Introduction to hierarchical tiling dynamical systems

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Lecture 2: Supertile construction methods

Recall from yesterday

- Objects: sequences on a finite alphabet; tilings on a finite prototile set
  - Letters/tiles serve as atoms in a model for quasicrystals
- Big ball metric
  - Two tilings are close if after a small translation they agree exactly on a big ball around the origin
  - You're standing at the origin and you can see the landscape around you clearly
  - If you want to see what is further away, you can shift/translate the part you are interested in
- Translation serves as the action for our dynamical systems
- We want to avoid any periodicity yet have a repetitive structure
  - Supertile construction techniques are a good way to do that

Classes of supertile methods

Supertile constructions

- Symbolic substitutions
- Constant-length $\mathbb{Z}^d$ substitutions
- Self-similar and self-affine tilings
  - "pseudo"-self-similar and -affine tilings
- Fusion rules
  - S-adic systems

Substitution for sequences

Symbolic substitutions

- A *substitution* is a map $\sigma : \mathcal{A} \to \mathcal{A}^*$
  - where $\mathcal{A}^*$ is the set of non-empty words on $\mathcal{A}$
- If $w = a_1 ... a_k \in \mathcal{A}^*$, then $\sigma(w) = \sigma(a_1)...\sigma(a_k)$
- Terminology: an *n-superword* is a word of the form $\sigma^n(a)$ for some $a \in \mathcal{A}$

Example

(A constant-length substitution.) Let $\sigma(a) = abb$ and $\sigma(b) = aaa$.

$$a \to abb \to abb\,aaa\,aaa \to abb\,aaa\,aaa\;\;abb\,abb\,abb\;\;abb\,abb\,abb \to \cdots,$$

The length is 3. There is a whole other lexicon for this. (sequence is 3-automatic, σ a non-erasing morphism,...)

Example

(Non-constant length) Choose a positive integer k and let $\sigma(a) = abbb$ and $\sigma(b) = a$. The first few supertiles are

$$a \to abbb \to abbb\,a\,a\,a \to abbb\,a\,a\,a\,abbb\,abbb\,abbb \to \cdots,$$

The subshift associated to the substitution σ

- Let $\mathcal{R} = \{\sigma^n(a)$ such that $a \in \mathcal{A}$ and $n \in \mathbb{N}\}$
- A sequence $\mathbf{x} \in \mathcal{A}^{\mathbb{Z}}$ is said to be *admitted* by σ if every subword of $\mathbf{x}$ is a subword of an element of $\mathcal{R}$.
- We define $\Omega_\sigma \subset \mathcal{A}^{\mathbb{Z}}$ to be the set of all sequences admitted by σ.
- Apologies to the computer scientists; we are using $\mathcal{R}$ as a sort of "language" for Ω_σ.
- Since all of the words in a shifted sequence are the same as those in the original, Ω_σ is a shift-invariant subset of $\mathcal{A}^{\mathbb{Z}}$

One-dimensional self-similar tilings

One-dimensional self-similar tilings

Example

(Obtaining a self-similar tiling from a symbolic substitution.)

- To make a tiling for $\sigma(a) = abbb$ and $\sigma(b) = a$, tiles t_a and t_b are made with (carefully chosen) lengths $|t_a|$ and $|t_b|$.
- We define a tile substitution $\mathcal{S}$:
  - $\mathcal{S}(t_a)$ is the tile t_a followed by 3 copies of t_b.
  - $\mathcal{S}(t_b)$ is just t_a.
- The lengths of the supertiles are $|\mathcal{S}(t_a)| = |t_a| + 3|t_b|$ and $|\mathcal{S}(t_b)| = |t_a|$.
- The ideal situation would be if there was an *inflation factor* $\lambda > 1$ such that $|\mathcal{S}(t_a)| = \lambda|t_a|$ and $|\mathcal{S}(t_b)| = \lambda|t_b|$.

Getting good tile lengths

If we know

$$|\mathcal{S}(t_a)| = |t_a| + 3|t_b| \qquad |\mathcal{S}(t_b)| = |t_a|$$

and we want

$$|\mathcal{S}(t_a)| = \lambda|t_a| \qquad |\mathcal{S}(t_b)| = \lambda|t_b|$$

we quickly see that λ must satisfy $3 = \lambda^2 - \lambda$. So we can let

$$\lambda = \frac{1+\sqrt{13}}{2}, \quad |t_a| = \lambda, \quad |t_b| = 1$$

Inflate-and-subdivide rule

The symbolic substitution becomes a tiling *inflate-and-subdivide* rule:

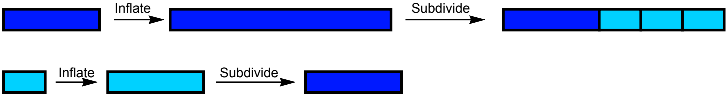


Figure: Inflation and subdivision for the example.

(a.k.a. tiling substitution rule, tiling inflation rule)

Tiling inflation rules in $\mathbb{R}$

- σ is a symbolic subs
- t_e is the tile corresponding to the symbol $e \in \mathcal{A}$.
- $\mathcal{S}(t_e)$ = patch of tiles for $\sigma(e)$ supported on $\lambda \,\mathrm{supp}(t_e)$.
- $\mathcal{S}$ is an 'inflate-and-subdivide rule'.

Extend $\mathcal{S}$ to be a map on $\Omega_{\mathcal{P}}$ as follows

- Let $\mathcal{T} \in \Omega_{\mathcal{P}}$ be a tiling and let $t \in \mathcal{T}$ be any tile
- $\mathcal{S}(t)$ = patch given by the substitution of the prototile of t, translated so that it occupies the set $\lambda \,\mathrm{supp}(t)$
- Apply $\mathcal{S}$ to all tiles in $\mathcal{T}$ simultaneously to get $\mathcal{S}(\mathcal{T})$

$$\mathcal{S}(\mathcal{T}) = \bigcup_{t \in \mathcal{T}} \mathcal{S}(t)$$

If $\mathcal{S}(\mathcal{T}) = \mathcal{T}$, then $\mathcal{T}$ is called a *self-similar tiling.*

Self-similar tiling for our example

Part of a self-similar tiling for our example:

If you imagine the origin at the far left, $\lambda(\mathcal{T})$ looks like

$\mathcal{S}(\mathcal{T}) = \mathcal{T}$, so $\mathcal{T}$ is self-similar.

Constant-length symbolic substitutions in $\mathbb{Z}^d$

Multidimensional constant-length symbolic substitutions

Goal: Construct substitutions for sequences in $\mathbb{Z}^d$.

- We want to replace each $a \in \mathcal{A}$ with a rectangular block of letters.
- Fix *lengths* $l_1, l_2, ...l_d$, positive integers with each $l_i > 1$.
- The *location set* $\mathcal{I}^d$ is:

$$\mathcal{I}^d = \{\vec{\jmath} = (j_1, ...j_d) \text{ such that } j_i \in 0, 1, ..., l_i - 1 \text{ for all } i = 1, ..d\}.$$

- The substitution is a map $\mathcal{S} : \mathcal{A} \times \mathcal{I}^d \to \mathcal{A}$.
- For any $e \in \mathcal{A}$ write $\mathcal{S}(e)$ a block of letters; we call it a *1-superblock* or *1-supertile*.

A two-dimensional Thue-Morse substitution

Let $l_1 = l_2 = 2$, so that the location set is

$$\mathcal{I}^2 = \{(0,0),(0,1),(1,0),(1,1)\}.$$

Define the substitution as:

$$\mathcal{S}(0) = \begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix}, \qquad \mathcal{S}(1) = \begin{matrix} 0 & 1 \\ 1 & 0 \end{matrix},$$

$$0 \to \begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix} \to \begin{array}{cc|cc} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ \hline 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{array} \to \begin{array}{cccc|cccc} 1 & 0 & 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 & 1 & 1 & 0 \\ \hline 0 & 1 & 1 & 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 & 0 & 1 \end{array}$$

Figure: The first three superblocks of type 0. The lines emphasize

Substitution as a block of maps

We can see $\mathcal{S}$ as a matrix $\left(p_{\vec{k}}\right)_{\vec{k}\in\mathcal{I}^2}$ of maps on $\mathcal{A}$.
If we denote by g_0 the identity map and g_1 the map switching 0 and 1, we obtain:

$$\mathcal{S}(*,\mathcal{I}^2) = \left(p_{\vec{k}}\right)_{\vec{k}\in\mathcal{I}^2} = \begin{matrix} g_1 & g_0 \\ g_0 & g_1 \end{matrix} . \qquad (1)$$

For example we see that $p_{(0,0)} = g_0$ and $p_{(0,1)} = g_1$.
We call this substitution **bijective** because each of the maps are bijections of the alphabet.

Bijective substitutions

- $\vec{k} \in \mathcal{I}^d$ represents a location in a 1-superblock
- $\mathcal{S}$ restricted to $\vec{k}$ is a map $p_{\vec{k}} : \mathcal{A} \to \mathcal{A}$
- (These maps determine the cocycle for the skew product representation of the system.)

Definition

Let the substitution $\mathcal{S}$ as defined in this section be written as $\mathcal{S} = \left(p_{\vec{k}}\right)_{\vec{k} \in \mathcal{I}^d}$. We say $\mathcal{S}$ is *bijective* if and only if each $p_{\vec{k}}$ is a bijection from $\mathcal{A}$ to itself.

Self-similar and self-affine tilings in $\mathbb{R}^d$

Self-similar and self-affine tilings in $\mathbb{R}^d$

- Consider a finite prototile set $\mathcal{P}$.
- If you are lucky, you may be able to make an inflation rule that acts as a substitution
  - (Unions of prototiles are related to prototiles via linear maps)
- There are two ways to formally define self-similar/self-affine tilings

One way to define self-affine/similar tilings

Definition

Let $\phi : \mathbb{R}^d \to \mathbb{R}^d$ be a diagonalizable linear transformation all of whose eigenvalues are greater than one in modulus. A tiling $\mathcal{T}$ is called *self-affine with expansion map* ϕ if

1. for each tile $t \in \mathcal{T}$, $\phi(\text{supp}(t))$ is the support of a union of $\mathcal{T}$-tiles, and
2. t and t' are equivalent up to translation if and only if $\phi(\text{supp}(t))$ and $\phi(\text{supp}(t'))$ support equivalent patches of tiles in $\mathcal{T}$.

If ϕ is a similarity the tiling is called *self-similar*. For self-similar tilings of $\mathbb{R}$ or $\mathbb{R}^2 \cong \mathbb{C}$ we obtain an *inflation constant* λ for which $\phi(z) = \lambda z$.

Another way to define self-similar tilings

Definition

Let $\mathcal{P}$ be a finite prototile set in $\mathbb{R}^d$ and let $\phi : \mathbb{R}^d \to \mathbb{R}^d$ be a diagonalizable linear transformation all of whose eigenvalues are greater than one in modulus. A function $\mathcal{S} : \mathcal{P} \to \mathcal{P}^*$ is called a *tiling inflation rule with inflation map* ϕ if for every $p \in \mathcal{P}$,

$$\phi(\mathrm{supp}(p)) = \mathrm{supp}(\mathcal{S}(p)).$$

Another way to define self-similar tilings, continued

We can extend $\mathcal{S}$ to tiles, patches, and tilings:

- If $t = p - x$ for $p \in \mathcal{P}$ and $x \in \mathbb{R}^d$ we define
$$\mathcal{S}(t) := \mathcal{S}(p) - \phi(x)$$
- $\mathcal{Q}$ patch or tiling:
$$\mathcal{S}(\mathcal{Q}) = \bigcup_{t \in \mathcal{Q}} \mathcal{S}(t)$$
- If a tiling $\mathcal{T}$ is invariant under $\mathcal{S}$ we call it a *self-affine tiling*
- Lingo: an *n-supertile* is a patch of the form $\mathcal{S}^n(t)$

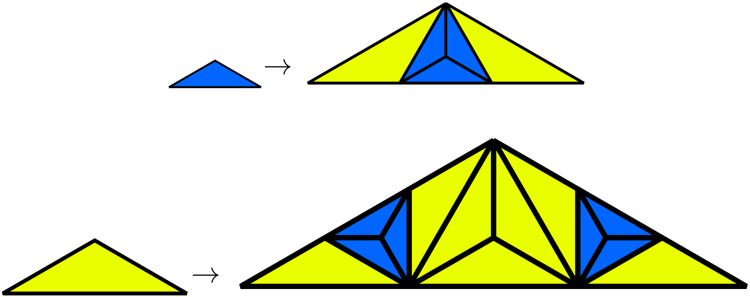
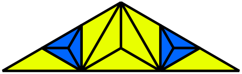
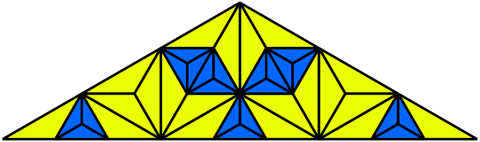
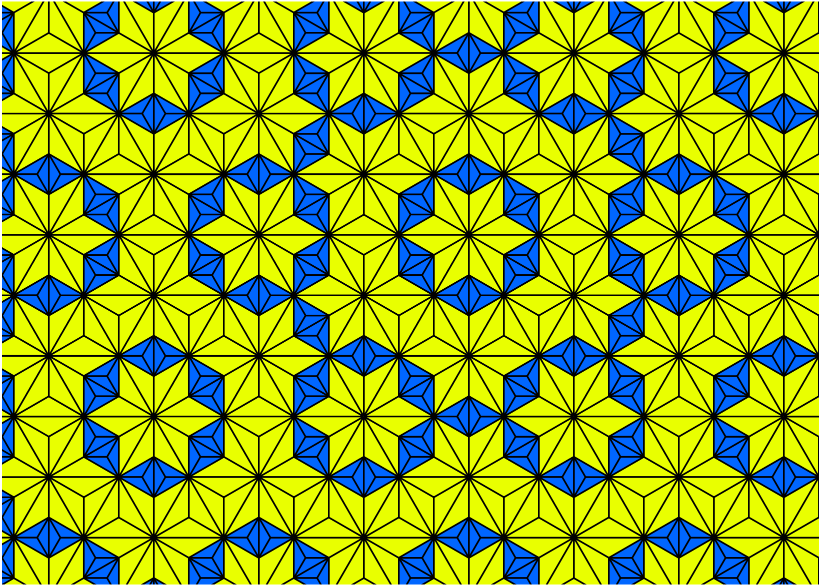


Figure: The T2000 inflate-and-subdivide rule.

Supertiles







Pseudo-self-similar tilings

We need to define what it means to expand $\mathcal{T}$ to obtain the tiling $\phi(\mathcal{T})$:

For every tile t in $\mathcal{T}$, $\phi(t)$ is defined to be a tile supported on $\phi(\mathrm{supp}(t))$

that carries the label of t

We define $\phi(\mathcal{T}) := \bigcup_{t \in \mathcal{T}} \phi(t)$.

Note that $\phi(\mathcal{T})$ is a tiling made using the prototile set $\phi(\mathcal{P})$.

Definition

Let $\mathcal{P}$ be a finite prototile set in $\mathbb{R}^d$ and let $\phi : \mathbb{R}^d \to \mathbb{R}^d$ be a diagonalizable linear transformation all of whose eigenvalues are greater than one in modulus. We say a tiling $\mathcal{T} \in \Omega_{\mathcal{P}}$ is *pseudo-self-similar with expansion* ϕ if $\mathcal{T}$ is locally derivable from $\phi(\mathcal{T})$.

$$\phi = \begin{pmatrix} 5/2 & \sqrt{3}/2 \\ -\sqrt{3}/2 & 5/2 \end{pmatrix}$$

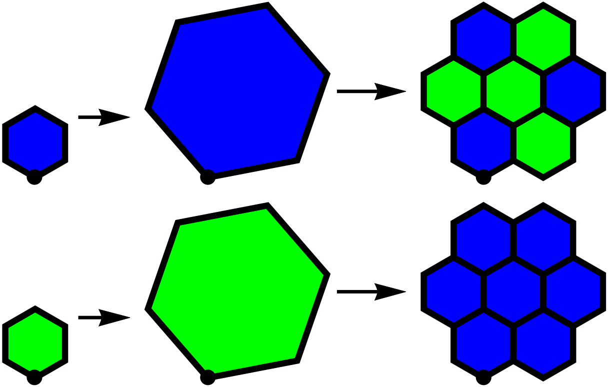


Figure: The inflate-and-subdivide rule for a hexagonal pseudo-self-similar tiling.

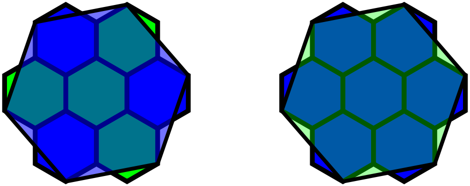


Figure: The inflated blue tile and its patch, left; the inflated green and its patch, right.

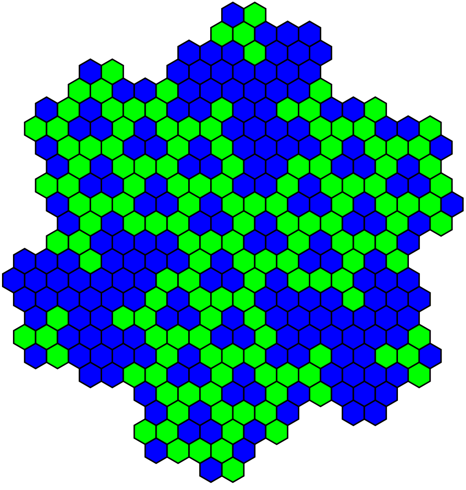


Figure: 2- and 3-supertiles for the blue prototile.

Fusion rules

Definition of a fusion rule $\mathcal{R}$

Definition. A *fusion* of a patch P_1 to another patch P_2 is a union of P_1 and P_2 that is connected and does not contain overlaps.

$$fusion = geometric\ concatenation$$

- *0-supertiles.* A finite collection $\mathcal{P}_0$ of tiles. These are "atoms".
- *1-supertiles.* A finite collection $\mathcal{P}_1$ of patches (fusions) of tiles from $\mathcal{P}_0$. These are "molecules".
- *2-supertiles.* A finite collection $\mathcal{P}_2$ of patches made by fusing together elements from $\mathcal{P}_1$.
- *n-supertiles.* For each $n > 0$, $\mathcal{P}_n$ is a finite set of patches that are fusions of $(n-1)$-supertiles.

Definiton of fusion rule

Notation: the set of n-supertiles is

$$\mathcal{P}_n = \{P_n(1), ..., P_n(j_n)\},$$

and we can think of our n-supertiles as patches of k-supertiles for any $k < n$.

All supertiles together form an atlas of patches called a *fusion rule:*

$$\mathcal{R} = \{\mathcal{P}_n\}_{n=0}^{\infty}$$

$\mathcal{T}$ is admitted by $\mathcal{R}$ if all its patches lie in elements of $\mathcal{R}$.

The tiling space $\Omega_{\mathcal{R}}$ is the set of all tilings admitted by $\mathcal{R}$.

A direct product fusion

$$\sigma(a) = abb, \sigma(b) = aa$$

$$\mathcal{P} = \{(a,a),(a,b),(b,a),(b,b)\}$$

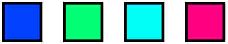
$$\mathcal{S}((a,a)) = \begin{array}{ccc} (a,b) & (b,b) & (b,b) \\ (a,b) & (b,b) & (b,b) \\ (a,a) & (b,a) & (b,a) \end{array}, \qquad \mathcal{S}((a,b)) = \begin{array}{ccc} (a,a) & (b,a) & (b,a) \\ (a,a) & (b,a) & (b,a) \end{array},$$

$$\mathcal{S}((b,a)) = \begin{array}{cc} (a,b) & (a,b) \\ (a,b) & (a,b) \\ (a,a) & (a,a) \end{array}, \qquad \mathcal{S}((b,b)) = \begin{array}{cc} (a,a) & (a,a) \\ (a,a) & (a,a) \end{array}$$

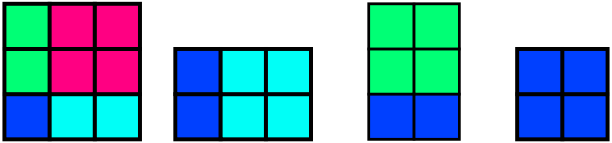
Substitute the first coordinate horizontally and the second coordinate vertically.

Direct product tiling

The prototiles look like

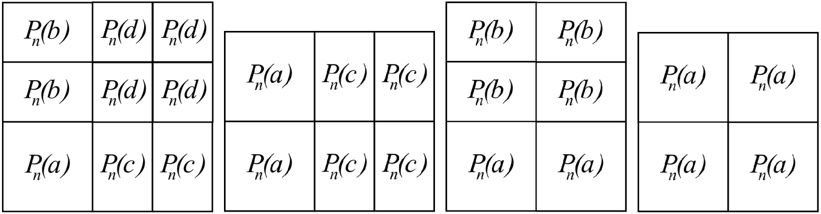
$\mathcal{P} = \{(a,a),(a,b),(b,a),(b,b)\} =$ 

and the 1-supertiles look like

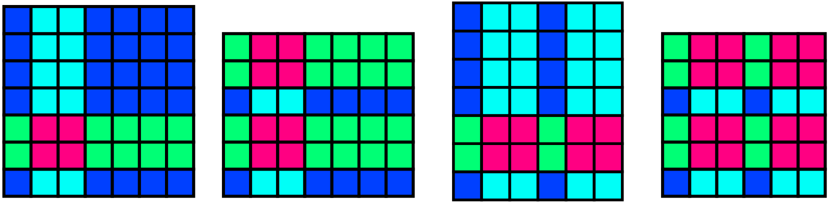


Template for DP concatenation

Make the $n+1$-supertile from the n-supertiles using these combinatorics:



The 2-supertiles come out to be

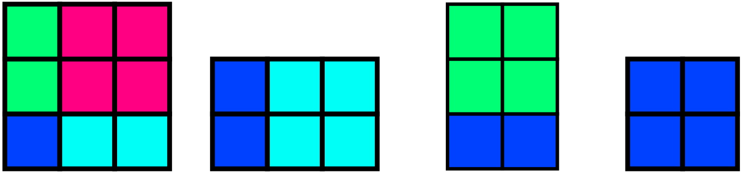


“Direct product variation” (DPV) tilings

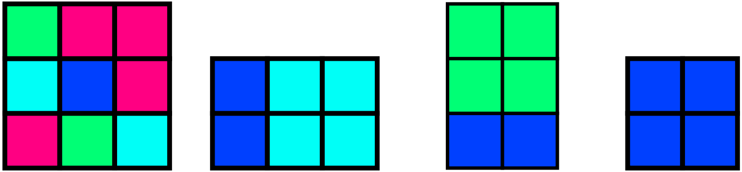
- The $\mathbb{Z}^2$ dynamical system for a DP is conjugate to the direct product of the one-dimensional systems.
- If we are careful we can we rearrange the substitution to obtain “Direct Product Variation” (DPV) tilings.
- Care must be taken so that the DPV substitution can be iterated to form legitimate patches and tilings.

Breaking the direct product structure

Start with a direct product:



The tile on the left has been carefully rearranged:



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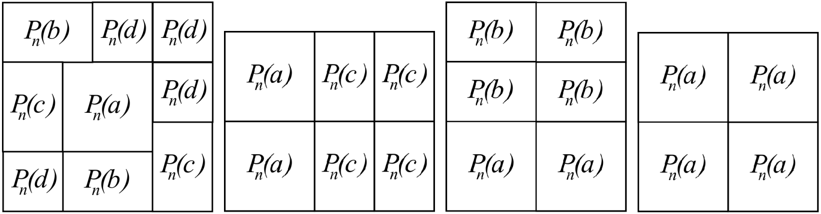

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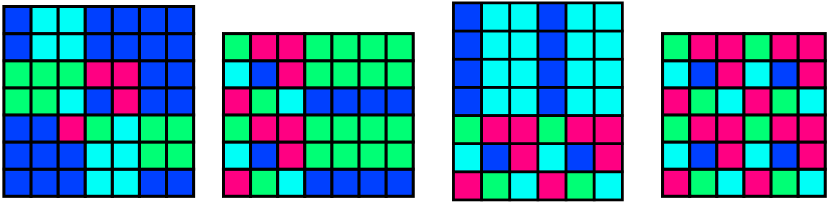
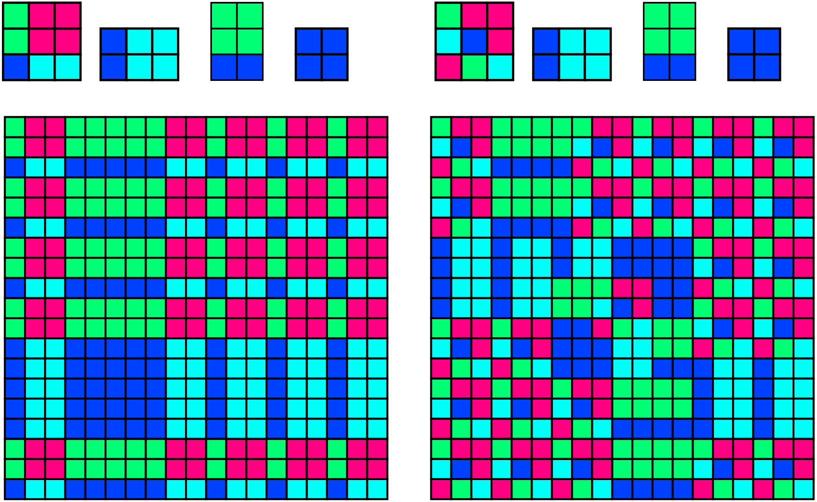
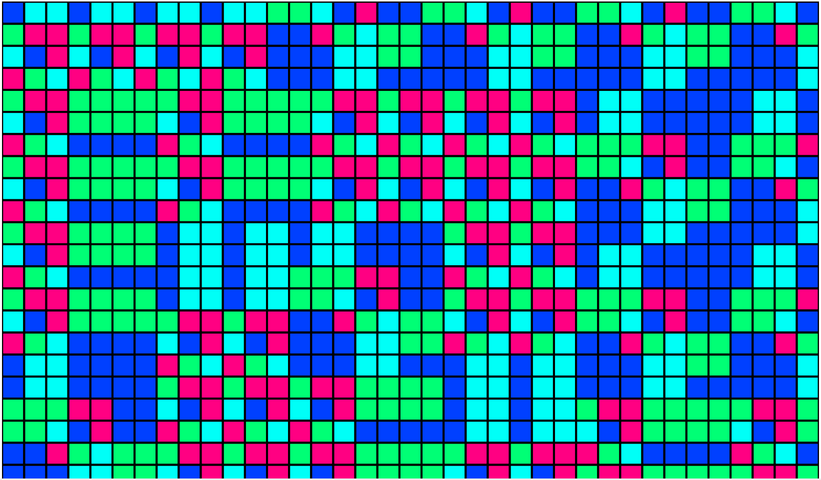


A comparison of the DP and DPV



A problem

Details of this DPV prevent us from seeing it as a substitution. Namely, given adjacent tiles, how should their supertiles fit together?



Example: an "algorithmic" fusion rule

This example doesn't have diagram to show how to put the supertiles together like the DPV, but there is a simple algorithm to determine the fusion.

- Inputs: n-supertiles A_n and B_n; fundamental nth-level translation vectors.
- Fixed: matrix $L = \begin{pmatrix} 2 & 1 \\ -1 & 1 \end{pmatrix}$.
- Outputs are $(n+1)$-supertiles and $(n+1)$-th level translations $\vec{k}_n$ and $\vec{l}_n$.

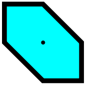
$$A_{n+1} = A_n \cup (B_n + \vec{k}_n) \cup (B_n + \vec{l}_n)$$

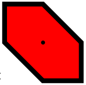
$$B_{n+1} = B_n \cup (A_n + \vec{k}_n) \cup (A_n + \vec{l}_n).$$

$$\vec{k}_{n+1} = L\vec{k}_n \qquad \vec{l}_{n+1} = L\vec{l}_n$$

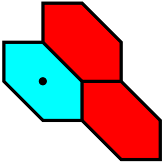
To run the algorithm, put in a prototile set and some initial vectors and see what happens.

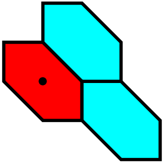
Inputs: hexagonal tiles

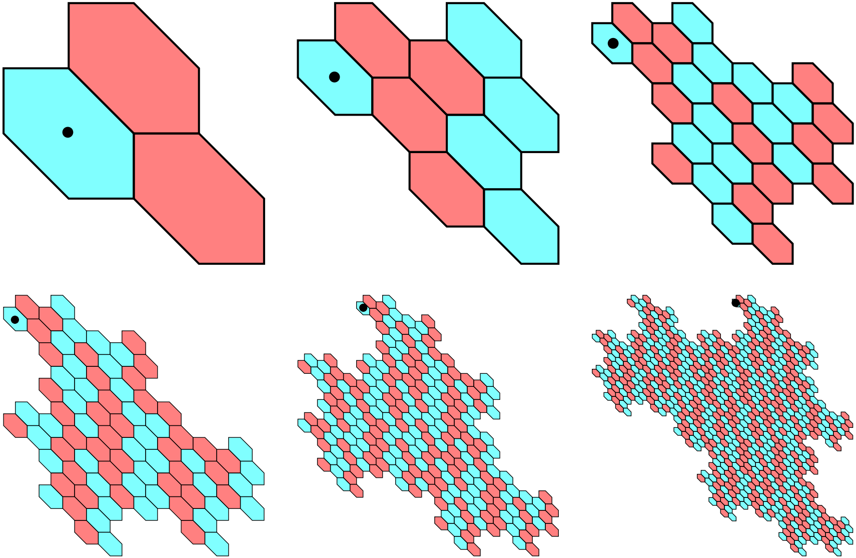
$A_0 =$ 

$B_0 =$ 

$\vec{k}_0 = (2, -1)$ and $\vec{l}_0 = (1, 1)$







The 10^n example—minimal but not uniquely ergodic.

$$\mathcal{P}_0 = \{a, b\}, \qquad \mathcal{P}_n = \{P_n(a), P_n(b)\},$$

where

$$P_1(a) = a^{10}b = aaaaaaaaaab \qquad P_1(b) = b^{10}a = bbbbbbbbbba$$

Let

$$P_2(a) = (P_1(a))^{100}P_1(b) \qquad P_2(b) = (P_1(b))^{100}P_1(a)$$

and in general

$$P_n(a) = (P_{n-1}(a))^{10^n}P_{n-1}(b) \qquad P_n(b) = (P_{n-1}(b))^{10^n}P_{n-1}(a)$$

S-adic systems

- Let $\mathcal{A}_0, \mathcal{A}_1, \mathcal{A}_2, \ldots$ be a family of finite alphabets,
- Let
$$\sigma_n : \mathcal{A}_{n+1} \to \mathcal{A}_n^*$$
- Let $\{a_n\}_{n=0}^{\infty}$ represent a sequence for which $a_n \in \mathcal{A}_n$ for all $n \in \mathbb{N}$

Definition

An infinite word $\mathbf{x} \in \mathcal{A}_0^{\mathbb{N}}$ admits the *S-adic expansion* $\{(\sigma_n, \mathcal{A}_n)\}_{n=0}^{\infty}$ if

$$\mathbf{x} = \lim_{n\to\infty} \sigma_0\sigma_1 \cdots \sigma_{n-1}(a_n).$$

S-adic constructions are fusions

- The prototile set is $\mathcal{A}_0$
- The 1-supertiles are constructed using the map $\sigma_0 : \mathcal{A}_1 \to \mathcal{A}_0$, where $a \in \mathcal{A}_1$:

$$\mathcal{P}_1 = \{\sigma_0(a) \text{ such that } a \in \mathcal{A}_1\}$$

- The 2-supertiles are given by $\sigma_0(\sigma_1(a)))$, where $a \in \mathcal{A}_2$:

$$\mathcal{P}_2 = \{\sigma_0\sigma_1(a) \text{ such that } a \in \mathcal{A}_2\}$$

  - The 2-supertiles are given by $\sigma_0(\sigma_1(a)))$, where $a \in \mathcal{A}_2$, . Notice that $\sigma_1(a)$ is a word in $\mathcal{A}_1^*$, and so we can apply σ_0 to each of its letters. Thus one can see $\sigma_0(\sigma_1(a))$ as the fusion of blocks of the form $\sigma_0(a')$ in the order prescribed by $\sigma_1(a)$
  - $$\mathcal{P}_n = \{\sigma_0\sigma_1 \cdots \sigma_{n-1}(a) \text{ such that } a \in \mathcal{A}_n\},$$

    an n-supertile is the fusion of $n-1$-supertiles $\sigma_0\sigma_1 \cdots \sigma_{n-2}(a')$ in the order prescribed by $\sigma_{n-1}(a)$.

Dynamics of supertile constructions

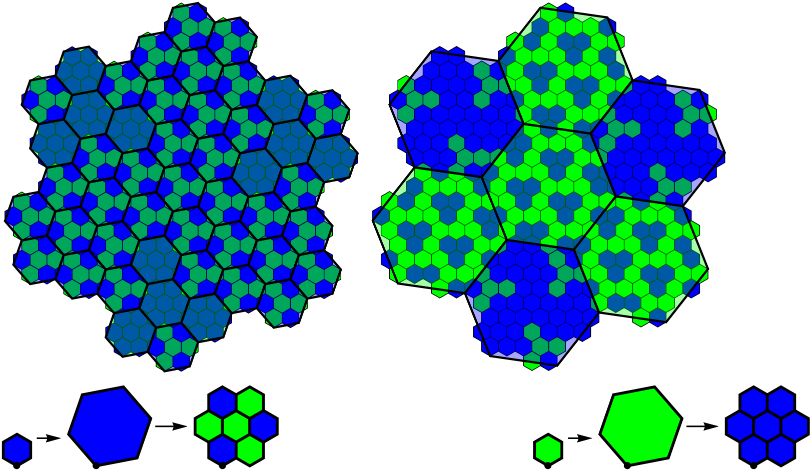
Tiling spaces from supertile methods

- **Let $\mathcal{R}$ be the set of all supertiles.** We can make $\Omega_{\mathcal{R}}$ as the set of all tilings allowed by $\mathcal{R}$.
  - This will always work.
- **If there is an invariant tiling $\mathcal{T}$**, we make the hull $\Omega_{\mathcal{T}}$.
  - This only works when the supertile rule can act as a map on the full tiling space, i.e. substitutions/inflations.
- For many supertile rules the two constructions give the same space.
- We use "Ω" from here forward to denote either type of supertiling space.

(It is possible for $\mathcal{R}$ not to admit any tilings, but we ignore that situation.)

Supertiling spaces and recognizability

- **Fact:** If $\mathcal{T} \in \Omega$, every tile in $\mathcal{T}$ must be in some n-supertile, either from the generating tiling or from $\mathcal{R}$.
- The n-supertiles might not all be unique, but
- All tiles in $\mathcal{T}$ itself can be composed into n-supertiles that overlap only on their boundary.
- A tiling $\mathcal{T}_n$ obtained by this composition, i.e. where the prototile set is considered at $\mathcal{P}_n$ rather than $\mathcal{P}$, is called an *n-supertiling* of $\mathcal{T}$.
- The space of all n-supertilings of Ω is denoted Ω_n



Supertiling spaces

Each n-supertile is constructed from $(n-1)$-supertiles; there is a unique *decomposition map* f_n taking Ω_n to Ω_{n-1}.

Definition

A supertile rule is said to be *recognizable* if the decomposition map from Ω_n to Ω_{n-1} is invertible for all n.

We tend to think of recognizability locally: we should be able to tell what $(n+1)$-supertile is at $\vec{x}$ by knowing the patch of n-supertiles in a ball around $\vec{x}$.

"I can tell what type of supertile I'm in by looking around me."