A class of two-dimensional model sets from one-dimensional substitutions

Natalie Priebe Frank

Vassar College

The pursuit of symmetry: celebrating Moody's 80th
Fields Institute, April 2022

A fan favorite: the Fibonacci substitution

Fibonacci symbolic substitution

The substitution:

$$a \to ab \quad b \to a$$

Iterate to obtain a 'language' for a shift space Σ :

$$a \to ab \to aba \to abaab \to abaababa \to abaababaabaab \to \ldots$$

Since Σ is shift-invariant, (Σ, σ) is a subshift of $\mathbb{Z}^{\mathcal{A}}$.

Fibonacci symbolic substitution

$$a \to ab \quad b \to a$$

Substitution matrix and expansion constant

$$M = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}, \quad \tau = \frac{1+\sqrt{5}}{2}.$$

The eigenvectors of this matrix give

- The relative frequencies of the letters
- The canonical tile lengths for a self-similar tiling.

Discrete dynamical spectrum given by

$$L = \mathbb{Z}[\tau]/\sqrt{5}.$$

Fibonacci natural tile lengths

Tile lengths are given by the left Perron-Frobenius eigenvector $(\tau, 1)$ of $M = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$

$$a = [0, \tau] \quad \text{and} \quad b = [0, 1]$$

we obtain an inflate-and-subdivide rule

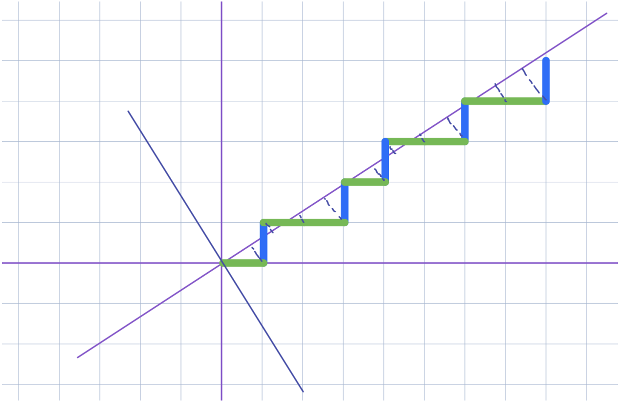
As before, the rule can be iterated to make the allowable tile patches

Cut-and-project scheme

Let $\vec{a} = \begin{pmatrix}1\\0\end{pmatrix}$ and $\vec{b} = \begin{pmatrix}0\\1\end{pmatrix}$. With $M = \begin{pmatrix}1 & 1\\1 & 0\end{pmatrix}$, the vectors

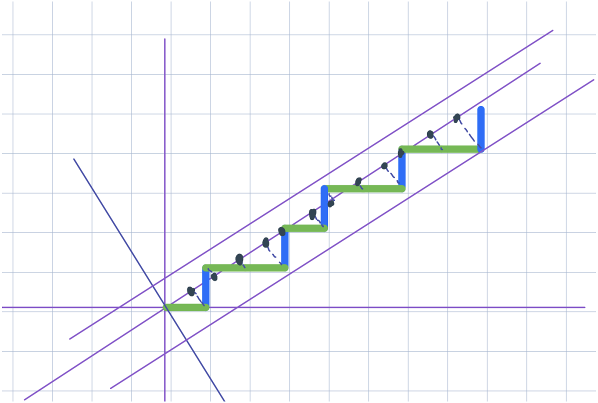
$$M^n\vec{a} \text{ and } M^n\vec{b}$$

are the population vectors of $\sigma^n(a)$ and $\sigma^n(b)$.



Cut-and-project scheme

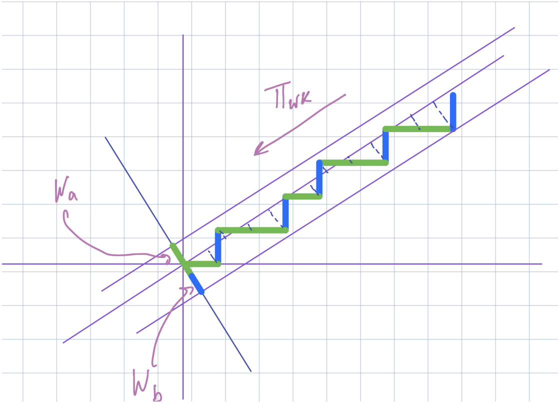
The vertices of the staircase project onto the right Perron-Frobenius eigenline to make the self-similar tiling.



Vertex type determines tile type.

Cut-and-project scheme

If you project the staircase onto the weak eigenline, you get the window.

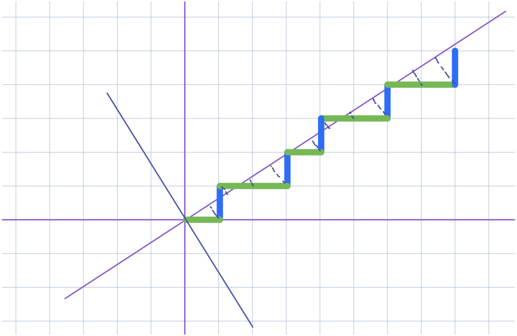


$$W = W_a \cup W_b$$

Renormalization

Λ_a, Λ_b points of type a, b resp. on the staircase

$$\Lambda = \Lambda_a \cup \Lambda_b$$



$$M\Lambda \subset \Lambda$$

Renormalization

- Every point on the staircase is in a copy of $\sigma(a)$ or $\sigma(b)$.
- Substituted words start at vertices of the form $M\vec{v}$, where $\vec{v}$ is already in the staircase.
- The a's appear as the first letter in $\sigma(a) = ab$ and as the first letter in $\sigma(b) = a$.
- The elements of Λ_b appear only as the second letter in $\sigma(a) = ab$.

$$\Lambda_a = M\Lambda_a \cup M\Lambda_b \quad \text{and} \quad \Lambda_b = M\Lambda_a + \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

Renormalization for the window

Points in the window are in one-to-one correspondence with points on the staircase (and therefore the tiling).

$$\Lambda_a = M\Lambda_a \cup M\Lambda_b \quad \text{and} \quad \Lambda_b = M\Lambda_a + \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

Projecting onto the window yields

$$W_a = \frac{-1}{\tau} W_a \cup \frac{-1}{\tau} W_b \quad \text{and} \quad W_b = \frac{-1}{\tau} W_a + proj_{wk} \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

This is an IFS whose attractor contains the window.

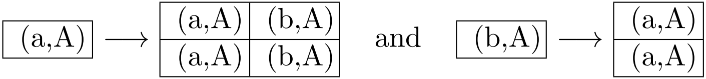
‘Fibinoid’ tilings

Fibinoid direct product substitution

Fibonacci $\times$ 'solenoid' or 2-adic substitution

$$a \to ab \quad b \to a \qquad \text{and} \qquad A \to AA$$

we end up with alphabet $\{(a, A), (b, A)\}$ and we obtain



Canonical tiles

We obtain tiles for an inflate-and-subdivide rule:

- The horizontal expansion factor is τ and the vertical expansion factor is 2

$$Q = \begin{pmatrix} \tau & 0 \\ 0 & 2 \end{pmatrix}.$$

- tile widths are τ and 1 and heights should be equal.
- The canonical tiles can be:

$(a, A) =$  $(b, A) =$ 

Fibinoid inflate-and-subdivide rule

The symbolic substitution

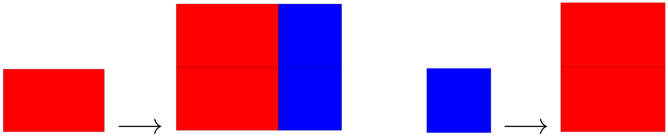
(a,A) ⟶

(a,A)	(b,A)
(a,A)	(b,A)

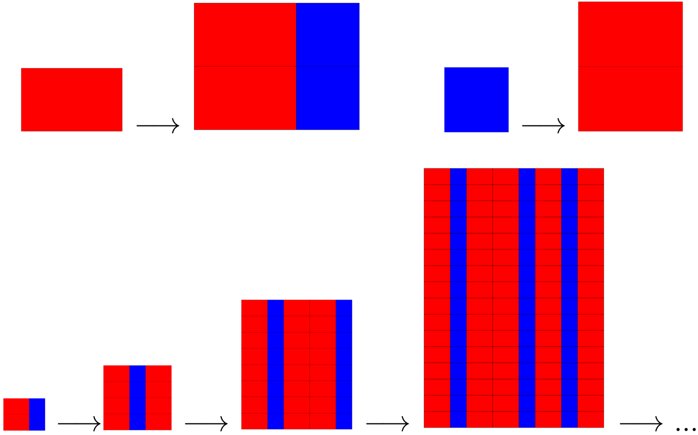
and (b,A) ⟶

(a,A)
(a,A)

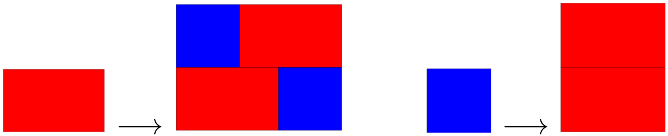
becomes the inflate-and-subdivide rule

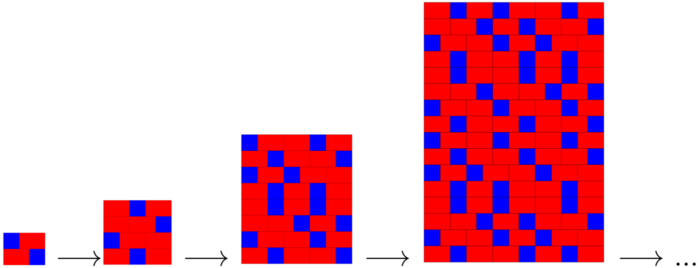


Iterating the Fibinoid DP

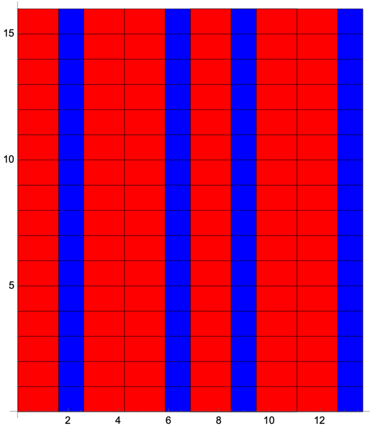


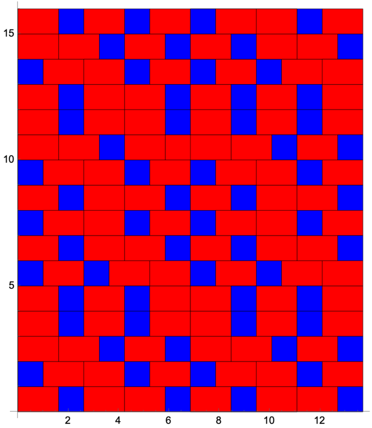
A Fibinoid variation



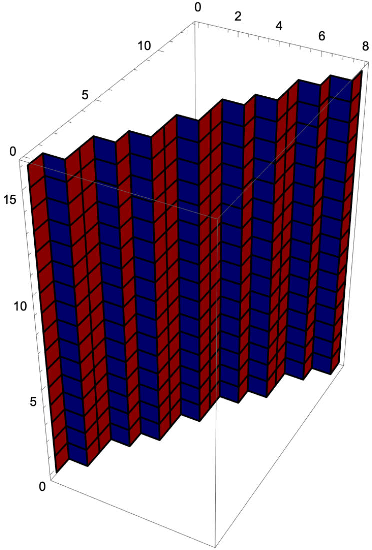


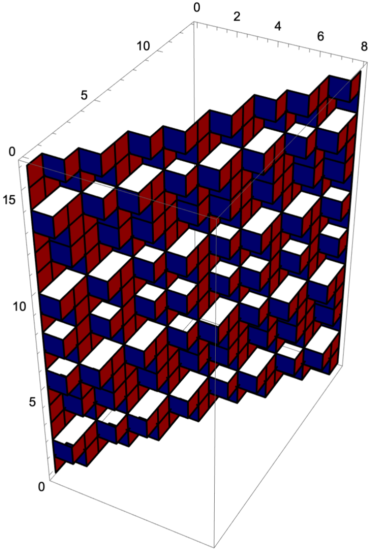
Fibinoid DP vs DPV





Fibinoid 3D stepped surfaces



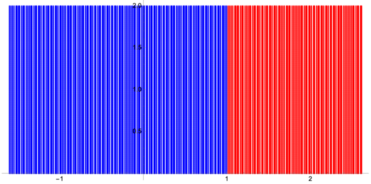


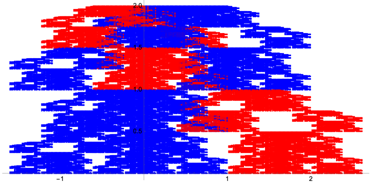
Fibinoid not-windows

- There are again renormalization relations for the points on the stepped surface
- If we adopt a fourth dimension to go with the vertical axis, we have renormalization given by the matrix

$$Q = \begin{pmatrix} \tau & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & -1/\tau & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

- This induces an IFS on the weak eigenspace as before.





Fibonacci DPVs

Fibonacci direct product substitution

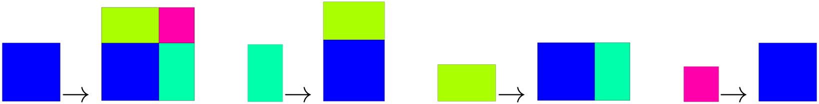
Fibonacci × Fibonacci: $\mathcal{A} = \{(a,a),(a,b),(b,a),(b,b)\}$

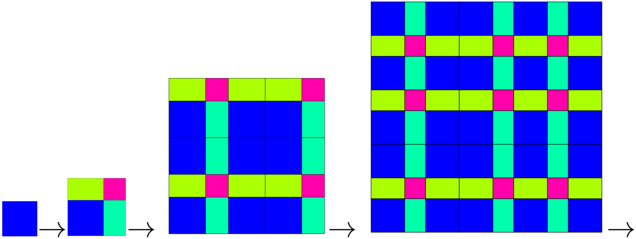
$$\boxed{\text{(a,a)}} \longrightarrow \begin{array}{|c|c|}\hline \text{(a,b)} & \text{(b,b)} \\ \hline \text{(a,a)} & \text{(b,a)} \\ \hline \end{array}, \qquad \boxed{\text{(a,b)}} \longrightarrow \begin{array}{|c|c|}\hline \text{(a,a)} & \text{(b,a)} \\ \hline \end{array},$$

$$\boxed{\text{(b,a)}} \longrightarrow \begin{array}{|c|}\hline \text{(a,b)} \\ \hline \text{(a,a)} \\ \hline \end{array}, \qquad \boxed{\text{(b,b)}} \longrightarrow \boxed{\text{(a,a)}}.$$

The canonical tiles have side lengths given by τ and 1.

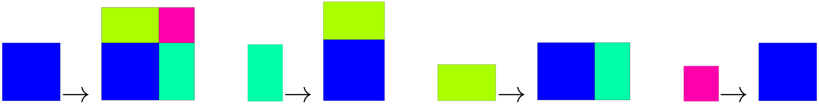
Fibonacci direct product substitution



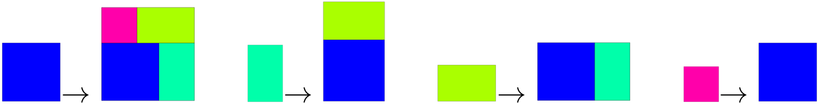


A variation

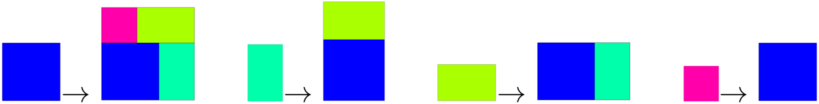
The DP:

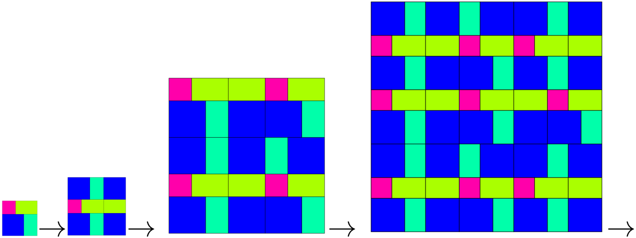


A DPV:



A variation





There are 48 total variations, including the DP itself.

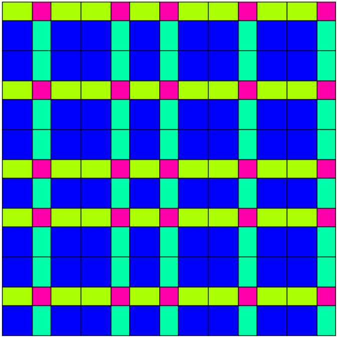
The measure-theoretic situation

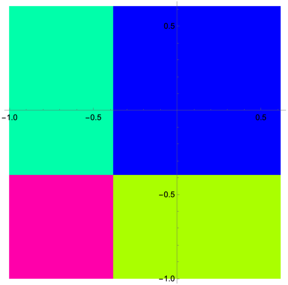
Theorem (Baake-F.-Grimm, 2021)

The 48 inflation tiling dynamical systems that emerge from the Fibonacci DPVs all have pure point dynamical spectrum.

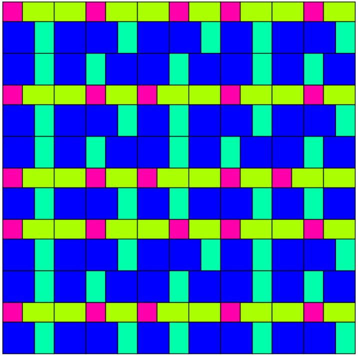
- The point spectrum is $L^* \times L^*$, where $L^* = \mathbb{Z}[\tau]/\sqrt{5}$.
- All 48 systems are measure-theoretically isomorphic.
- They are all model sets
- We can compute* windows with an IFS derived from the substitution as an action on $\mathbb{Z}^4$.

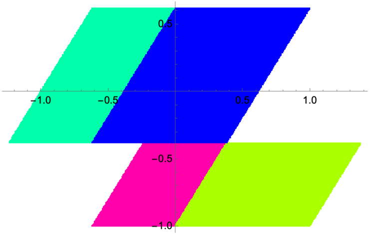
Fibonacci DP



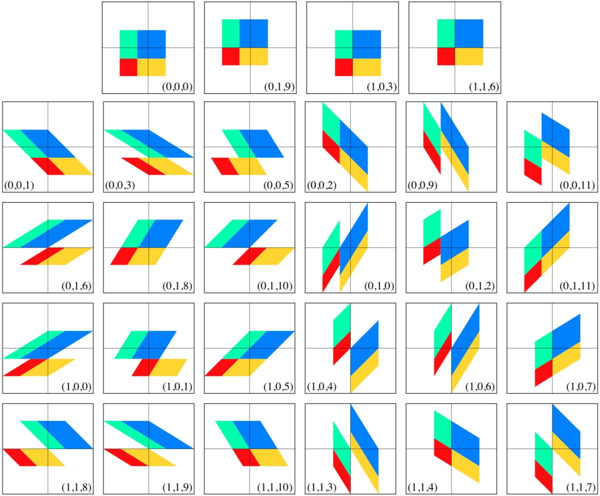


Fibonacci DPV–skewed window





The 28 polygonal windows



Fibonacci DPVs with polygonal windows are topologically conjugate

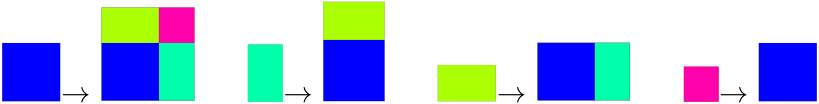
Theorem (Baake, Gähler, Mazáč, preprint 2022)

The 28 inflation tiling dynamical sustems that emerge from the DPVs with polygonal windows form one class of topologically conjugate dynamical systems.

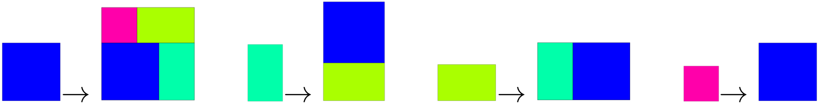
- When two systems have windows with different slopes, they cannot be MLD
- So these topological conjugacies require far-flung information

Some DPVs with fractal windows

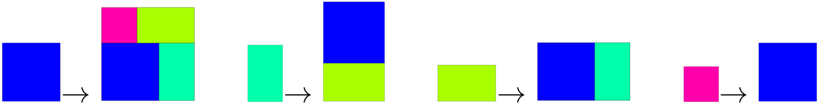
Recall the direct product substitution:



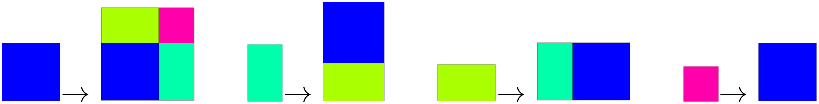
Type ‘island’



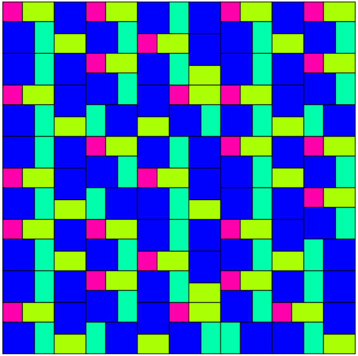
Type ‘cross’

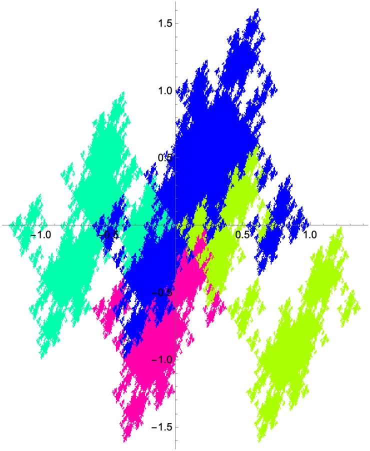


Type ‘castle’

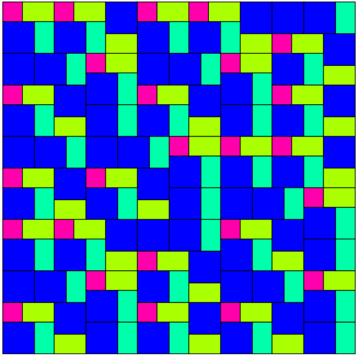


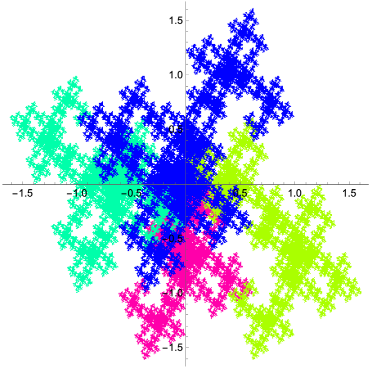
Rauzy fractal 'island'



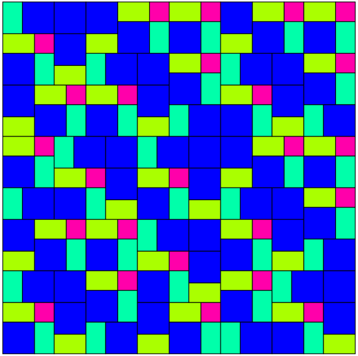


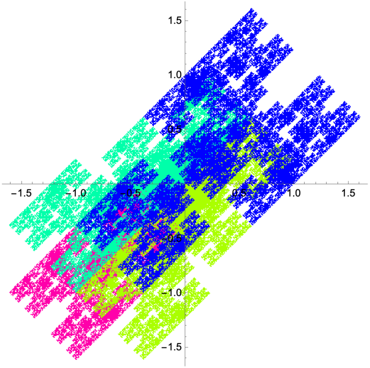
Rauzy fractal 'cross'



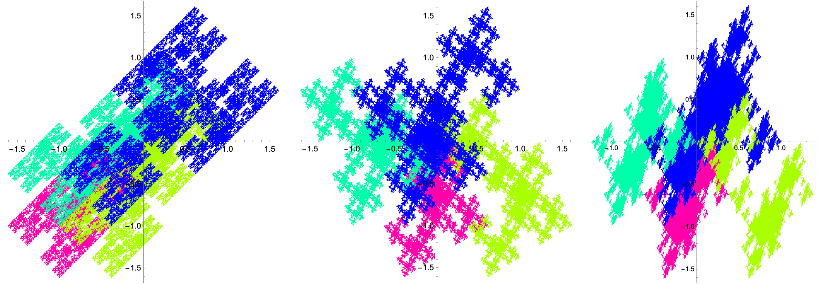


Rauzy fractal 'castle'





Rauzy fractal window comparison



$dim_H = 1.875$ $dim_H = 1.756$ $dim_H = 1.561$

Calculated by Bernd Sing

General DPVs in two dimensions

Direct products and variations

Given substitutions σ_1 and σ_2 on alphabets $\mathcal{A}_1$ and $\mathcal{A}_2$.
The *direct product substitution* σ is defined on $\mathcal{A} = \mathcal{A}_1 \times \mathcal{A}_2$ as

$$\sigma(a, b) = \sigma_1(a) \times \sigma_2(b),$$

a rectangular word of size $|\sigma_1(a)| \times |\sigma_2(b)|$ in which:

- Each row has $\sigma_1(a)$ in coordinate 1, read left to right
- Each column has $\sigma_2(b)$ in coordinate 2, read bottom to top

Canonical tiles are rectangles with dimensions given by the natural tile lengths.

DPVs and their canonical tilings.

A *direct product variation* substitution is a substitution rule obtained from a direct product substitution through rearrangement within some or all of the supertiles.

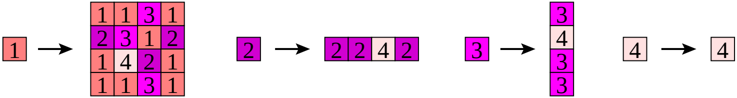
We do not attempt to classify which rearrangements will result in a substitution rule.

Park-Robinson 1991: Chacon $\mathbb{Z}^2$ action

The Chacon substitution (orig. Chacon "cut-and-stack"):

$$a \to aaba \quad b \to b$$

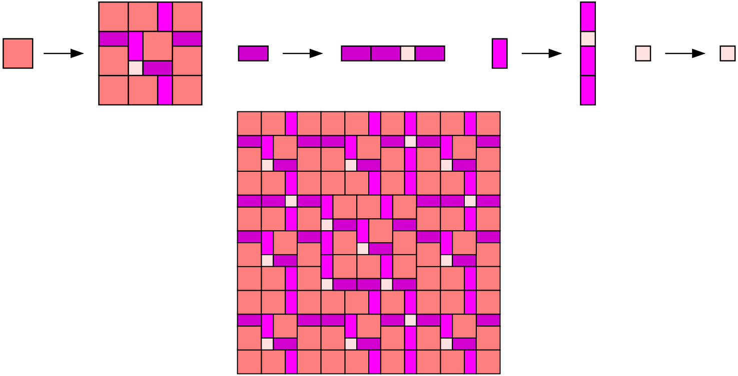
A variation on Chacon $\times$ Chacon:



There aren't really canonical tiles for this one

Park-Robinson 1991: Chacon $\mathbb{Z}^2$ action

Park-Robinson's version:

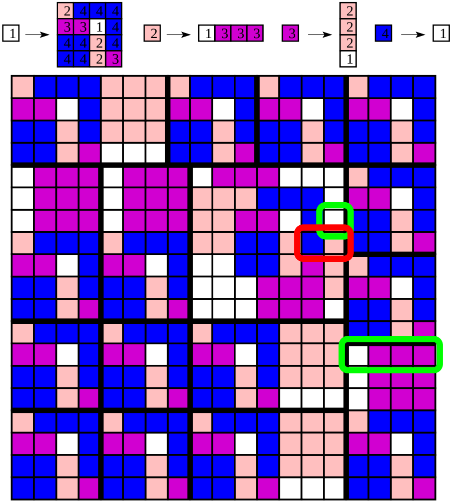


Theorem (Park-Robinson, 1991)

The Chacon Z^2-action is weakly mixing but not mixing.

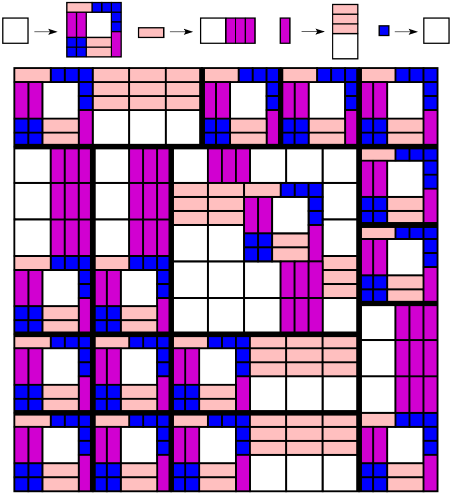
A non-Pisot DPV–the symbolic case

$$a \to abbb \quad b \to a$$



A non-Pisot DPV with canonical tiles

$$a \to abbb \quad b \to a$$



Finite local complexity

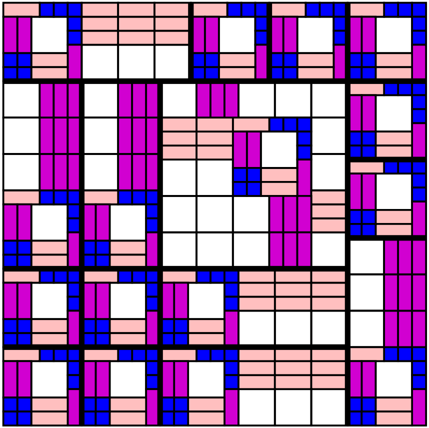
A tiling is said to be of *finite local complexity (FLC)* if it contains only a finite number of two-tile patches up to translation. If not, it is of *infinite local complexity (ILC)*.

Theorem (F.-Robinson, 2008)

Suppose that S is a property-(C) substitution on a finite set of fractagonal prototiles. If the length expansion of S is a Pisot number, then any tiling admitted by S is locally finite.

Corollary

Let σ be a one-dimensional substitution with a Pisot expansion factor. Every canonical DPV tiling constructed from $\sigma \times \sigma$ is locally finite.



Theorem (F.-Robinson, 2008)

The tiling above is of infinite local complexity.

Some questions

- There are partial results on the dffraction/dynamical spectrum questions
- There is no general theory for the windows
- The question of FLC for a general $\sigma_1 \times \sigma_2$ hasn't been looked at systematically.
  - If the expansion factors for both σ_1 and σ_2 are Pisot, one would expect FLC.
  - If one or both of the expansion factors is not Pisot, there can be FLC or ILC:
    - The direct product substitution will always have finite local complexity
    - In the presence of a (strongly) non-Pisot expansion factor, is it possible for *any* nontrivial variation to maintain FLC?