Introduction to hierarchical tiling dynamical systems

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Research School: Tiling Dynamical Systems

Lecture 1: Symbolic and tiling dynamical systems

The 2011 Nobel Prize in Chemistry

Daniel Shectman

"For the discovery of quasicrystals"

‘Impossible’ diffraction image

- Shectman’s colleague at U.S. NIST made an aluminum-magnesium alloy
- Shechtman did a diffraction analysis and found contradictory properties
  - it had bright spots indicative of a periodic (crystal) atomic structure
  - had symmetries impossible for such a structure

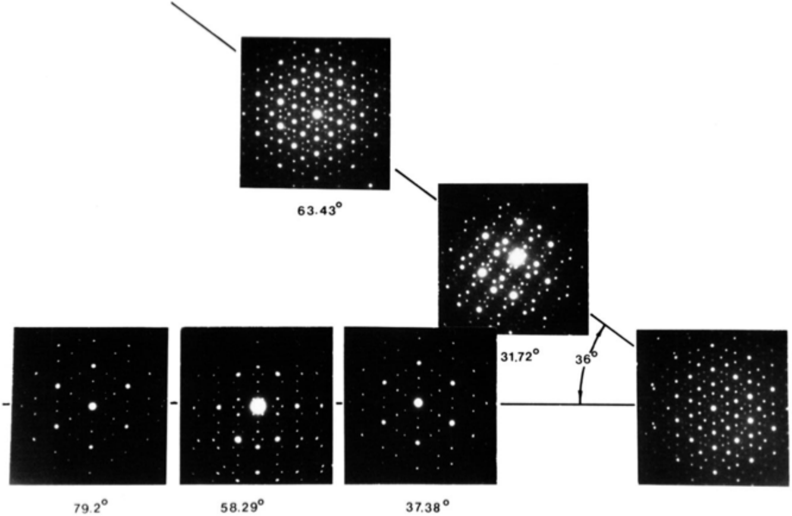
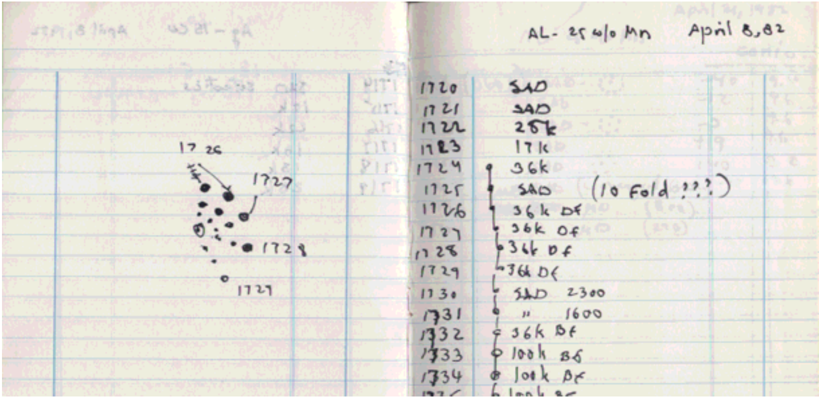


FIG. 2. Selected-area electron diffraction patterns taken from a single grain of the icosahedral phase. Rotations match those in Fig. 1.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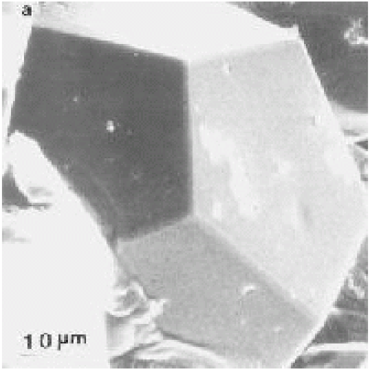
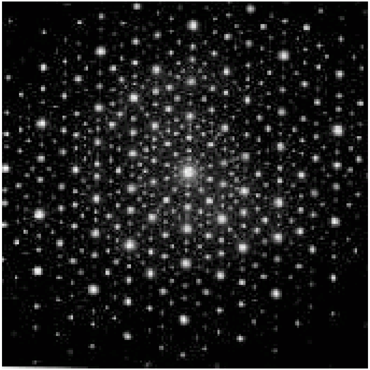
Discovery of Quasicrystals



Single grain of icosahedral Al–Pd–Mn phase

source: A. P. Tsai



Diffraction image of Al6Mn

www.ph.melb.edu.au/diffraction/image/fivefold.html

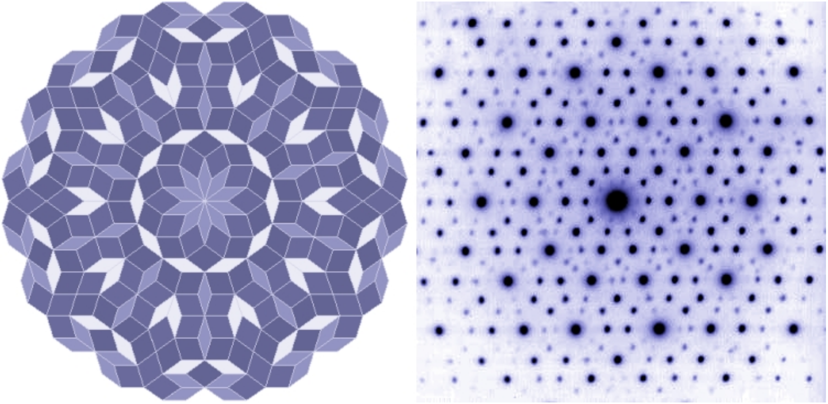


Image source: Oxford Dept. of Chemistry http://www.xtl.ox.ac.uk/tag/penrose-tiling.html

Left: A patch of a Penrose tiling. Right: An electron diffraction pattern of Zn-Mg-Ho alloy.

Connections

- Diffraction image of Penrose tiling (computed by Mackay in 1982) looked like Shechtman's images
- Tilings like the Penrose tiling might be good models for quasicrystals
- Penrose tiles have matching rules like the Robinson tiling; go to Jeandel's lectures!
- The Penrose tiling is made with a supertile construction method, go to my lecture 2!
- (Note: there was already a field of one-dimensional supertile construction methods: substitution)
- We can analyze tilings via their (dynamical or diffraction) spectrum, see my lecture 3.

Related yet omitted topics

- tiling cohomology
- the projection method for constructing tilings
- K-theory of C^*-algebras of tilings
- the spectrum of Schrodinger operators modeled on tilings
- tilings of hyperbolic space or other spaces

Four types of objects of interest

The four types of spaces under consideration

- Sequences in $\mathbb{Z}$
- Sequences in $\mathbb{Z}^d$
- Tilings in $\mathbb{R}$
- Tilings in $\mathbb{R}^d$

Sequences

- $\mathcal{A}$ = some finite set= *alphabet*.
- *sequence* $\mathbf{x} : \mathbb{Z} \to \mathcal{A}$
- The set of all sequences is $\mathcal{A}^{\mathbb{Z}}$.
- Metric (a "big ball" metric)

$$N(\mathbf{x}, \mathbf{y}) = \min\{n \geq 0 \text{ such that } \mathbf{x}(j) \neq \mathbf{y}(j) \text{ for some } |j| = n\}$$

$$d(\mathbf{x}, \mathbf{y}) = \exp(-N(\mathbf{x}, \mathbf{y}))$$

Idea: $\mathbf{x}$ and $\mathbf{y}$ are close if they agree on a large ball centered at the origin.

Shift dynamical systems

- $j \in \mathbb{Z}$ and $\mathbf{x} \in \mathcal{A}^{\mathbb{Z}}$
- Shift $\mathbf{x}$ by j to get $\mathbf{x} - j \in \mathcal{A}^{\mathbb{Z}}$

$$(\mathbf{x} - j)(k) = \mathbf{x}(k + j)$$

- Notice that $(\mathbf{x} - j)(0) = \mathbf{x}(j)$
- $(\mathcal{A}^{\mathbb{Z}}, \mathbb{Z})$ is the *full shift* on $|\mathcal{A}|$ symbols

Subshifts

- Closed, shift-invariant subset $\Omega \subset \mathcal{A}^{\mathbb{Z}}$
- Ω is called a *shift space* or *subshift of* $\mathcal{A}^{\mathbb{Z}}$
- $(\Omega, \mathbb{Z})$ is called a *shift dynamical system*

EXAMPLE. Let $\Omega = \{...0101.0101...\ ,\ ...1010.1010...\}$,

- Convention is to put the decimal to the left of $\mathbf{x}(0)$
- Ω is finite iff the sequences it contains are periodic

Sequences in $\mathbb{Z}^d$

- (Multidimensional) sequence $\mathbf{x}\colon \mathbb{Z}^d \to \mathcal{A}$
- $\mathcal{A}^{\mathbb{Z}^d}$ is the set of all sequences in $\mathbb{Z}^d$ (the "full shift")
- Metric for "big ball topology":

$$N(\mathbf{x}, \mathbf{y}) = \min\{n \geq 0 \text{ such that } \mathbf{x}(\vec{\jmath}) \neq \mathbf{y}(\vec{\jmath}) \text{ for some } |\vec{\jmath}| = n\}$$

$$d(\mathbf{x}, \mathbf{y}) = \exp(-N(\mathbf{x}, \mathbf{y}))$$

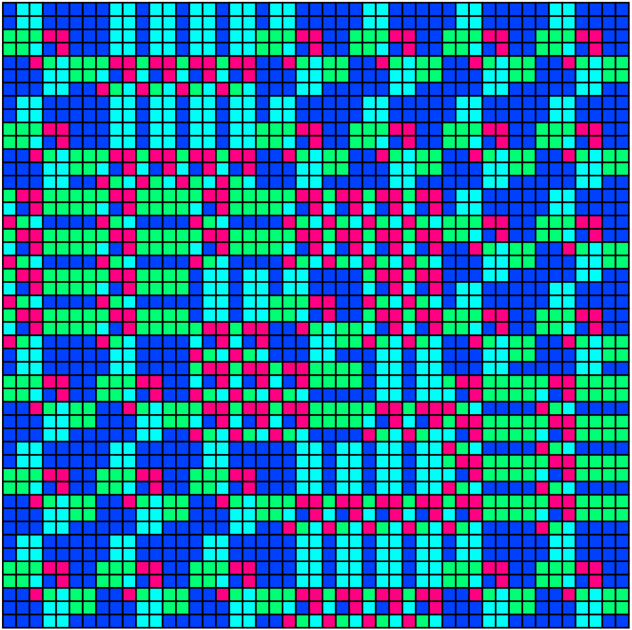
$\mathbb{Z}^d$ dynamics

- $\vec{\jmath} \in \mathbb{Z}^d$ and $\mathbf{x} \in \mathcal{A}^{\mathbb{Z}^d}$
- Shift $\mathbf{x}$ by $\vec{\jmath}$ to get $\mathbf{x} - \vec{\jmath} \in \mathcal{A}^{\mathbb{Z}^d}$

$$(\mathbf{x} - \vec{\jmath})(\vec{k}) = \mathbf{x}(\vec{k} + \vec{\jmath})$$

- Notice that $(\mathbf{x} - \vec{\jmath})(\vec{0}) = \mathbf{x}(\vec{\jmath})$
- $(\mathcal{A}^{\mathbb{Z}}, \mathbb{Z})$ is the *full shift*; a closed translation-invariant subset Ω is a *subshift*

Often we visualize sequences as tilings...



Tilings of $\mathbb{R}$

- Given an alphabet $\mathcal{A}$, choose a closed interval for each symbol in $\mathcal{A}$
- For $\mathbf{x} \in \mathcal{A}^{\mathbb{Z}}$ we could make a tiling by placing the intervals end-to-end in the order specified by $\mathbf{x}$
- The precise location of the endpoint of the tile corresponding to $\mathbf{x}(0)$ matters
- Thus any $\mathbf{x}$ corresponds to infinitely many tilings

A sequence in $\mathbb{Z}$ and its tiling in $\mathbb{R}$

$$\mathbf{x} = ...abbbaaaa.bbbabbbabbb...$$

$\mathcal{T} =$ 

Prototiles, patches, tilings

- *prototile* $p \in \mathcal{P}$ labelled topological disk
- Topological disk = *support* of p (denoted $\text{supp}(p)$)
- We always have a finite *prototile set* $\mathcal{P}$
- A $\mathcal{P}$-*tile* or just *tile* t is denoted $t = p - \vec{v}$:
  translate the support: $\text{supp}(t) = \text{supp}(p) - \vec{v}$
  keep the label: the label of t is the label of p

Prototiles, patches, tilings

- $\mathcal{P}$-*patch* or just *patch* is a set of tiles:
  1. Intersect at most on their boundaries
  2. Supports form a connected set
- $\mathcal{P}$-*tiling* or just *tiling* is an infinite set of tiles

$$\mathcal{T} = \{t_i \text{ such that } i \in \mathbb{Z}\}$$

  1. Tiles intersect at most on their boundaries
  2. Supports cover all of $\mathbb{R}^d$

The action of translation

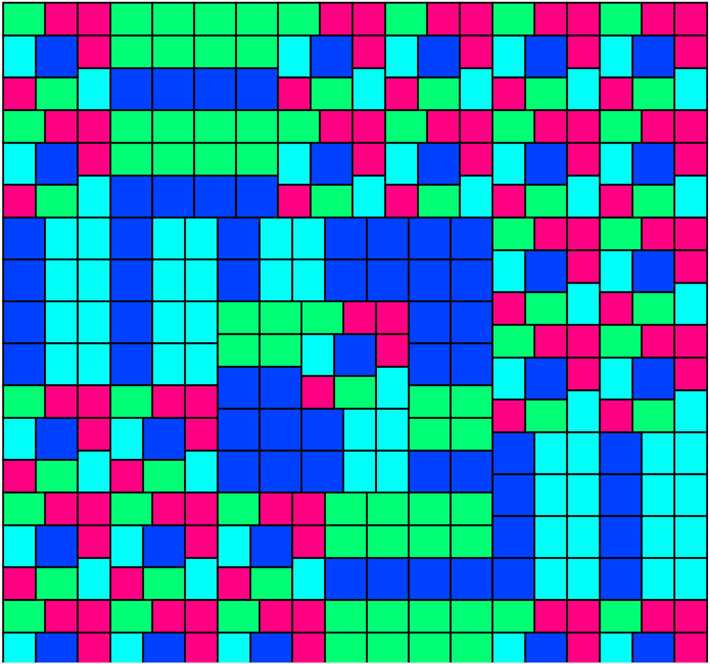
- To get the tile $t - \vec{v}$, translate the support of t by $\vec{v}$ and keep the label
- $\mathcal{T} - \vec{v}$ is the set of tiles $\{t - v \text{ such that } t \in \mathcal{T}\}$
- Note the origin in $\mathcal{T} - \vec{v}$ corresponds to $\vec{v}$ in $\mathcal{T}$
- $\mathcal{T}$ is *nonperiodic* if there is no $\vec{v}$ for which $\mathcal{T} - \vec{v} = \mathcal{T}$
  - If $d > 1$, $\mathcal{T}$ can be periodic but not *fully periodic*, if the directions of periodicity do not form a basis for $\mathbb{R}^d$.

Finite local complexity

Definition

We say a tiling $\mathcal{T} \in \Omega_{\mathcal{P}}$ has *finite local complexity (FLC)* if it contains only finitely many two-tile patches up to translation.

For the purposes of this work, we assume finite local complexity in all tilings and tiling spaces unless otherwise stated.



Tiling spaces and tiling dynamical systems

The “big ball” metric

Definition

Let $R(T, T')$ be the supremum of all $r \geq 0$ such that there exists $\vec{x}, \vec{y} \in \mathbb{R}^d$ with

1. $|\vec{x}| < 1/2r$ and $|\vec{y}| < 1/2r$, and
2. On the ball of radius r around the origin, $(T - \vec{x}) \cap B_r(0) = (T' - \vec{y}) \cap B_r(0)$.

We define

$$d(T, T') := \min\left\{\frac{1}{R(T, T')}, 1\right\}$$

Tiling spaces

- $\Omega_{\mathcal{P}}$ = the space of all $\mathcal{P}$-tilings
  - Note: Elements of $\mathcal{A}^{\mathbb{Z}}$ are infinite sequences, likewise elements of $\Omega_{\mathcal{P}}$ are infinite tilings of $\mathbb{R}^d$.
- A tiling space Ω is a closed, translation-invariant subset of $\Omega_{\mathcal{P}}$
- We write $(\Omega, \mathbb{R}^d)$ for the dynamical system under the action of translation
- There are two types of tiling spaces we are particularly interested in:
  - The "hull" of a tiling $\mathcal{T}$
  - The set of all tilings made of specified patches

Two tiling space constructions

To study a particular tiling: The *hull of the tiling* $\mathcal{T}$ is the orbit closure of $\mathcal{T}$:

$$\Omega_{\mathcal{T}} = \overline{\{\mathcal{T} - \vec{v} \text{ for all } \vec{v}\}}$$

To restrict the patch types: Let $\mathcal{R}$ be a set of $\mathcal{P}$-patches.

We say that $\mathcal{T} \in \Omega_{\mathcal{P}}$ is *allowed* by $\mathcal{R}$ if every patch in $\mathcal{T}$ is translation-equivalent to a subpatch of an element of $\mathcal{R}$.

The tiling space $\Omega_{\mathcal{R}}$ is the set of all allowed tilings.

Cylinder sets for sequence spaces

Suppose w is a finite word in $\mathcal{A}^*$ supported on some subset $U \subset \mathbb{Z}$. The *cylinder set* Ω_w *generated by* w is given by

$$C_w = \{\mathbf{x} \in \Omega \text{ such that } \mathbf{x}(U) = w\}$$

- cylinder sets are both closed and open in the metric topology
- for any $\epsilon > 0$ and any $\mathbf{x} \in \Omega$, the ball of radius ϵ around x is a cylinder set for a word around the origin in $\mathbf{x}$.
- cylinder sets form a basis for the topology in shift space

Cylinder sets for tilings

- Let P be a $\mathcal{P}$-patch, let $U \subset \mathbb{R}^d$, and let $\Omega \subset \Omega_{\mathcal{P}}$ be a tiling space.
- $\Omega_{P,U} = \{\mathcal{T} \in \Omega \text{ such that } P - u \subset \mathcal{T} \text{ for some } u \in U\}$
- $\Omega_{P,U}$ is the set of all tilings in Ω that contain a copy of P translated by an element of U.
- if $U = B_\epsilon(0)$, where this is the open ball around the origin and ϵ is sufficiently small, the cylinder set is open.
- We can get a countable basis for the topology by discretizing $\epsilon_n \to 0$, since there are only a countable number of patches of any size up to translation.

Basic topology of tiling spaces

Lemma. Under mild conditions, Ω is connected. Each tiling in Ω defines a path component that is homeomorphic to $\mathbb{R}^d$, and there are uncountably many path components.

Lemma. If $\Omega \subset \Omega_{\mathcal{P}}$ is closed and of finite local complexity, then Ω is complete and compact.

Topological conjugacies of shift spaces: sliding block codes

- Let $\mathcal{A}$ and $\mathcal{A}'$ be finite alphabets and suppose Ω is a shift space in $\mathcal{A}^{\mathbb{Z}}$
- Choose nonnegative integers m and n (to make a "window")
- let $B_{m,n}$ denote the set of all words of length $m+n+1$ that appear in Ω
- Let $\Phi : B_{m,n} \to \mathcal{A}'$ be any map
- *sliding block code* $\phi : \Omega \to (\mathcal{A}')^{\mathbb{Z}}$ is defined as:

$$y_i = \Phi(x_{i-m}x_{i-m+1}\cdots x_{i+n-1}x_{i+n}) = (\phi(\mathbf{x}))_i.$$

Homeomorphisms of shift maps

Theorem (Curtis-Lyndon-Hedlund)

Suppose Ω and Ω' are shift spaces, not necessarily on the same alphabet, and let $\theta : \Omega \to \Omega'$. Then θ is a sliding block code if and only if it is shift-commuting and continuous.

Thus sliding block codes are the only factor maps and conjugacies of shift dynamical systems.

Topological conjugacies of tiling spaces: local derivability

Notation If $\mathcal{T} \in \Omega_{\mathcal{P}}$ and $U \subset \mathbb{R}^d$,the patch of tiles in $\mathcal{T}$ whose supports intersect U is denoted

$$\mathcal{T} \cap U$$

Definition

A continuous surjective mapping between tiling spaces $Q : \Omega \to \Omega'$ is a *local mapping* if there is $r > 0$ such that for any $x \in \mathbb{R}^d$ and $\mathcal{T}_1, \mathcal{T}_2 \in \Omega$,

$$\text{if } \mathcal{T}_1 \cap B_r(x) = \mathcal{T}_2 \cap B_r(x), \text{ then } Q(\mathcal{T}_1) \cap \{x\} = Q(\mathcal{T}_2) \cap \{x\}$$

That is, any time $\mathcal{T}_1$ and $\mathcal{T}_2$ match in a ball around $\vec{x}$, they map to the same tile at $\vec{x}$

Mutual local derivability

- If a local mapping $Q\colon \Omega \to \Omega'$ exists we say $Q(\mathcal{T})$ is *locally derivable* from $\mathcal{T}$.
- If Q is invertible we say $\mathcal{T}$ and $Q(\mathcal{T})$ are *mutually locally derivable*
- We also use this terminology for their tiling spaces.

Lemma

If Ω and Ω' are mutually locally derivable tiling spaces, then their dynamical systems are topologically conjugate.

No Curtis-Lyndon-Hedlund theorem for tilings

Homeomorphisms of topologically conjugate tiling dynamical systems can be local maps

but they don't have to be.

This fact was uncovered in the late 1990s.

Nonlocal homeomorphisms for tilings need the nonlocal information in $\mathcal{T}$ to determine the precise location of the origin in $\mathbb{Q}(\mathcal{T})$.

Natural examples arise in supertile constructions and we will show how their nonlocal maps work when we have more definitions.