Flow views and infinite interval exchange transformation for substitution tilings

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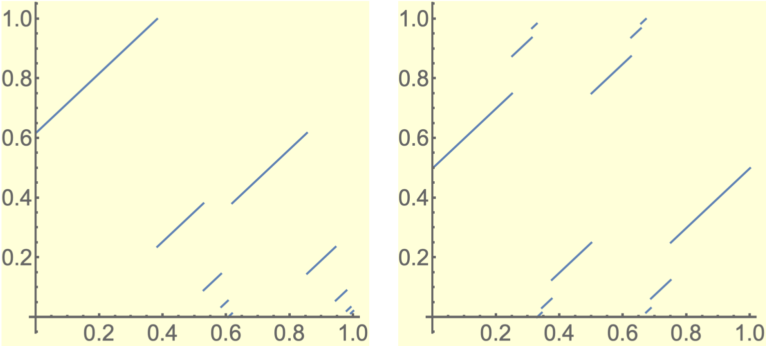
Canonical IIETs for substitutions.

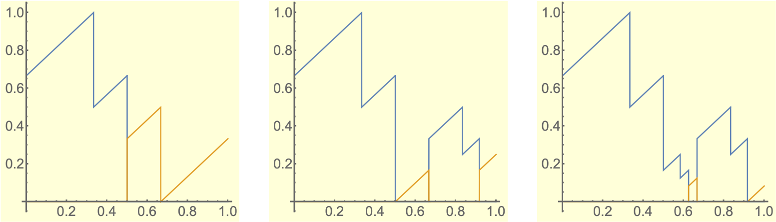
I'll show you how to construct an infinite interval exchange transformation (IIET) $\mathfrak{F}$ to represent any minimal and recognizable substitution subshift in $\mathbb{Z}$.



(a) $\mathfrak{F}_7$ for the **Fibonacci** $\alpha \to \alpha\beta$, $\beta \to \alpha$.

(b) $\mathfrak{F}_5$ for **Thue–Morse** $\alpha \to \alpha\beta$, $\beta \to \beta\alpha$.

Efficiency. All but ϵ of $[0,1]$ is contained in $\mathcal{O}(|\ln(\epsilon)|)$ intervals in the domain of $\mathfrak{F}$.



The graphs of $\mathfrak{F}_1, \mathfrak{F}_2$, and $\mathfrak{F}_3$ for $\mathcal{S}_{PD}$.

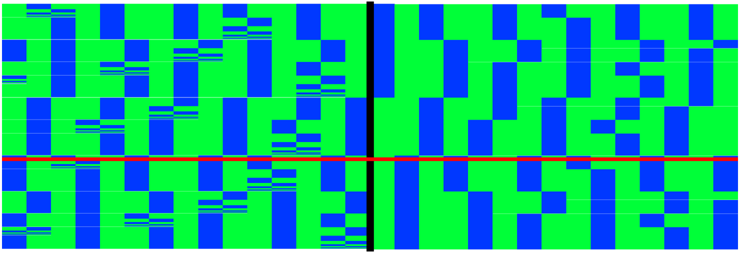
Flow view: a picture of the substitution subshift

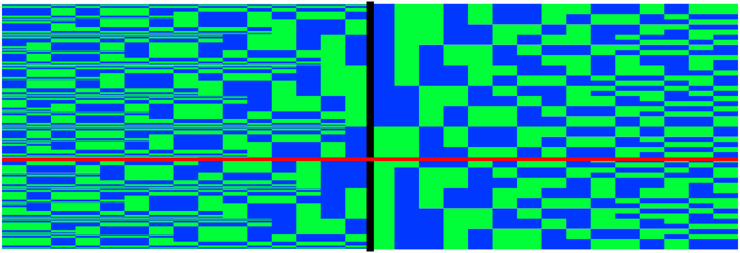
Definition

A ***flow view*** is the graph of a conjugacy $\Phi : \mathbf{\Sigma} \to [0, 1]$ between σ and $\mathfrak{F}$.

- It literally graphs the a.e. one-to-one correspondence between $[0, 1]$ and the subshift by showing each $\boldsymbol{\tau} \in \mathbf{\Sigma}$ (in colored unit interval tiles) at a height of $\Phi(\boldsymbol{\tau})$.
- The IIET can be understood as a shift on the flow view.

Flow views for Fibonacci and Thue–Morse subshifts.

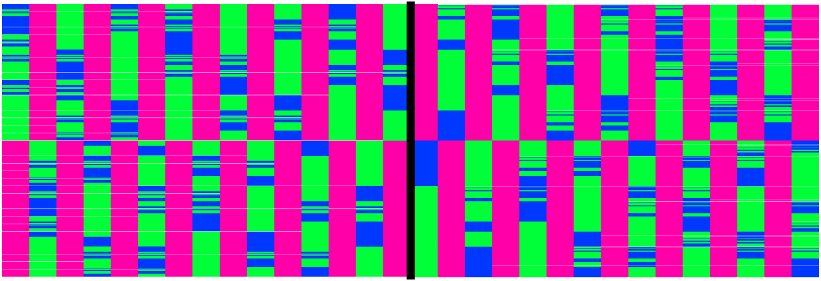




The red line highlights the $\boldsymbol{\tau} \in \boldsymbol{\Sigma}$ for which $\Phi(\boldsymbol{\tau}) = 1/e$.

Advantage: the shift is in the flow view

Shifting moves the central interval representing $[0, 1]$ one unit to the right.

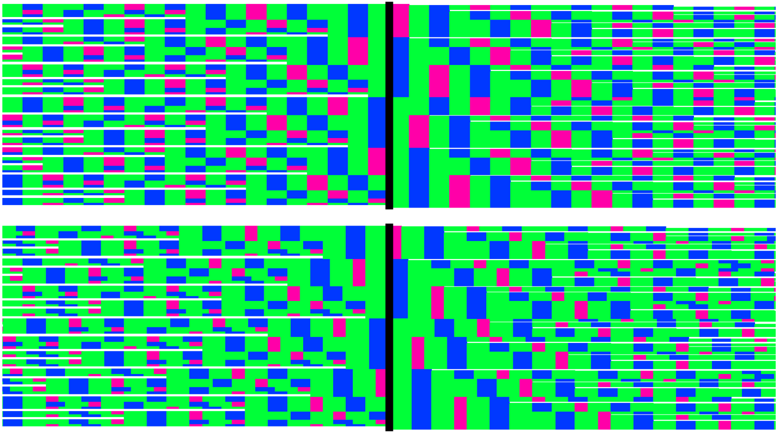


This fun example is of constant length 3 with height 2 and no coincidences.

Advantage: it generalizes

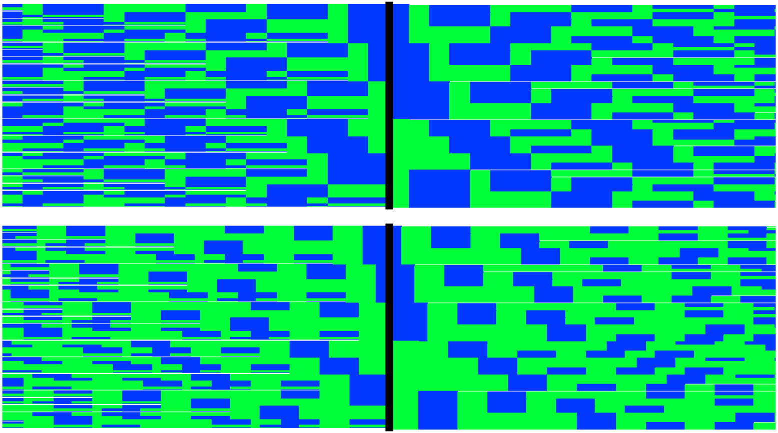
- A straightforward adaptation to the proof makes IIETs for a large class of **S-adic systems**
- Self-similar and fusion tilings of $\mathbb{R}$ are suspensions; the IIET represents the first return map to a transversal.
- The construction works in some higher dimensional situations to produce commuting IIETs on $[0, 1]$

Flow view comparison: Tribonacci



Top: unit length tiles. Bottom: natural length tiles. For the tiling flow, the IIET is the first return map to the transversal of all tilings with an endpoint at 0.

Flow views for $A \to A\,B\,B\,B,\;,B \to A.$

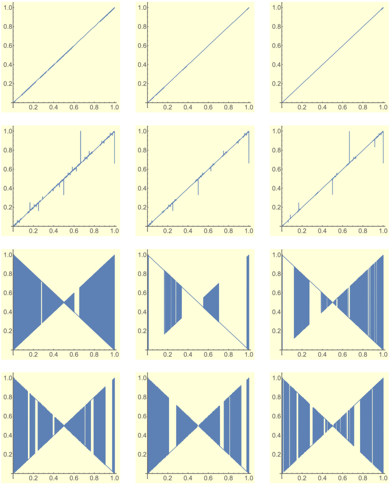


Top: unit length tiles. Bottom: natural length tiles.

Advantage: applications and connections

- Spectral theory. Φ is a particularly nice element of $L^2(\boldsymbol{\Sigma}, \mu)$
- Self-similar functions. The graphs of the IIETs always show some form of it.
- IIETs provide an unlimited stable of translation surfaces that are probably of infinite genus and retain some kind of self-inducing properties
- There are tons of questions this whole theory brings up. If you are/have a student that is interested, I am happy to share.
    - Side note: mathematica users could use/help develop my package for these images

Spectral theory: $\mathfrak{F}_{20}^{j}$, where $j = 2^8, 2^9$, and 2^{10}.



Dyadic odometer, period-doubling, Thue–Morse, and Rudin–Shapiro.

Three main ingredients

1. A system for associating each $\boldsymbol{\tau} \in \boldsymbol{\Sigma}$ with an address $\mathbf{a}(\boldsymbol{\tau}) = (\mathsf{a}_1, \mathsf{a}_2, \ldots)$,
   - $\mathsf{a}_n \in \mathsf{A} \neq \mathcal{A} \subset \mathcal{A} \times \mathbb{N}$
   - the label a_n represents how the $(n-1)$-supertile sits inside its n-supertile
2. A function ϕ on the alphabet A.
   - This function depends on a choice of dual substitution
   - Uses the frequency vector of $\mathcal{S}$ as a modified length vector
3. A function $\Phi(\mathbf{a}) = \Phi_0(\mathsf{a}_1) + \sum_{n=1}^{\infty} \phi(\mathsf{a}_n)\lambda^{-n}$,
   - λ is the expansion constant
   - $\Phi_0(\mathsf{a}_1)$ depends on the letter at the origin
   - Note: a_1 is the most significant digit

Setting: symbolic and substitution dynamical systems

- Finite ***alphabet*** $\mathcal{A}$
- ***substitution rule*** map $\mathcal{S}: \mathcal{A} \to \mathcal{A}^+$
- For each $\alpha \in \mathcal{A}$ we write $\mathcal{S}(\alpha) = \alpha_1\alpha_2 ... \alpha_l$
- $\mathcal{S}^n(\alpha)$ is called an n***-supertile of type*** α

$$\mathcal{S}^n(\alpha) = \mathcal{S}^{n-1}(\alpha_1)\,\mathcal{S}^{n-1}(\alpha_2)\ldots\mathcal{S}^{n-1}(\alpha_l). \qquad (1)$$

  - This is a *fusion* perspective, so it will work for S-adic systems
  - The word $\mathcal{S}^n(\alpha)$ is assumed to begin at $1 \in \mathbb{Z}$ for any $n \geq 1$.

The substitution subshift $\mathbf{\Sigma}$

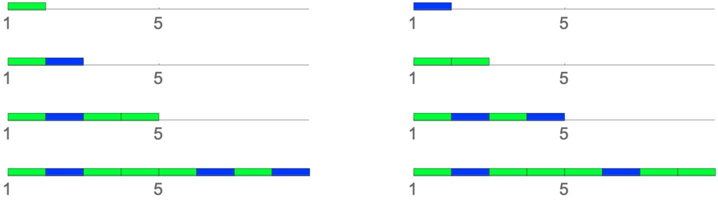
Use $\mathcal{L} = \{\mathcal{S}^n(\alpha), n \in \mathbb{N}$ and $\alpha \in \mathcal{A}\}$ as a 'language':

> *We say $\boldsymbol{\tau} \in \mathcal{A}^{\mathbb{Z}}$ is **admitted** by $\mathcal{S}$ if and only if every finite subword of $\boldsymbol{\tau}$ is a subword of an element of $\mathcal{L}$.*

Definition

The set $\mathbf{\Sigma} = \{\boldsymbol{\tau} \in \mathcal{A}^{\mathbb{Z}}$ admitted by $\mathcal{L}\}$, if nonempty, is endowed with the subspace topology, shift σ, and a shift-invariant Borel probability μ to become the ***substitution subshift*** $(\mathbf{\Sigma}, \sigma, \mu)$.

Period-doubling substitution, our illustrating example



The A tile (top left) and the B tile (top right) with 1-, 2, and 3-supertiles below.

Measure stuff

- ***transition matrix*** of $\mathcal{S}$ is the matrix M for which M_{ij} is the number of α_i's in $\mathcal{S}(\alpha_j)$.
- The ***"PF eigenvalue"*** $\lambda \geq |\lambda'|$ for any other eigenvalue.
  - We call λ the ***expansion factor*** of $\mathcal{S}$.
  - In the non-primitive case it may not be unique
- Supertile lengths are given by $[1\,1\ldots 1]M^n = [|S^n(\alpha_1)|\,|S^n(\alpha_2)|\ldots|S^n(\alpha_{|\mathcal{A}|})|]$..
- A left eigenvector for λ represents the *natural lengths* for self-similar tilings.
- A right probability eigenvector $\vec{r}$ for λ represents relative frequencies of letters in $\mathcal{A}$, at least in a subspace of $\mathbf{\Sigma}$.
- There is an invariant measure μ such that $\mu([\alpha_j]) = \vec{r}(j)$ for all $j = 1, 2, ..., |\mathcal{A}|$.
- Handy fact: $\mu(\mathcal{S}^n([\alpha_j])) = \vec{r}(j)/\lambda^n$.

Recognizability

$\mathcal{S}$ is ***Recognizable:***

there is $R > 0$ s.t. if $\boldsymbol{\tau}, \boldsymbol{\tau}' \in \boldsymbol{\Sigma}$ and $\boldsymbol{\tau}[n-R, ..n+R] = \boldsymbol{\tau}'[n-R, ..n+R]$, then $\boldsymbol{\tau}(n)$ and $\boldsymbol{\tau}'(n)$ are in exactly the same location of the same supertile.

- For $\boldsymbol{\tau} = \{\alpha_n\}_{n\in\mathbb{Z}}$, define $\mathcal{S}(\boldsymbol{\tau})$ to be ...$\mathcal{S}(\alpha_{-1})\mathcal{S}(\alpha_0)\mathcal{S}(\alpha_1)$... where $\mathcal{S}(\alpha_1)$ starts at 1.
- Recognizability extends to supertiles of any level

The canonical partitition sequence of n-cylinders

- For $\alpha \in \mathcal{A}$ let $[\alpha] = \{\boldsymbol{\tau} \in \boldsymbol{\Sigma} \text{ with } \boldsymbol{\tau}(1) = \alpha\}$.
- Define the ***n*-cylinder** to be the set of all tilings with a true n-supertile of type α starting at 1:

$$\mathcal{S}^n([\alpha]) = \{\mathcal{S}^n(\boldsymbol{\tau}), \boldsymbol{\tau} \in [\alpha]\}$$

- For each $n = 0, 1, 2, ...,$

$$\mathcal{B}_n = \left\{\sigma^k(\mathcal{S}^n([\alpha])), \alpha \in \mathcal{A} \text{ and } 1 \leq k < |\mathcal{S}^n(\alpha)|\right\} \tag{2}$$

  forms a partition of $\boldsymbol{\Sigma}$.
- Sets of the form $\sigma^k(\mathcal{S}^n([\alpha]))$ with $1 \leq k < |\mathcal{S}^n(\alpha)|$ are called ***n*-cylinders**.
  - There may be a difference between $[\mathcal{S}(\alpha)]$ and $\mathcal{S}([\alpha])$.

Alphabet for addresses

- The ***domain*** of $\mathcal{S}$ is the subset of $\mathcal{A} \times \mathbb{N}$ given by

$$\mathsf{A} = \{\mathsf{a} := (\alpha, j) \text{ such that } 1 \leq j \leq |\mathcal{S}(\alpha)|\}.$$

  - The projection maps $\pi_{\mathcal{A}}(\mathsf{a})$ and $\pi_{\mathbb{N}}(\mathsf{a})$ are used when needed.
- Two crucial uses:
  1. To specify the word $\mathcal{S}(\mathsf{a}) := \sigma^j(\mathcal{S}(\alpha))$.
  2. To identify the the letter ***in the*** j***th position of*** $\mathcal{S}(\alpha)$, which is $\mathcal{S}(\alpha)(j) = \sigma^j(\mathcal{S}(\alpha))(0) = \mathcal{S}(\mathsf{a})(0)$.

1-addresses for elements of $\boldsymbol{\Sigma}$

Premise: The 1-supertile **at the origin** in any $\boldsymbol{\tau} \in \boldsymbol{\Sigma}$ has a unique label in A by recognizability. So do all n-supertiles, being concatenations of $(n-1)$-supertiles

The 1-***address of*** $\boldsymbol{\tau}$ is given by $\mathbf{a}_1(\boldsymbol{\tau}) = \mathsf{a} \in \mathsf{A}$, where $\boldsymbol{\tau}(0)$ is in the jth spot of the 1-supertile $\mathcal{S}(\alpha)$. Equivalently, $\boldsymbol{\tau} \in \sigma^j(\mathcal{S}[\alpha])$.

We define the 1-***cylinder*** of $\mathsf{a} = (\alpha, j)$ to be

$$[\mathcal{S}(\mathsf{a})] = \{\boldsymbol{\tau} \in \boldsymbol{\Sigma} \text{ such that } \mathbf{a}_1(\boldsymbol{\tau}) = \mathsf{a}\} = \sigma^j(\mathcal{S}([\alpha])).$$

First hint of the dual substitution

By recognizability, the origin is inside a unique nested sequence of n-supertiles in any $\boldsymbol{\tau} \in \boldsymbol{\Sigma}$.

- The 0-cylinder of type $\alpha \in \mathcal{A}$ is the union of 1-cylinders that have α at the origin.
- The set of all positions α appears in 1-supertiles is

$$\begin{aligned} \mathsf{T}(\alpha) &= \{\mathsf{b} \in \mathsf{A} \,|\, \mathcal{S}(\mathsf{b})_0 = \alpha\} \\ &= \{(\beta, j) \in \mathsf{A} \,|\, \alpha \text{ is the } j\text{th letter of } \mathcal{S}(\beta)\}. \end{aligned}$$

- If $\pi_{\mathcal{A}}(\mathsf{a}) = \alpha$ we write $\mathsf{T}(\mathsf{a}) = \mathsf{T}(\alpha)$.
- For each $\alpha \in \mathcal{A}$ we have $\sigma([\alpha]) = \bigcup_{\mathsf{b} \in \mathsf{T}(\alpha)} [\mathcal{S}(\mathsf{b})]$.

2-addresses for elements of $\boldsymbol{\Sigma}$

- The position of $\boldsymbol{\tau}(0)$'s 1-supertile inside of its 2-supertile is uniquely determined and can be labeled by A.
- The ***2-address*** is $\mathbf{a}_2(\boldsymbol{\tau}) = (\mathsf{a}_1, \mathsf{a}_2)$ if
  1. $\boldsymbol{\tau}(0)$ is in a 1-supertile of type $\pi_{\mathcal{A}}(\mathsf{a}_1) = \alpha_1$ in position $\pi_{\mathbb{N}}(\mathsf{a}_1) = j_1$, and
  2. that 1-supertile is contained in a 2-supertile of type $\pi_{\mathcal{A}}(\mathsf{a}_2) = \alpha_2$ at position $\pi_{\mathbb{N}}(\mathsf{a}_2) = j_2$.
- There is an appropriate $k \in \{1, 2, ..., |\mathcal{S}^2(\alpha_2)|\}$ for which $\mathcal{S}(\mathsf{a}_1, \mathsf{a}_2) = \sigma^k(\mathcal{S}^2(\alpha_2))$.
- All of these things are true for n-supertiles and addresses.

n-addresses, -cylinders, and -supertiles

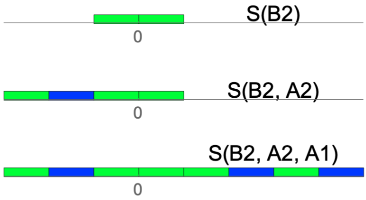
- We say $\mathsf{a} = (\mathsf{a}_1, \mathsf{a}_2, ...) \in \mathsf{A}^{\mathbb{N}} \cup \mathsf{A}^{\infty}$ is an ***address*** if $\mathsf{a}_k \in \mathsf{T}(\alpha_{k-1})$ for all $1 \leq k \leq |\mathsf{a}|$.
- The set of all addresses of lengths n, ∞, or "any" are denoted $\mathbf{A}_n, \mathbf{A}_\infty$, and $\mathbf{A}$, respectively.
- For $\boldsymbol{\tau} \in \mathbf{\Sigma}$, the ***$n$-address of $\boldsymbol{\tau}$***, denoted $\mathsf{a}_n(\boldsymbol{\tau})$, is the address of $\boldsymbol{\tau}(0)$'s n-supertile.
- When $n < \infty$ and $\mathsf{a} = (\mathsf{a}_1 \mathsf{a}_2, ..., \mathsf{a}_n)$, we define
  - the ***n-supertile addressed by a*** to be $\mathcal{S}(\mathsf{a}) = \sigma^j(\mathcal{S}^n(\pi_{\mathcal{A}}(\mathsf{a})))$for the appropriate value of j
  - the ***n-cylinder*** denoted $[\mathcal{S}(\mathsf{a})] = \sigma^j(\mathcal{S}^n([\pi_{\mathcal{A}}(\mathsf{a})]))$

Building a supertile from an address string.

Recall $\mathcal{S}_{PD}(A) = A\,B$ and $\mathcal{S}_{PD}(B) = A\,A$.

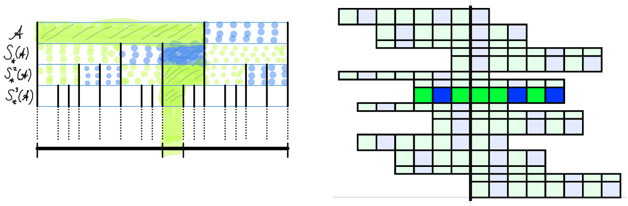
Instructions for placing the supertile $\mathcal{S}_{PD}(B2, A2, A1)$:

1. Place $\mathcal{S}_{PD}(B)$ so that the origin is in the 2nd spot.
2. Slide a copy of $\mathcal{S}^2_{PD}(A)$ to match its 2nd 1-supertile to the one in place already.
3. Move a copy of $\mathcal{S}^3_{PD}(A)$ to match its 1st 2-supertile to the existing one.

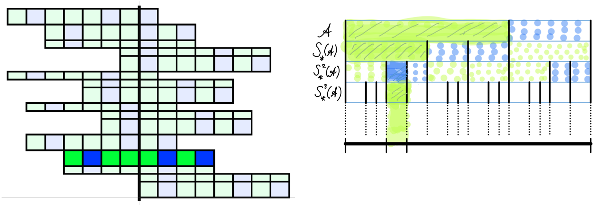


Building a supertile from an address string.

From Σ to $[0, 1]$: the big picture



(a) The shaded path leads to the correct interval for $\mathcal{S}_{PD}(B2, A2, A1)$.

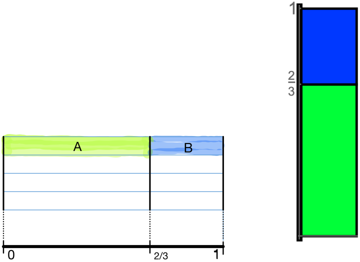


(b) The shifted supertile $\sigma(\mathcal{S}_{PD}(B2, A2, A1))$ has address

Definition of Φ: Initial partition

- $\sum_{\alpha\in\mathcal{A}} \mu([\alpha]) = 1$
- for each $\alpha \in \mathcal{A}$, choose a left endpoint $\Phi_0(\alpha) \in [0,1)$ such that the intervals $\mathbf{I}(\alpha) := [\Phi_0(\alpha), \Phi_0(\alpha) + \mu([\alpha]))$ cover $[0,1)$.
- The initial partition is $\mathcal{I}_0 = \{\mathbf{I}(\alpha), \alpha \in \mathcal{A}\}$.

$\Phi_0(A) = 0, \Phi_0(B) = 2/3$.



The initial partition for the alphabet $\mathcal{S}_{PD}$, in dual subdivision graph (left) and flow view (right).

Making ϕ

For $\mathsf{b} = (\beta, j) \in \mathsf{A}$ we have

$$\mu([\mathcal{S}(\mathsf{b})]) = \mu(\mathcal{S}([\beta])) = \mu([\beta])/\lambda$$

$$\mu([\alpha]) = \sum_{\mathsf{b} \in \mathsf{T}(\alpha)} \mu([\mathcal{S}(\beta)]) = \sum_{\mathsf{b} \in \mathsf{T}(\alpha)} \mu([\beta])/\lambda. \qquad (3)$$

For each $\alpha \in \mathcal{A}$, choose a function $\phi : \mathsf{T}(\mathsf{a}) \to [0, \mu([\alpha])$ for which

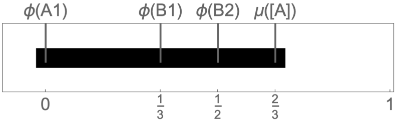
$$[0, \mu([\alpha])) = \bigcup_{\mathsf{b} \in \mathsf{T}(\mathsf{a})} \left[\phi(\mathsf{b}),\, \phi(\mathsf{b}) + \frac{\mu([\beta])}{\lambda}\right),\ \text{where } \pi_{\mathcal{A}}(\mathsf{b}) = \beta. \qquad (4)$$

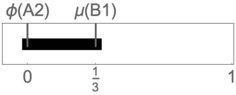
Making ϕ

For period-doubling, $\mathsf{A} = \{A1, A2, B1, B2\}$.

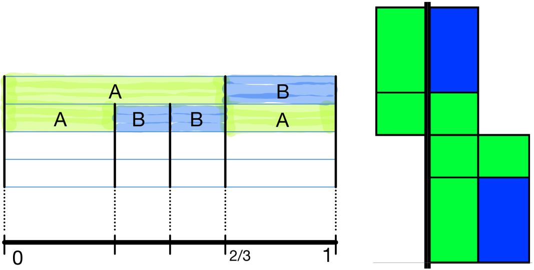
Since $\mathsf{T}(A) = \{A1, B1, B2\}$ and $\mathsf{T}(B) = \{A2\}$, we choose the dual substitution $\mathcal{S}_* : A \to ABB$ and $B \to A$.

$\phi(A1) = 0, \quad \phi(B1) = 1/3, \quad \phi(B2) = 1/2,$ and $\phi(A2) = 0,$ and so
$\Phi_1(A1) = 0, \quad \Phi_1(B1) = 1/3, \quad \Phi_1(B2) = 1/2,$ and $\Phi_1(A2) = 2/3.$





The definition of ϕ.



The first dual subdivision and the level-1 flow view for $\mathcal{S}_{PD}$.

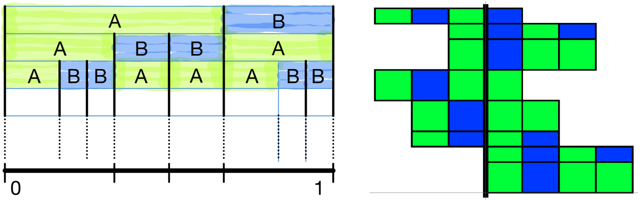
- The refinement $\mathcal{I}_2$ is given by sets of the form $\mathbf{I}(\mathsf{a}_1, \mathsf{a}_2)$, where $(\mathsf{a}_1, \mathsf{a}_2) \in \mathbf{A}_2$.
- Each $\mathbf{I}(\mathsf{a}_1) \in \mathcal{I}_1$ is partitioned by $\{\mathbf{I}(\mathsf{a}_1, \mathsf{a}_2), \mathsf{a}_2 \in \mathsf{T}(\mathsf{a}_1)\}$ placed in the order given by $\mathcal{S}_*(\alpha)$.
- Suppose $\pi_{\mathcal{A}}(\mathsf{a}_i) = \alpha_i, i = 1, 2$.

$$\mu([\mathcal{S}(\mathsf{a}_1)]) = \sum_{\mathsf{a}_2 \in \mathsf{T}(\mathsf{a}_1)} \mu([\mathcal{S}(\mathsf{a}_1, \mathsf{a}_2)]) = \sum_{\mathsf{a}_2 \in \mathsf{T}(\mathsf{a}_1)} \mu([\pi_{\mathcal{A}}(\mathsf{a}_2)])/\lambda^2.$$

- Because the interval $[0, \mu([\mathcal{S}(\alpha_1)]))$ is scaled by $1/\lambda$ from $[0, \mu([\alpha_1]))$, we use $\phi(\mathsf{a}_2)/\lambda$ to partition it.
- Take $\Phi_1(\mathcal{S}(\mathsf{a}_1))$ and add on $\phi(\mathsf{a}_2)/\lambda$:

$$\Phi_2(\mathsf{a}_1, \mathsf{a}_2) = \Phi_1(\mathsf{a}_1) + \phi(\mathsf{a}_2)/\lambda$$

- $\mathbf{I}(\mathsf{a}_1, \mathsf{a}_2) = [\Phi_2(\mathsf{a}_1, \mathsf{a}_2), \Phi_2(\mathsf{a}_1, \mathsf{a}_2) + \mu([\pi_{\mathcal{A}}(\mathsf{a}_2)])/\lambda^2)$.



The level-2 dual subdivision and flow view for $\mathcal{S}_{PD}$.

nth level flow view

- For notational convenience $\mathbf{A}_0 = \mathsf{A}$ and $\Phi_0 : \mathsf{A} \to [0,1]$ is defined as $\Phi_0(\mathsf{a}) := \Phi_0(\pi_{\mathcal{A}}(\mathsf{a}))$.
- If $\mathbf{a} = (\mathsf{a}_1, \mathsf{a}_2, ... \mathsf{a}_n)$, then

$$\Phi_n(\mathbf{a}) = \Phi_{n-1}(\mathsf{a}_1 ..., \mathsf{a}_{n-1}) + \phi(\mathsf{a}_n)/\lambda^{n-1} = \Phi_0(\mathsf{a}_1) + \sum_{k=1}^{n} \phi(\mathsf{a}_k)/\lambda^{k-1} \tag{5}$$

- The interval corresponding to $\mathbf{a}$ is thus

$$\mathbf{I}(\mathbf{a}) = [\Phi_n(\mathbf{a}), \Phi_n(\mathbf{a}) + \mu(\pi_{\mathcal{A}}(\mathsf{a}_n)))/\lambda^n) \quad = [\Phi_n(\mathbf{a}), \Phi_n(\mathbf{a}) + \mu([\mathcal{S}(\mathbf{a}$$

  making the Lebesgue measure of $\mathbf{I}(\mathbf{a})$ is equal to $\mu([\mathcal{S}(\mathbf{a})])$.
- We define the ***canonical partition sequence of*** $[0,1)$ ***given by*** $\mathcal{S}_*$ to be $\mathcal{I}_n = \{\mathbf{I}(\mathsf{a}_1, \mathsf{a}_2, ... \mathsf{a}_\mathsf{n}),\ \text{such that}\ (\mathsf{a}_1, \mathsf{a}_2, ... \mathsf{a}_\mathsf{n}) \in \mathbf{A}_n\}$.

Definition

The ***coordinate map given by*** $\mathcal{S}_*$ is the map $\Phi : \mathbf{\Sigma} \to [0,1]$ given by

$$\begin{aligned}\Phi(\boldsymbol{\tau}) &= \lim_{n\to\infty} \Phi_n(\mathsf{a}_n(\boldsymbol{\tau})) \\ &= \Phi_0(\mathsf{a}_1) \ + \sum_{k=1}^{\infty} \phi(\mathsf{a}_k)/\lambda^{k-1}, \text{ where } \mathbf{a}(\boldsymbol{\tau}) = (\mathsf{a}_1, \mathsf{a}_2, \ldots).\end{aligned}$$

The ***flow view given by*** $\mathcal{S}_*$ is the graph of a canonical isomorphism Φ, with each $\boldsymbol{\tau} \in \mathbf{\Sigma}$ shown at the height $\Phi(\boldsymbol{\tau})$. The nth level flow view is the graph of Φ_n.

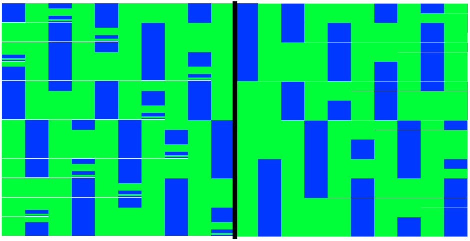
- For any $\mathsf{a} = (\mathsf{a}_1, \mathsf{a}_2, ...) \in \mathbf{A}$ define $\mathbf{n}(\mathsf{a})$ to be the first index at which an element of a can be increased
- the smallest k for which $\mathsf{a}_{[1,...k]} \neq \overline{\mathsf{a}_k}(\alpha)$ for any α.

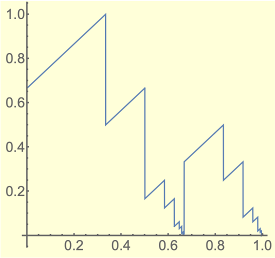
Definition

The ***Vershik*** map $\mathcal{V} : \mathbf{A} \to \mathbf{A}$ is defined for any a for which $\mathbf{n}(\mathsf{a}) = N < \infty$ with $\mathsf{a} = (\overline{\mathsf{a}_{N-1}}(\alpha), (\alpha_N, j_N), \mathsf{a}_{N+1} ...)$ to be

$$\mathcal{V}(\mathsf{a}) = \left(\underline{\mathsf{a}_{N-1}}(\beta), (\alpha_N, j_N + 1), \mathsf{a}_{N+1}, ...\right), \text{ where } \beta \text{ is the } (j_N{+}1)\text{th le} \tag{6}$$

$$\mathfrak{F}(x) = x - \Phi_N(\mathsf{a}_N(x)) + \Phi_N(\mathcal{V}(\mathsf{a}_N(x))). \tag{7}$$





The final results for period-doubling. The blue vertical lines in the IIET connect the ends of jump discontinuities.

Proposition

Given $\boldsymbol{\tau} \in \boldsymbol{\Sigma}$ *with* $\mathsf{a}_\infty(\boldsymbol{\tau}) = \{\mathsf{a}_\mathsf{n}\}_{n=1}^{\infty}$, *the map*

$$\Phi(\boldsymbol{\tau}) = \Phi_0(\boldsymbol{\tau}(0)) + \sum_{n=1}^{\infty} \frac{\phi(\mathsf{a}_n)}{\lambda^{n-1}} = \Phi_K(\mathsf{a}_K(\boldsymbol{\tau})) + \sum_{n=K+1}^{\infty} \frac{\phi(\mathsf{a}_n)}{\lambda^{n-1}} \quad (8)$$

is uniformly continuous everywhere and bijective almost everywhere.

Theorem

Let $\mathcal{S}$ be a recognizable substitution and let $\Phi : (\mathbf{\Sigma}, \mu) \to ([0,1], m)$ be a canonical isomorphism. For $x \in [0,1]$ with $\mathbf{n}(x) = N < \infty$ we define

$$\mathfrak{F}(x) = x - \Phi_N(\mathsf{a}_N(x)) + \Phi_N(\mathcal{V}(\mathsf{a}_N(x))). \tag{9}$$

Then $\mathfrak{F}$ is defined for m-almost every x and Φ is a measurable conjugacy between $(\mathbf{\Sigma}, \sigma, \mu)$ and $([0,1], \mathfrak{F}, m)$.

Corollary

For any $n \in \mathbb{N}$ there is an exchange of $n(|\mathsf{A}| - |\mathcal{A}|) + |\mathcal{A}|$ intervals that is equal to $\mathfrak{F}$ on all but $|\mathcal{A}|$ intervals of total measure $\leq \lambda^{-n}$.

Proof: $\mathfrak{F}(\Phi(\boldsymbol{\tau})) = \Phi(\sigma(\boldsymbol{\tau}))$ a.e.

Let $\boldsymbol{\tau} \in \boldsymbol{\Sigma}$ with $\mathbf{n}(\boldsymbol{\tau}) = M < \infty$ so that $\Phi(\boldsymbol{\tau})$ lies in $\mathbf{I}(\mathsf{a}_M(\boldsymbol{\tau}))$. We can write $\Phi(\boldsymbol{\tau}) = \Phi_M(\mathsf{a}_M(\boldsymbol{\tau})) + \sum_{n=M+1}^{\infty} \frac{\phi(\mathsf{a}_n)}{\lambda^{n-1}}$. We have

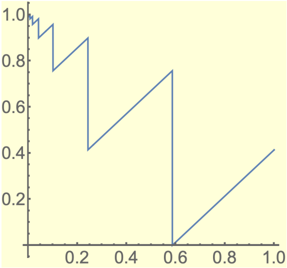
$$
\begin{aligned}
\mathfrak{F}(\Phi(\boldsymbol{\tau})) &= \Phi(\boldsymbol{\tau}) - \Phi_M(\mathsf{a}_M(\boldsymbol{\tau})) + \Phi_M(\mathcal{V}(\mathsf{a}_M(\boldsymbol{\tau}))) \\
&= \left(\Phi(\mathsf{a}_M(\boldsymbol{\tau})) + \sum_{n=M+1}^{\infty} \frac{\phi(\mathsf{a}_n)}{\lambda^{n-1}} \right) - \Phi_M(\mathsf{a}_M(\boldsymbol{\tau})) + \Phi_M(\mathcal{V}(\mathsf{a}_M(\boldsymbol{\tau} \\
&= \Phi_M(\mathcal{V}(\mathsf{a}_M(\boldsymbol{\tau}))) + \sum_{n=M+1}^{\infty} \frac{\phi(\mathsf{a}_n)}{\lambda^{n-1}} \\
&= \Phi_M(\mathsf{a}_M(\sigma(\boldsymbol{\tau}))) + \sum_{n=M+1}^{\infty} \frac{\phi(\mathsf{a}_n)}{\lambda^{n-1}} = \Phi(\sigma(\boldsymbol{\tau})),
\end{aligned}
$$

with the last two equalities following from lemma 11 and the fact that $\boldsymbol{\tau}$ and $\sigma(\boldsymbol{\tau})$ are tail equivalent with $\mathsf{a}_{[M+1,\infty)}(\boldsymbol{\tau}) = \mathsf{a}_{[M+1,\infty)}(\sigma(\boldsymbol{\tau}))$.

Proposition

Suppose there are $\beta, \gamma \in \mathcal{A}$ such that $\mathcal{S}(\alpha)$ begins with β and ends with γ for all $\alpha \in \mathcal{A}$. Then there is a canonical IIET of $(\mathbf{\Sigma}, \sigma, \mu)$ and a constant $\kappa \in [0, 1)$ for which

$$\mathfrak{F}(x) = \lambda(\mathfrak{F}(x/\lambda) + \kappa) \text{ for a.e. } x \in [0, 1]. \qquad (10)$$



Shazam! A self-similar IIET for $A \to BBA$, $B \to BA$.

Key lemmas

Lemma

If $(\mathbf{\Sigma}, \sigma)$ is minimal and μ is shift invariant, the subset

$$\mathbf{\Sigma}_0 = \{\boldsymbol{\tau} \in \mathbf{\Sigma} \,|\, \textit{each } \mathsf{a} \in \mathcal{A} \textit{ appears infinitely often in } \mathsf{\mathbf{a}}(\boldsymbol{\tau})\} \tag{11}$$

has full measure.

$$\mathfrak{LEs} = \{\Phi_n(\mathsf{\mathbf{a}}), \mathsf{\mathbf{a}} \in \mathbf{A}_{\mathbb{N}} \text{ and } n \in \mathbb{N}\}. \tag{12}$$

Remark

What sequences map to $\mathfrak{LEs}$?88 include sequences whose supertile sequence at the origin only covers a half-line, but there are others. A problematic such case appears in the proof of theorem **??**. *88understand the relationship between $\mathbf{\Sigma}_0$ and $\mathfrak{LEs}$.*

Lemma

For all $\boldsymbol{\tau} \in \mathbf{\Sigma}$ with $\mathbf{n}(\mathsf{\mathbf{a}}(\boldsymbol{\tau})) < \infty$, $\mathsf{\mathbf{a}}(\sigma(\boldsymbol{\tau})) = \mathcal{V}(\mathsf{\mathbf{a}}(\boldsymbol{\tau}))$.

tion Lebesgue measure is the push-forward of μ under Φ and so

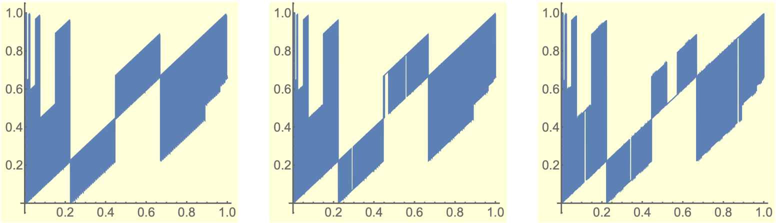
Corollary

For all integrable $f : [0,1] \to \mathbb{C}$, $\int_0^1 f(x)dm = \int_{\boldsymbol{\Sigma}} f(\Phi(\boldsymbol{\tau}))d\mu$.

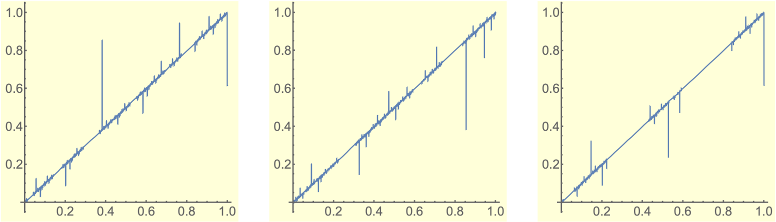
At points where Φ is one-to-one its inverse is continuous in the following sense.

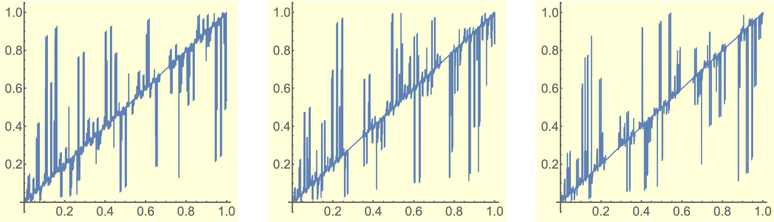
Corollary

Let $x_0 \in [0,1)/\mathfrak{L}\mathfrak{E}\mathfrak{s}$. *For every* $\delta > 0$ *there exists an* $\epsilon' > 0$ *such that if* $|x - x_0| < \epsilon'$, *then* $d(\Phi^{-1}(x), \Phi^{-1}(x_0)) < \delta$ *for any element of* $\Phi^{-1}(x)$.

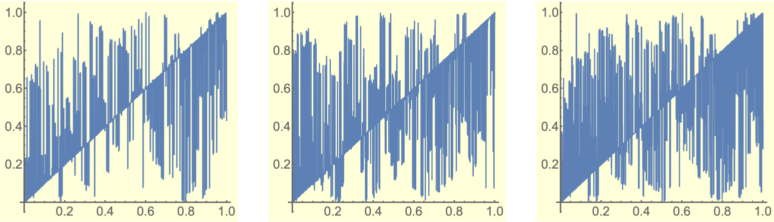


The IIET $\mathfrak{F}_{20}^{j}$ for the Chacon substitution $\mathcal{S}_C(0) = 0\,0\,1\,0$ and $\mathcal{S}_C(1) = 1$, where $j = 121, 364$, and 1093. This substitution is weakly mixing.

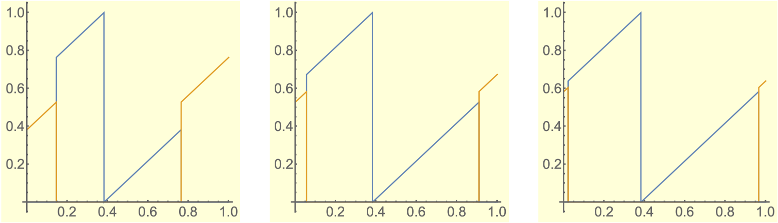




The IIET $\mathfrak{F}_{20}^{j}$ for the 'tribonacci' substitution
$A \to AB$, $B \to AC$, $C \to A$, for $j = 204, 574$, and 927.



The IIET $\mathfrak{F}_{20}^{j}$ for the substitution $A \to ABBB$, $B \to A$, for



The first three approximants for the IIET of $\mathcal{S}^2_{fib}$.