Topological analysis of substitution tiling spaces

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Vassar College

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Question.

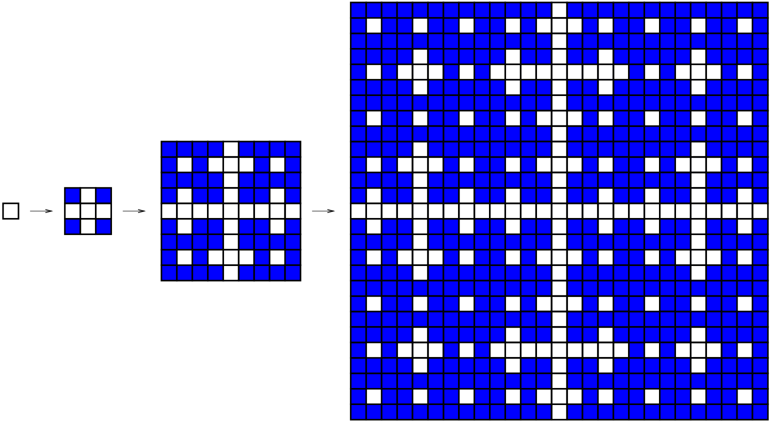
Suppose you had an INFINITE SUPPLY
of blue and white SQUARE TILES.
How might you form an infinite tiling of the plane $\mathbb{R}^2$?

An interesting answer.

Use a tiling substitution!

An example of a *substitution rule*.

Iterate the substitution to get arbitrarily large patches:



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- In this talk we'll see that substitution tiling spaces are Cantor set fiber bundles that can be seen as inverse limits and that their cohomology can be computed.

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- Standing assumptions: finite local complexity, nonperiodicity

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- σ can be applied to any patch of tiles by applying σ to each tile t in the patch and placing the result atop $L(t)$.
- We call $\sigma(t)$ a supertile, $\sigma^2(t)$ a 2-supertile, and $\sigma^n(t)$ an n-supertile

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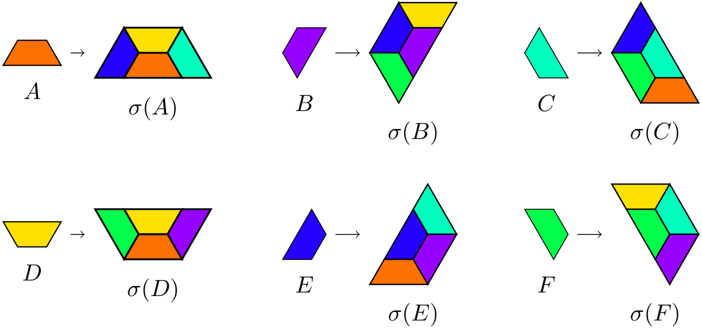

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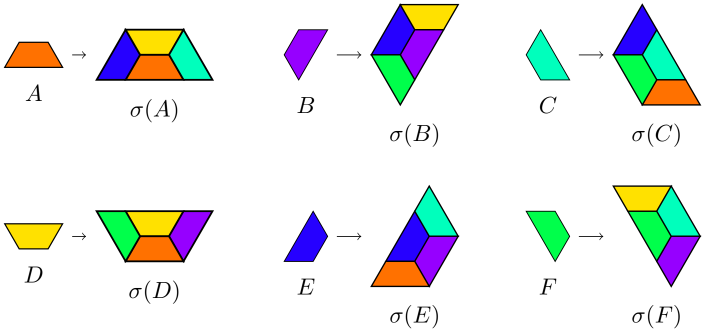
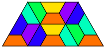


The half-hex substitution rule.

The prototile set $\mathcal{P}$ contains six tile types rather than one because our framework considers tiles the same only if they are translates of one another.

A few half-hex supertiles

2-, 3-, and 4-supertiles



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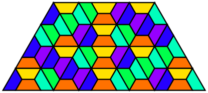

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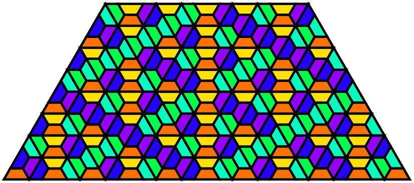

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- The set of all n-supertiles acts as the 'language' of Ω
- If $T \in \Omega$, then $T - \vec{x} \in \Omega$ for any translation $\vec{x} \in \mathbb{R}^d$

The big ball metric

How to measure the distance between tilings

Let T and T' be tilings of $\mathbb{R}^d$ from a prototile set $\mathcal{P}$.

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We define

$$d(T, T') = \min\left\{\frac{1}{R(T, T')}, 1\right\}$$

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Substitution tiling spaces: topological basics

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- Simplicial, singular, and cellular cohomology don't work well either, since they study path connected components.
- Čech cohomology is computable and we will do the half-hex.

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Theorem (SW)

A tiling space that satisfies certain WLOG[1] hypotheses is a fiber bundle over the torus, with totally disconnected fiber.

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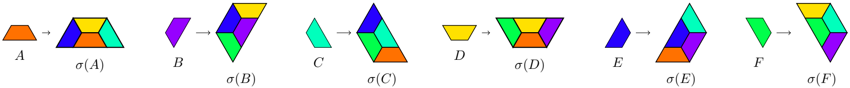

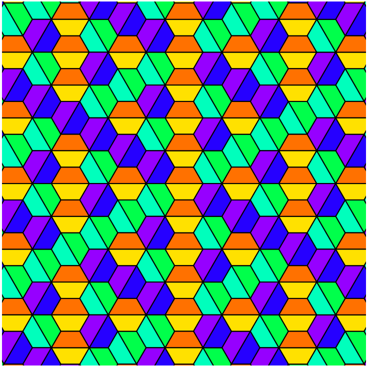

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Substitution tiling spaces as inverse limits

Overview

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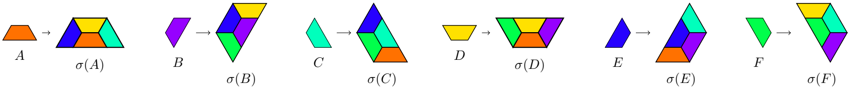

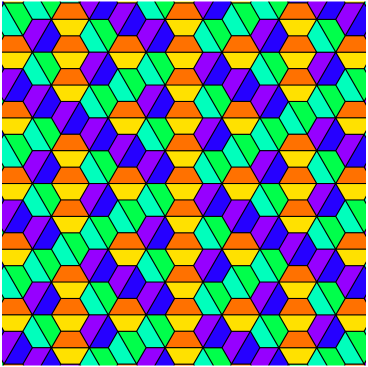

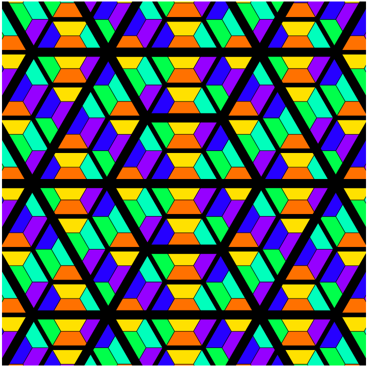

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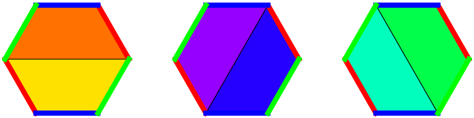
  - Since every $(n+1)$-supertile is composed of n-supertiles, the CW complex for the $(n+1)$-supertiles maps onto that of the n-supertiles.
- Every tiling $T \in \Omega$ corresponds to an element of the inverse limit by noting the location of 0 in each of its n-supertiles.

Γ_0 is the CW complex given by all prototiles, with edges identified if they meet in a tiling in Ω.



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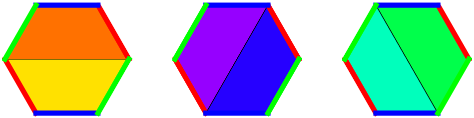

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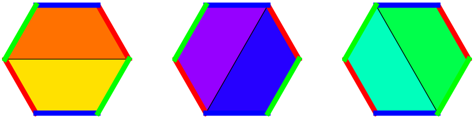

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- Conversely, a point interior to Γ_0 unambiguously tells how to place a tile at the origin.

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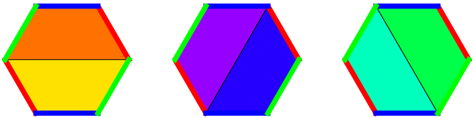

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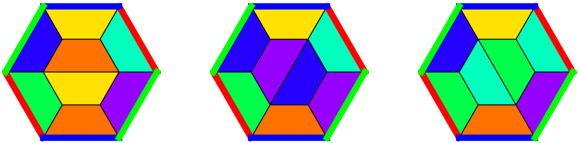
- Conversely, a point interior to Γ_0 unambiguously tells how to place a tile at the origin.
- A branch point in Γ_0 yields a few choices of patches.

Γ_1 is the CW complex given by all 1-supertiles, with superedges identified if they meet in a tiling in Ω.



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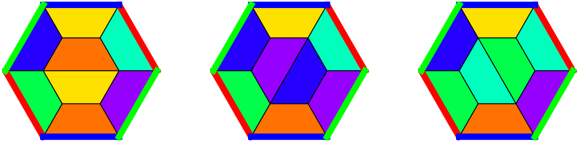

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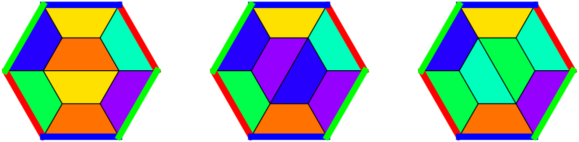

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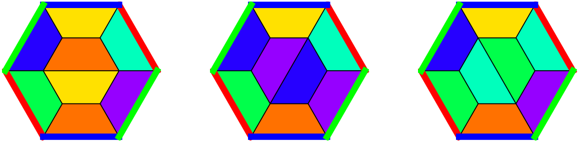

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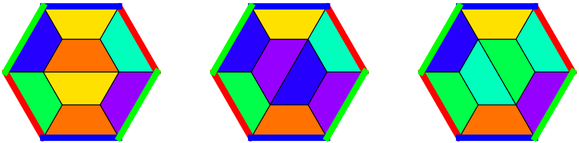

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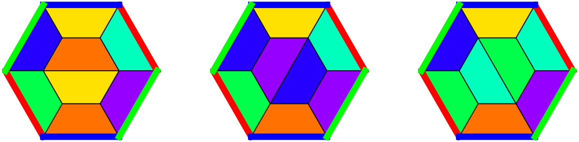

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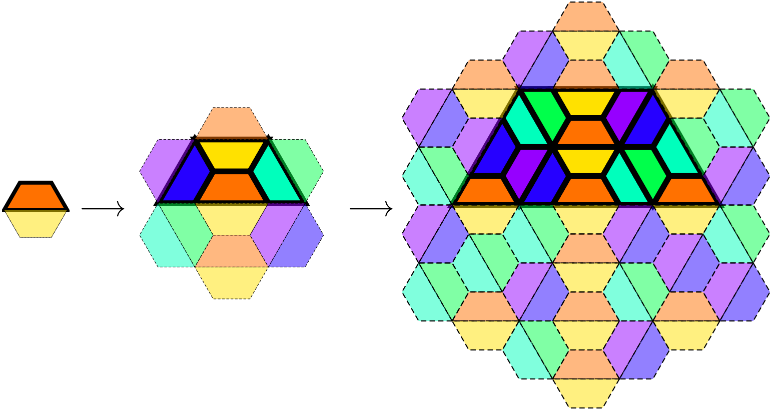
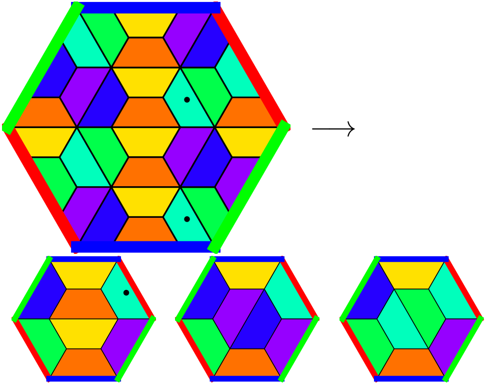
We make Γ_n in exactly the same fashion, using n-supertiles. This gives instructions for tiling larger and larger regions.

A substitution forces the border if there is an N such that every N-supertile determines the tiles immediately adjacent to it.



The half-hex substitution forces the border with $N = 2$.



The forgetful map ϕ_1 on part of Γ_2

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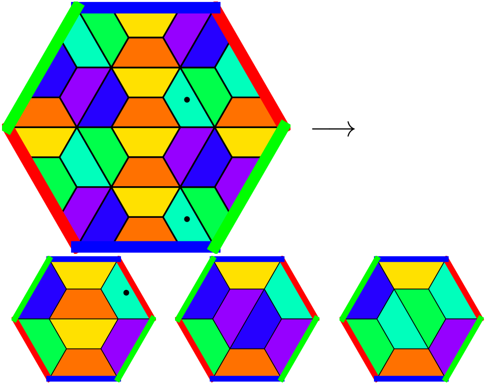

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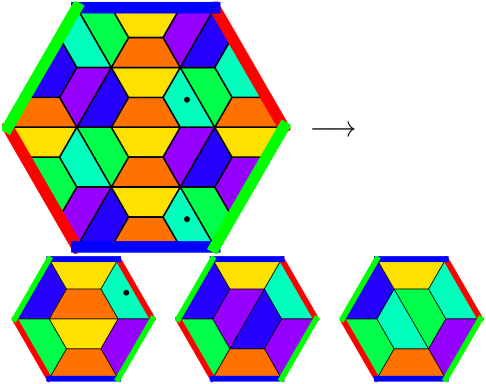

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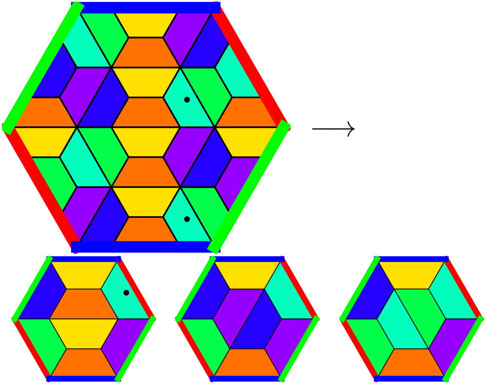

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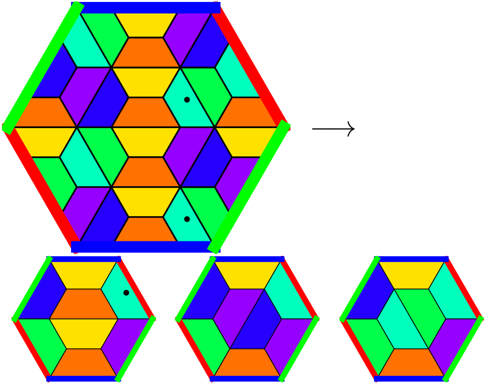

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Theorem (AP)

When the substitution forces the border, Ω and $\varprojlim(\Gamma_n, \phi_n)$ are homeomorphic.

(If it doesn't force the border we use a trick called “collaring”)

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(We need it but can avoid computing it directly)

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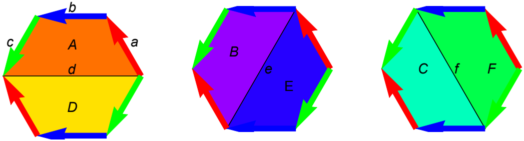
- Take the direct limit of the matrix to get the cohomology of the inverse limit and thus of Ω.



The labelled CW complex Γ

After a bit of linear algebra we obtain that

$$H^0(\Gamma, \mathbb{Z}) = \mathbb{Z} \qquad H^1(\Gamma, \mathbb{Z}) = \mathbb{Z}^2 \qquad H^2(\Gamma, \mathbb{Z}) = \mathbb{Z}^3$$

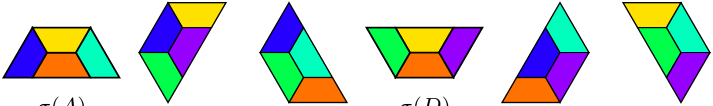
Let $A^*, B^*, C^*, D^*, E^*, F^*$ represent the dual cochains to the 2-chains A, B, C, D, E, F. The equivalence relation in the quotient for H^2 gives $A^* = D^*, B^* = E^*$, and $C^* = F^*$.

Generators for $H^2(\Gamma, \mathbb{Z})$ are A^*, B^*, C^*

Top Čech cohomology of the half-hex

Forgetful map as substitution

The six 2-cells of Γ_1, with the map onto Γ_0 indicated:



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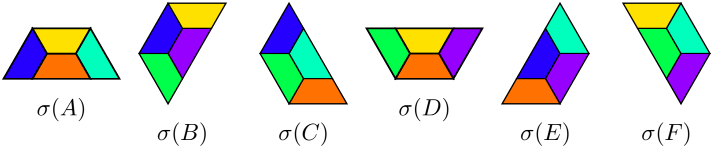

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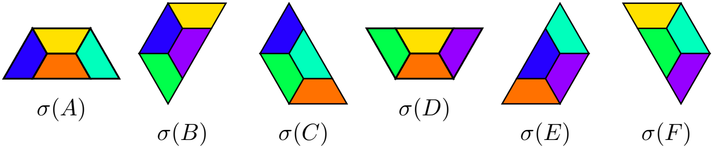

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The forgetful map ϕ on 1-chains is computed to be:

$$A \to A + C + D + E \quad B \to B + D + E + F \quad C \to A + C + E + F$$
$$D \to A + B + D + F \quad E \to A + B + C + E \quad F \to B + C + D + F$$

We compute the pullback $\phi^* : C^2(\Gamma_n) \to C^2(\Gamma_{n+1})$:

$$\phi^*(\omega)(q) = \omega(\phi(q)),\ \omega \in C^2(\Gamma_n) \text{ and } q \in C_2(\Gamma_{n+1})$$

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Pullback of the forgetful map

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Consider $A^* \in C^2(\Gamma_n)$ and any $q \in C_2(\Gamma_{n+1})$.

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- Thus $\phi^*(A^*) = A^* + C^* + D^* + E^* \cong 2A^* + B^* + C^*$

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Using similar logic for the other two generators of $H^2(\Gamma_n, \mathbb{Z})$ we find that in the basis A^*, B^*, C^*, ϕ^* acts as the matrix
$\begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix}$

$$\check{H}^2(\Omega, \mathbb{Z}) = \varinjlim (H^2(\Gamma, \mathbb{Z}), \phi^*) = \varinjlim \left(\mathbb{Z}^3, \begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix} \right)$$

The eigenvalues are $4, 1, 1$ and in the final analysis we arrive at:

$$\check{H}^2(\Omega, \mathbb{Z}) = \mathbb{Z}[1/4] \oplus \mathbb{Z} \oplus \mathbb{Z}$$

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The two copies of $\mathbb{Z}$ are generated by cochains that can tell an A from a B and a B from a C.

Top Čech cohomology of the half-hex

Intuitive interpretation

$$\check{H}^2(\Omega, \mathbb{Z}) = \mathbb{Z}[1/4] \oplus \mathbb{Z} \oplus \mathbb{Z}$$

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The two copies of $\mathbb{Z}$ are generated by cochains that can tell an A from a B and a B from a C.

Taken together the cohomology has the ability to recognize types of supertiles of all orders, up to the identification $A^* = D^*, B^* = E^*$, and $C^* = F^*$.

First Čech cohomology of the half-hex

In a similar fashion we can compute that

$$\check{H}^1(\Omega, \mathbb{Z}) = \varinjlim(H^1(\Gamma, \mathbb{Z}), \phi^*) = \varinjlim \left(\mathbb{Z}^2, \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} \right)$$

And ultimately

$$\check{H}^1(\Omega, \mathbb{Z}) = \mathbb{Z}[1/2] \oplus \mathbb{Z}[1/2].$$

This is harder to interpret but reflects the linear expansion factor of two on edges.

References

[AP] J. Anderson and I.F. Putnam, Topological invariants for substitution tilings and their C^*-algebras, *Ergodic Th. and Dynam. Sys.* **18** (1998), 509–537.

[SW] L. Sadun and R.F. Williams, Tiling spaces are Cantor set fiber bundles, *Ergodic Theory and Dynamical Systems* **23** (2003), 307–316.

[Sa] L. Sadun, *Topology of tiling spaces*, Univ. Lecture Ser. **46**, Amer. Math. Soc., Providence (2008).

[Sol1] B. Solomyak, *Dynamics of self-similar tilings*, Ergodic Th. and Dynam. Sys. **17** (1997), 695–738.

[Sol2] B. Solomyak, *Nonperiodicity implies unique composition for self-similar translationally finite tilings*, Disc. Comp. Geom. **20** (1998), 265–279.

[TilEncy] *The Tilings Encyclopedia*, Web address: http://tilings.math.uni-bielefeld.de/