3(a)$ we get:

$$\begin{pmatrix} 1 & 1 \\ 3 & 0 \end{pmatrix} \begin{pmatrix} 4 \\ 3 \end{pmatrix} = \begin{pmatrix} 7 \\ 12 \end{pmatrix}$$

Transition matrices for general fusions

- The *transition matrix* $M_{n-1,n}$ is the $|\mathcal{P}_{n-1}| \times |\mathcal{P}_n|$ matrix whose (k, l) entry is the number of supertiles equivalent to $P_{n-1}(k)$ (i.e. the number of $(n-1)$ supertiles of type k) in the supertile $P_n(l)$.
- If there is more than one fusion of $\mathcal{P}_{n-1}$-supertiles that can make $P_n(l)$, we fix a preferred one to be used in all computations.

Transition matrices for the 10^n fusion

$$P_n(a) = (P_{n-1}(a))^{10^n} P_{n-1}(b) \qquad P_n(b) = (P_{n-1}(b))^{10^n} P_{n-1}(a)$$

has transition matrix

$$M_{n-1,n} = \begin{pmatrix} 10^n & 1 \\ 1 & 10^n \end{pmatrix}$$

Transition matrices for general fusions

- Notice that the matrix product $M_{n,N} = M_{n,n+1}M_{n+1,n+2}\cdots M_{N-1,N}$ is well-defined when $N > n$.
- The entries of $M_{n,N}$ reveal the number of n-supertiles of every type of N-supertile.

Repetitivity and minimality

Definition

A general dynamical system is called *transitive* if there is a dense orbit and *minimal* if every orbit is dense.

Definition

A tiling $\mathcal{T}$ is said to be *repetitive* if for every finite patch P in $\mathcal{T}$ there is an $R = R(P) > 0$ such that $\mathcal{T} \cap B_R(x)$ contains a translate of P for every $x \in \mathbb{R}^d$.
It is *linearly repetitive* if there is a $C > 0$ such that for any $\mathcal{T}$-patch P there is a translate of P in any ball of radius $C\, diam(P)$ in $\mathcal{T}$.

The tiling space of a repetitive tiling

Lemma

Let $\mathcal{T} \in \Omega_{\mathcal{P}}$ and let $\Omega_{\mathcal{T}}$ denote its hull under the translation group G. The tiling dynamical system $(\Omega_{\mathcal{T}}, G)$ is minimal if and only if $\mathcal{T}$ is repetitive.

(Versions of this work for both symbolic and tiling dynamical systems.)

Primitivity

Definition

A symbolic or tiling substitution rule is defined to be *primitive* if and only if there is an N such that all of the entries of M^N are strictly positive.

A fusion rule is defined to be *primitive* if and only if for every $n \in \mathbb{N}$ there exists an N such that the entries of $M_{n,n+N}$ are strictly positive.

A fusion rule is *strongly primitive* if there is an $N \in \mathbb{N}$ such that for all $n \in \mathbb{N}$ the entries of $M_{n,n+N}$ are strictly positive.

General result: Primitivity implies minimality

Theorem ([**Kamae**, **Queffelec**] for symbolic substitutions, [**Praggastis**] for self-affine tilings, [**Frank.Sadun.fusion**] for fusions)

Let Ω be the space of tilings allowed by a supertile construction and let G be its group of translations. If the supertile construction is primitive, then (Ω, G) is minimal.

General result: Bounded fusion systems are not strongly mixing

Theorem ([Frank.Sadun.fusion])

The dynamical system of a strongly primitive fusion rule with a bounded number of supertiles at each level and bounded transition matrices, and with group $G = \mathbb{Z}^d$ or $\mathbb{R}^d$, cannot be strongly mixing.

Constructing a strongly mixing fusion requires a certain degree of unboundedness, either in the number of supertiles at each level or in the entries of the transition matrices between consecutive levels.

Ergodic measures for general tiling spaces

Definition

A Borel probability measure μ on Ω is said to be *invariant* with respect to translation if $\mu(A - g) = \mu(A)$ for all Borel measurable sets A and all $g \in G$.

Definition

We say μ is *ergodic* with respect to translation if whenever A is a translation-invariant set, then $\mu(A)$ equals 0 or 1.

Recall from lecture 1: Cylinder sets

- Let P be a $\mathcal{P}$-patch, let $U \subset \mathbb{R}^d$, and let $\Omega \subset \Omega_{\mathcal{P}}$ be a tiling space.
- $\Omega_{P,U} = \{\mathcal{T} \in \Omega \text{ such that } P - u \subset \mathcal{T} \text{ for some } u \in U\}$
- $\Omega_{P,U}$ is the set of all tilings in Ω that contain a copy of P translated by an element of U.

Ergodic measures and frequency

Let P be a $\mathcal{P}$-patch and let U be a fixed and very small ball

Denote by $\mathbb{I}_{P,U}$ the indicator function for $\Omega_{P,U}$

Let μ be some ergodic measure for translation. From elementary measure theory along with the ergodic theorem we know that for μ-almost every $\mathcal{T}_0 \in \Omega$,

$$\begin{aligned}\mu(\Omega_{P,U}) &= \int_\Omega \mathbb{I}_{P,U}(\mathcal{T})d\mu(\mathcal{T}) \\ &= \lim_{r\to\infty} \frac{1}{Vol(B_r(0))} \int_{B_r(0)} \mathbb{I}_{P,U}(\mathcal{T}_0 - x)dx\end{aligned}$$

Ergodic measures and frequency

$\mu(\Omega_{P,U}) = \lim_{r\to\infty} \frac{1}{Vol(B_r(0))} \int_{B_r(0)} \mathbb{I}_{P,U}(\mathcal{T}_0 - x)dx$

- For every copy of P in $\mathcal{T}_0 \cap B_r(0)$ that isn't too close to the boundary of $B_r(0)$ the indicator function will be 1 over a set of size $Vol(U)$
- Patches on the boundary of $B_r(0)$ can also contribute, but as $r \to \infty$ their effect is negligible.
- The term on the right becomes $\lim_{r\to\infty} \frac{\#(P \in \mathcal{T}_0 \cap B_r(0))}{Vol(B_r(0))} Vol(U)$.
- For μ-almost every $\mathcal{T}_0$ we get the same answer and so we can say that μ defines a frequency measure on the set of $\mathcal{P}$-patches as:

$$\mu(\Omega_{P,U}) = Vol(U) freq_\mu(P).$$

An extemely useful theorem

Theorem (Perron-Frobenius Theorem)

Let M be an irreducible matrix.

- *Then M has positive left and right eigenvectors $\vec{l}$ and $\vec{r}$ with corresponding eigenvalue $\theta > 0$ that is both geometrically and algebraically simple.*
- *If θ' is another eigenvalue for M then $|\theta'| \leq \theta$. Any positive left or right eigenvector for M is a multiple of $\vec{l}$ or $\vec{r}$.*
- *If M is primitive and $\vec{l}$ and $\vec{r}$ are normalized so that $\vec{l} \cdot \vec{r} = 1$, it is true that $\lim_{n\to\infty} \frac{M^n}{\theta^n} = \vec{r}\vec{l}$.*

Volumes and frequencies for self-affine tilings

Corollary (Corollary 2.4 of [**Solomyak**])

Suppose $\mathcal{T}$ is a self-affine tiling with expansion map ϕ for which the transition matrix M is primitive. Then the Perron eigenvalue of M is $|det\phi|$. Writing the prototile set as $\mathcal{P} = \{p_1, p_2, ..., p_m\}$, the left eigenvector can be obtained by $\vec{l} = (Vol(p_j))_{j=1}^m$. Moreover,

$$\lim_{n\to\infty} |det\phi|^{-n} M_{ij}^n = r_i Vol(p_j).$$

The n-supertile frequencies are given by $\frac{1}{|det(\phi)|^n}\vec{r}$ and we can get the frequencies of everything else from that information.

Case study: ergodic measures for constant-length $\mathbb{Z}^d$ substitutions

If $\mathcal{S}$ is a primitive, nonperiodic substitution with size $l_1 \cdot l_2 \cdots l_d = K$ and ϕ is its natural expanding map, the corollary implies that

- The largest eigenvalue of M must be equal to $|\det \phi| = K$.
- The left Perron eigenvector $\vec{l}$ must the the tile volumes, which are all 1.
- This implies that $\sum_{i=1}^{|\mathcal{A}|} r_i = 1$ and that
-

$$\lim_{n\to\infty} K^{-n} M^n = \begin{pmatrix} r_1 & r_1 & \cdots & r_1 \\ r_2 & r_2 & \cdots & r_2 \\ \vdots & \vdots & & \vdots \\ r_{|\mathcal{A}|} & r_{|\mathcal{A}|} & \cdots & r_{|\mathcal{A}|} \end{pmatrix}.$$

Case study, continued

- We can compute frequencies by looking at $\mathcal{S}^n(a_1)$ as $n \to \infty$.
- Use the notation $N_{a_i}(B)$ to denote the number of occurrences of the letter a_i in a block B.
- Note that K^n is the volume of the substituted block $\mathcal{S}^n(a_1)$
- $$freq(a_i) = \lim_{n \to \infty} \frac{N_{a_i}(\mathcal{S}^n(a_1))}{K^n}$$
- But $N_{a_i}(\mathcal{S}^n(a_1)) = M^n_{i1}$,
- So $freq(a_i) = r_i$.

Lemma

Let $\mathcal{S}$ be a primitive and nonperiodic substitution of constant length $l_1 \cdot l_2 \cdots l_d = K$ in $\mathbb{Z}^d$. Then M has the property that $\sum_{j=1}^{|\mathcal{A}|} M_{ij} = K$ for all $i \in 1, 2, ...|\mathcal{A}|$ if and only if the frequency of any letter $a_i \in \mathcal{A}$ is $1/|\mathcal{A}|$.

Proof.

If $\sum_{j=1}^{|\mathcal{A}|} M_{ij} = K$, then a right eigenvector for M is given by $\vec{r} = (1/|\mathcal{A}|, ..., 1/|\mathcal{A}|)$. Since $\vec{l} \cdot \vec{r} = 1$ it must be the (unique) right Perron eigenvector for M. Since $freq(a_i) = r_i$ the result follows.

Conversely, the vector $\vec{r}$ defined as above again is a right eigenvector and we have that

$$(M\vec{r})_i = \sum_{j=1}^{|\mathcal{A}|} M_{ij}/|\mathcal{A}| = (K\vec{r})_i = K/|\mathcal{A}|$$

and the result follows. □

A proposition

Proposition *Let $\mathcal{S}$ be a primitive and nonperiodic substitution of constant length $l_1 \cdot l_2 \cdots l_d = K$ in $\mathbb{Z}^d$. Then M has the property that $\sum_{j=1}^{|\mathcal{A}|} M_{ij} = K$ for all $i \in 1, 2, ...|\mathcal{A}|$ if and only if the frequency of any letter $a_i \in \mathcal{A}$ is $1/|\mathcal{A}|$.*

Proof.
If $\sum_{j=1}^{|\mathcal{A}|} M_{ij} = K$, then a right eigenvector for M is given by $\vec{r} = (1/|\mathcal{A}|, ..., 1/|\mathcal{A}|)$. Since $\vec{l} \cdot \vec{r} = 1$ it must be the (unique) right Perron eigenvector for M. Since $freq(a_i) = r_i$ the result follows. Conversely, the vector $\vec{r}$ defined as above again is a right eigenvector and we have that

$$(M\vec{r})_i = \sum_{j=1}^{|\mathcal{A}|} M_{ij}/|\mathcal{A}| = (K\vec{r})_i = K/|\mathcal{A}|$$

and the result follows. □

Corollary

If $\mathcal{S}$ is a primitive, nonperiodic, bijective substitution of constant length in $\mathbb{Z}^d$, then the frequency of any letter $a_i \in \mathcal{A}$ is $1/|\mathcal{A}|$.

Sketch.

The row sum for row i is the number of times we see a_i in the substitution of any tile. Because the substitution is bijective, for any given location in the substitution we know that a_i appears exactly once. That means that the number of times a_i appears is the number of spots in the substitution, which is K. □

Ergodic measures for general fusions

Well-defined supertile frequencies

- By recognizability we know that every tiling $\mathcal{T} \in \Omega$ has a unique n-supertiling $\mathcal{T}_n \in \Omega_n$.
- The frequency of the n-supertile $P_n(j) \in \mathcal{P}_n$ *as an n-supertile* is denoted $\tilde{f}_{P_n(j)}$.
- We know that $P_n(j)$ as a patch of ordinary tiles may have a larger frequency $\bar{f}_{P_n(j)}$ in $\mathcal{T}$.
- Let $\rho_n = (\rho_n(1), ..., \rho_n(j_n)) \in \mathbb{R}^{j_n}$ represent the n-supertile frequencies.

Definition

Let ρ be a sequence of vectors $\{\rho_n\}$ described above. We say that ρ is *volume-normalized* if for all n we have $\sum_{i=1}^{j_n} \rho_n(i) Vol(P_n(i)) = 1$. We say that it has *transition consistency* if $\rho_n = M_{n,N}\rho_N$ whenever $n < N$. A transition-consistent sequence ρ that is normalized by volume is called a sequence of *well-defined supertile frequencies*.

Measures from well-defined supertile frequencies

Theorem

[**Frank.Sadun.fusion**] *Let $\mathcal{R}$ be a recognizable, primitive, van Hove fusion rule with tiling dynamical system $(\Omega, \mathbb{R}^d)$. There is a one-to-one correspondence between the set of all invariant Borel probability measures on $(\Omega, \mathbb{R}^d)$ and the set of all sequences of well-defined supertile frequencies with the correspondence that, for all patches P,*

$$freq_\mu(P) = \lim_{n\to\infty} \sum_{i=1}^{j_n} \#\left(P \ in \ P_n(i)\right) \rho_n(i) \tag{1}$$

Dynamical spectrum

The Koopman operator

- Let (Ω, G, μ), with G our translations and μ an ergodic probability measure.
- For each $\vec{\jmath} \in G$ there is a unitary operator $U^{\vec{\jmath}} : L^2(\Omega, \mu) \to L^2(\Omega, \mu)$ defined by

$$U^{\vec{\jmath}}(f)(\mathcal{T}) = f(\mathcal{T} - \vec{\jmath}).$$

- The spectrum of the Koopman operator is called the *dynamical spectrum* of Ω.

The spectral measure of $f \in L^2(\Omega, \mu)$

- For $\vec{\jmath} \in G$ we define $\hat{f}(\vec{\jmath}) = \int_\Omega f(\mathcal{T} - \vec{\jmath})\overline{f(\mathcal{T})}d\mu(\mathcal{T})$.
- There is a positive real-valued measure σ_f on $\mathbb{T}^d$ with these Fourier coefficients.
- Letting $z^{\vec{\jmath}} := z_1^{j_1} \cdots z_d^{j_d}$ we have

$$\hat{f}(\vec{\jmath}) = \int_\Omega f(\mathcal{T} - \vec{\jmath})\overline{f(\mathcal{T})}d\mu(\mathcal{T}) = \int_{\mathbb{T}^d} z^{\vec{\jmath}} d\sigma_f(z).$$

Spectral breakdown of L^2

- Each $f \in L^2(\Omega, \mu)$ generates a cyclic subspace of L^2:

$$Z(f) = \overline{\text{span}\left\{U^{\vec{\jmath}} f \text{ such that } \vec{\jmath} \in G\right\}}$$

- We try to find generating functions $f_i, i = 1, 2, \ldots$ for which

$$L^2(\Omega, \mu) = \bigoplus Z(f_i).$$

- It is possible to find functions $f_1, f_2, f_3, \ldots$ such that $L^2(\Omega, \mu) = \bigoplus Z(f_i)$ and for which $\sigma_{f_1} \gg \sigma_{f_2} \gg \sigma_{f_3} \gg \ldots$.
- The spectral type of f_1 is known as the *maximal spectral type* of the system.

Eigenfunctions

- We say f is an eigenfunction if there exists an $\vec{\alpha} \in \mathbb{R}^d$ for which $U^{\vec{\jmath}} f = \exp(2\pi i \vec{\alpha} \cdot \vec{\jmath}) f$ for all $\vec{\jmath} \in G$.
- That is, for all $\mathcal{T} \in \Omega$ and all $\vec{\jmath} \in G$ we find $f(\mathcal{T} - \vec{\jmath}) = \exp(2\pi i \vec{\alpha} \cdot \vec{\jmath}) f(\mathcal{T})$.

Eigenfunctions make point measures

Example

Let's compute the spectral measure of an eigenfunction f of (Ω, μ) with eigenvalue α. For $\vec{\jmath} \in G$ we have

$$\hat{f}(\vec{\jmath}) = \int_\Omega \exp(2\pi i \alpha \cdot \vec{\jmath}) \overline{f(\mathcal{T})} f(\mathcal{T}) d\mu(\mathcal{T}) = \exp(2\pi i \alpha \cdot \vec{\jmath}),$$

since eigenfunctions are of almost everywhere constant modulus that can be taken to be 1.
Thus the spectral measure of f is a measure σ_f on $\mathbb{T}^d$ with these Fourier coefficients.
The measure on $\mathbb{T}^d$ with these coefficients is the atomic measure supported on α, and so $\sigma_f = \delta_\alpha$.

Conditions for the presence of discrete spectrum

Theorem (Theorem 5.1 of [Solomyak])

(i) Let $\mathcal{T}$ be a nonperiodic self-affine tiling of $\mathbb{R}^d$ with expansion map ϕ. Then $\alpha \in \mathbb{R}^d$ is an eigenvalue of the measure-preserving system (Ω, μ) if and only if

$$\lim_{n\to\infty} e^{2\pi i(\phi^n(x)\cdot\alpha)} = 1 \quad \textit{for all} \quad x \in \Xi(\mathcal{T}). \tag{2}$$

Moreover, if this equation holds, the eigenfunction can be chosen continuous.
(ii) Let $\mathcal{T}$ be a self-similar tiling of $\mathbb{R}$ with expansion constant λ. The tiling dynamical system is not weakly mixing if and only if $|\lambda|$ is a real Pisot number. If λ is real Pisot and $\mathcal{T}$ is nonperiodic, there exists nonzero $a \in \mathbb{R}$ such that the set of eigenvalues contains $a\mathbb{Z}[\lambda^{-1}]$.

[Theorem 5.1 of [**Solomyak**]], continued.

Theorem

(iii) Let $\mathcal{T}$ be a nonperiodic self-similar tiling of $\mathbb{R}^2 \equiv \mathbb{C}$ with expansion constant $\lambda \in \mathbb{C}$. Then the tiling dynamical system is not weakly mixing if and only if λ is a complex Pisot number. Moreover, if λ is a non-real Pisot number there exists nonzero $a \in \mathbb{C}$ such that the set of eigenvalues contains $\{(\alpha_1, \alpha_2) : \alpha_1 + i\alpha_2 \in a\mathbb{Z}[\lambda^{-1}]\}$.

Diffraction

Delone multiset for the tiling $\mathcal{T}$

- Let $\mathcal{T}$ be a tiling with prototile set $\mathcal{P} = \{p_1, ..., p_m\}$
- Mark a special point in p_i for each $i = 1, .., m$
- Let Λ_i be the Delone set of all copies of that point in $\mathcal{T}$.
- $\mathbf{\Lambda} = \Lambda_1 \times \Lambda_2 \times \cdots \times \Lambda_m$

Dirac comb for the tiling

- $\mathbf{\Lambda}$ represents our set of scatterers from $\mathcal{T}$ and we have kept track of the type of each scatterer.
- Choose scattering strengths $a_i \in \mathbb{C}$ for $i \leq m$.
- The Dirac delta function at x thought of as a probability measure with support x is δ_x.
- We have the *weighted Dirac comb*

$$\omega = \sum_{i \leq m} a_i \delta_{\Lambda_i} = \sum_{i \leq m} a_i \sum_{x \in \Lambda_i} \delta_x$$

The autocorrelation measure

The autocorrelation is defined to be

$$\begin{aligned}\gamma_\omega &= \lim_{R\to\infty} \frac{1}{Vol(B_R(0))} \left(\omega|_{B_R(0)} * \tilde{\omega}|_{B_R(0)}\right) \\ &= \sum_{i,j\leq m} a_i\overline{a_j} \sum_{z\in\Lambda_i-\Lambda_j} freq(z)\delta_z,\end{aligned}$$

where the freqency is computed as the limit, if it exists, as the average number of times z is a return vector per unit area:

$$freq(z) = \lim_{R\to\infty} \frac{1}{Vol(B_R(0))} \#\{x \in \Lambda_i \cap B_R(0) \text{ and } x - z \in \Lambda_j\}$$

The diffraction measure

Definition

If the autocorrelation measure γ_ω exists, the *diffraction measure* of $\mathcal{T}$ is the Fourier transform $\widehat{\gamma_\omega}$.

The measure $\widehat{\gamma_\omega}$ tells us how much intensity is scattered into a given volume.

$$\widehat{\gamma_\omega} = (\widehat{\gamma_\omega})_{pp} + (\widehat{\gamma_\omega})_{sc} + (\widehat{\gamma_\omega})_{ac}.$$

The pure point part tells us the location of the 'Bragg peaks'; the degree of disorder in the solid is quantified by the continuous parts.

The singular continuous part is rarely observed in physical experiments [**BaakeGrimmAperiodicOrder**].