FUSION: A GENERAL FRAMEWORK FOR HIERARCHICAL TILINGS OF $\mathbb{R}^d$

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Abstract. We introduce a formalism for handling general spaces of hierarchical tilings, a category that includes substitution tilings, Bratteli-Vershik systems, S-adic transformations, and multi-dimensional cut-and-stack transformations. We explore ergodic, spectral and topological properties of these spaces. We show that familiar properties of substitution tilings carry over under appropriate assumptions, and give counter-examples where these assumptions are not met. For instance, we exhibit a minimal tiling space that is not uniquely ergodic, with one ergodic measure having pure point spectrum and another ergodic measure having mixed spectrum. We also exhibit a 2-dimensional tiling space that has pure point measure-theoretic spectrum but is topologically weakly mixing.

Contents

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1. Introduction

Hierarchical structures are ubiquitous in the real world. Typically there are a finite number of levels, ranging from the tiny (say, subatomic particles) to the huge (say, clusters of galaxies). In many cases the smallest level is so small that it makes sense to extrapolate mathematically to infinitely small hierarchical structures – fractals. In this paper we consider the complementary situation where the smallest scale may not be small, but the largest scale is so large that it makes sense to extrapolate to infinite size.

There is an extensive literature devoted to expanding hierarchies, dating back to the 1800s [49], with applications to dynamics dating back to the early 1900s [41]. Most of the aperiodic sets of tiles that were discovered over the years, from Berger [11] to Robinson [54] to Penrose [31] to Goodman-Strauss [32] and others, used hierarchy as means of proving aperiodicity. Tiles group into clusters that group into larger clusters, etc., so that the resulting patterns exhibit structure at arbitrarily large length scales and cannot be periodic.

In most of the literature, it is assumed that the hierarchies have essentially the same structure at each level, so that the system can be described by a single substitution map. Indeed, there has been tremendous progress on substitution sequences, substitution subshifts, and substitution tilings. However, there is much to be said about hierarchical systems where the structure is *not* necessarily repeated at each level.

The idea of studying general hierarchical systems can be seen in the cut-and-stack formalism of ergodic theory. The first example of Chacon [14], which exhibited a weakly mixing system that was not strongly mixing, is a fusion of the sort discussed in this paper. Over the years the technique has been used to construct many interesting examples, and it has been shown [5] that all interval exchange transformations, and indeed all aperiodic measure preserving transformations, can be obtained by cutting and stacking. Cutting and stacking has been generalized to higher dimensions for $\mathbb{Z}^d$ actions [55, 36], for $\mathbb{R}^d$-actions on rectangular domains [17], and for general locally compact second countable groups [18] and amenable groups. Progress has recently been made on nonstationary Bratteli-Vershik systems [24, 23, 12], most of which can be viewed as a discrete 1-dimensional version of the fusion tilings described in this paper [10].

This paper provides a framework for studying the ergodic theory and topology of hierarchical tilings. Our formalism encompasses, among other things, substitution tilings and substitution subshifts, cut-and-stack transformations, S-adic transformations [22], and stationary and non-stationary Bratteli-Vershik systems [24, 12].

Taken to extremes, our formalism can be made *too* general. Without simplifying assumptions, essentially any tiling space can be viewed as a fusion, and almost any sort of dynamical behavior is possible. For instance, Jewett [35] and Krieger [39] showed that any ergodic measurable automorphism of a non-atomic Lebesgue space system can be realized topologically as a uniquely ergodic map on a Cantor set; in most cases these can be viewed as subshifts, and hence as fusion tiling spaces. Downarowicz [21] showed that there exist Cantor dynamical systems whose invariant measures match an arbitrary Choquet simplex.

In this paper we identify appropriate hypotheses that preserve the essential properties of substitutions while applying to more general systems. Certain properties, like minimality or unique ergodicity, hold under very general conditions. Others, like finitely generated (rational Čech) cohomology or pure point spectrum or (on the other extreme) topological weak mixing, require stronger assumptions.

In addition, we develop a number of examples that show how these properties can be lost when the assumptions are too weak. We hope that these examples will help to classify fusion tilings, and to better organize our understanding of tilings in general.

Some of our proofs are quite simple, yet determining how to apply the techniques of substitution systems to fusions is far from trivial. The key tools for studying substitution systems are Perron-Frobenius theory and the existence of a self-map that can be iterated arbitrarily many times. Neither of these work for general fusions. The new methods devised in this paper provide us with more insight into how properties of tiling *spaces* are related to properties of *tilings*. Some properties of a hierarchical tiling space are directly related to the geometry of the individual tiles. Others come from the details of how the tiles are assembled into bigger and bigger clusters. Still others can be deduced from coarser numerical data, such as from the matrices that count how many of each kind of tile appear in each kind of cluster. Because the hierarchy in fusion rules is less rigid than that of their substitutive counterparts, combinatorics, geometry, algebra, and topology can have effects that need to be teased apart. The challenge is to understand which properties come from which information, and to organize that information effectively.

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2. Definitions

In this work a *prototile* is a labelled, closed topological disk in $\mathbb{R}^d$. The label, which can be thought of as a color or a marking, is necessary when we wish to distinguish between prototiles that are geometrically similar. In general we assume that we have a finite set $\mathcal{P}$ of prototiles to use as building blocks for our tilings. (This assumption is useful but not entirely necessary. In a separate work [29] we consider tilings built from an infinite but compact set $\mathcal{P}$.) We also assume that we have fixed a closed subgroup G of the Euclidean group $E(d)$ that contains a full rank lattice of translations; this group G will be used to construct our tiles, patches, and tilings and can also serve as the group action of our dynamical system. (The two standard translation subgroups that appear in tiling theory are $\mathbb{Z}^d$ and $\mathbb{R}^d$.) It is possible to act on a prototile by an isometry in G by applying the isometry to the closed set defining the prototile and carrying the labelling information along unchanged. A prototile which has been so moved is called a *tile*. We will abuse notation by denoting the application of an isometry $g \in G$ to a prototile p as $g(p)$; when the isometry is translation by $\vec{v} \in \mathbb{R}^d$ we denote the translated tile by $p + \vec{v}$. A $\mathcal{P}$*-patch* (or *patch*, for short) of tiles is a connected, finite union of tiles that only overlap on their boundaries; the *support* of the patch is the closed set in $\mathbb{R}^d$ that it covers. Two tiles or patches are considered *equivalent* or *copies* of one another if there is an element of G taking one to the other. A *tiling* $\mathbf{T}$ of $\mathbb{R}^d$ is a collection of tiles that completely cover $\mathbb{R}^d$ and overlap only on their boundaries.

A tiling is said to have *finite local complexity (FLC)* with respect to the group G if it contains only finitely many connected two-tile patches up to motions from G. Most of the literature on tiling dynamical systems uses finite local complexity as a key assumption. This work in this paper is limited to FLC fusion tilings. Fusion tilings with infinite local complexity (ILC) will be considered in [29].

2.1. Fusion tilings.

Given two $\mathcal{P}$-patches P_1 and P_2 and two isometries g_1 and g_2 in G, if the patches $g_1(P_1)$ and $g_2(P_2)$ overlap only on their boundaries, and if the union $g_1(P_1) \cup g_2(P_2)$ forms a $\mathcal{P}$-patch, we call that union the *fusion* of P_1 to P_2 via g_1 and g_2. When we do not wish to specify the isometries we may call it a fusion of P_1 to P_2. Notice that there will be many ways to fuse two patches together and that we may attempt to fuse any finite number of patches together. We may even fuse a patch to copies of itself. Patch fusion is simply a version of concatenation for geometric objects.

The idea behind a "fusion rule" is an analogy to an atomic model: we have atoms, and those atoms group themselves into molecules, which group together into larger and larger structures. In this analogy we think of prototiles as atoms and patches as molecules. Let $\mathcal{P}_0 = \mathcal{P}$ be our prototile set, our "atoms". The first set of "molecules" they form will be defined as a set of finite $\mathcal{P}$-patches $\mathcal{P}_1$, with notation $\mathcal{P}_1 = \{P_1(1), P_1(2), ..., P_1(j_1)\}$. Next we construct the structures made by these "molecules": the set $\mathcal{P}_2$ will be a set of finite patches that are fusions of the patches in $\mathcal{P}_1$. That is $\mathcal{P}_2 = \{P_2(1), P_2(2), ..., P_2(j_2)\}$ is a set of patches, each of which is a fusion of patches from $\mathcal{P}_1$. While the elements of $\mathcal{P}_2$ are technically $\mathcal{P}$-patches, we can also think of them as $\mathcal{P}_1$-patches by considering the elements of $\mathcal{P}_1$ as prototiles. We continue in this fashion, constructing $\mathcal{P}_3$ as a set of patches that are fusions of patches from $\mathcal{P}_2$ and in general constructing $\mathcal{P}_n$ as a set of patches which are fusions of elements of $\mathcal{P}_{n-1}$. The elements of $\mathcal{P}_n$ are called n-fusion supertiles or n-supertiles, for short.[1] We collect them together into an atlas of patches we call our *fusion rule*:

$$\mathcal{R} = \{\mathcal{P}_n, n \in \mathbb{N}\} = \{P_n(j) \mid n \in \mathbb{N} \text{ and } 1 \leq j \leq j_n\}.$$

A patch is *admitted by* $\mathcal{R}$ if a copy of it can be found inside some supertile $P_n(j)$ for some n and j. A tiling $\mathbf{T}$ of $\mathbb{R}^d$ is said to be a *fusion tiling with fusion rule* $\mathcal{R}$ if every patch of tiles contained in $\mathbf{T}$ is admitted by $\mathcal{R}$. We denote by $X_{\mathcal{R}}$ the set of all $\mathcal{R}$-fusion tilings. Given a fusion rule, we can obtain another fusion rule $\mathcal{R}'$ with $j'_n = j_{n+1}$ and $P'_n(j) = P_{n+1}(j)$. We simply ignore the lowest level and treat the 1-fusion supertiles as our basic tiles. The resulting tiling space is denoted $X^1_{\mathcal{R}}$. Likewise, $X^k_{\mathcal{R}}$ is the space of tilings obtained from $\mathcal{R}$ in which the k-fusion supertiles are considered the smallest building blocks.

Standing assumption (for this entire paper): If none of the supertiles in $\mathcal{R}$ have inner radii approaching infinity then $X_{\mathcal{R}}$ will be empty, so for that reason we restrict our attention to fusion rules that have nontrivial tiling spaces.

When $d = 1$ and $G = \mathbb{Z}$, with all tiles having unit length, fusion tilings correspond to Bratteli-Vershik systems, modulo complications having to do with edge sequences that have no predecessors or no successors. See [10] for more about the correspondence. (In addition to subshifts, Bratteli-Vershik systems can model non-expansive maps on Cantor sets; these can also be viewed as 1-dimensional fusion tilings, albeit with infinitely many tile types [29].)

Example 2.1. *The Chacon transformation.* In [14] there is an early example of a transformation that is weakly mixing but not strongly mixing. The original cutting-and-stacking construction is a self-map on an interval; the stacking portion can be seen as a sort of fusion.

[1] If we wish, we can also add labels to the supertiles, so that the information carried in an n-supertile is more than just its composition as a patch in a tiling. This generalization is useful for collaring constructions, as in Section 5.

However for the purposes of an immediate example we use the fact that the Chacon space can be viewed symbolically using the substitution rule

$$a \to aaba \qquad b \to b,$$

which can be iterated by substituting each letter and concatenating the blocks. If we begin with an a we have:

$$a \to aaba \to aaba\,aaba\,b\,aaba \to \ldots$$

In order to make a Chacon tiling of $\mathbb{R}$ we only need to assign closed intervals to the symbols a and b and place them on the line according to the symbols in a Chacon sequence.

We can view a Chacon tiling of $\mathbb{R}$ as a fusion tiling as follows. Consider l_a and l_b to be two positive numbers and let a denote a prototile with support $[0, l_a]$ and b denote a prototile with support $[0, l_b]$. (If $l_a = l_b$ then we use the symbols a and b as labels to tell the tiles apart). We define $P_1(a) = a \cup (a + l_a) \cup (b + 2l_a) \cup (a + 2l_a + l_b)$ and $P_1(b) = b$. The length of $P_1(a)$ is $3l_a + l_b$. To make $P_2(a)$ we simply fuse three copies of $P_1(a)$ and one copy of $P_1(b)$ together in the correct order, and of course $P_2(b) = b$ still. The length of the new a supertile is three times that of the previous a supertile plus the length of b. We continue recursively to construct all of the n-fusion supertiles.

2.1.1. *Transition matrices and the subdivision map.* Given a fusion rule $\mathcal{R}$ there is a family of *transition matrices* that keep track of the number and type of $(n-1)$-fusion supertiles that combine to make the n-supertiles. The transition matrix for level n, denoted $M_{n-1,n}$, has entries $M_{n-1,n}(k, l)$ = the number of $(n-1)$-supertiles of type k, that is, equivalent to $P_{n-1}(k)$, in the n-supertile of type l, $P_n(l)$. If there is more than one fusion of $\mathcal{P}_{n-1}$-supertiles that can make $P_n(l)$, we fix a preferred one to be used in this and all other computations. For levels $n < N \in \mathbb{N}$, we likewise define the transition matrix from n- to N-supertiles as $M_{n,N} = M_{n,n+1}M_{n+1,n+2}\cdots M_{N-1,N}$. The (i, j) entry of $M_{n,N}$ is the number of n-supertiles of type i in the N-supertile of type j. Another way to think about this is to imagine a "population vector" $v \in \mathbb{Z}^{j_N}$ of a patch of N-supertiles: the entries represent the number of N-fusion supertiles of each type appearing in the patch. Then $M_{N-1,N}v$ gives the population of this patch in terms of $(N-1)$-supertiles, $M_{N-2,N-1}M_{N-1,N}v$ gives the population in terms of $(N-2)$-supertiles, and $M_{n,N}v$ gives the population of this patch in terms of n-supertiles.

Any self-affine substitution tiling, in any dimension, can be viewed as a fusion tiling. An n-supertile is what we get by applying the substitution n times to an ordinary tile, and can be decomposed into $(n-1)$-supertiles according to the pattern of the substitution. For such tilings, the matrix $M_{n,N}$ is just the $(N-n)$th power of the usual substitution matrix. However, there is an important difference in perspective between substitutions and fusions.

A substitution can be viewed as a map from a tiling space to itself, in which all tiles are enlarged and then broken into smaller pieces. This map can be repeated indefinitely. In a fusion tiling, we can likewise break each n-fusion tile into level $(n-1)$-supertiles using the *subdivision map* σ_n, which is a map from $X_{\mathcal{R}}^n$ to $X_{\mathcal{R}}^{n-1}$. Unlike the substitution map for self-affine tilings, it cannot go from $X_{\mathcal{R}}$ to itself, and this map cannot be repeated more than n times. Once you are down to the atomic level (i.e., ordinary tiles), you cannot subdivide further! The proofs of theorems about substitution tilings often involve taking an arbitrary tiling and applying a substitution, or sometimes its inverse, enough times to achieve a desirable result. For general fusion tilings, this line of reasoning usually does not work.

2.1.2. *Induced fusions.* Let $\{N(n)\}_{n=1}^{\infty}$ be an increasing sequence of positive integers. The *induced fusion on* $N(n)$ *levels*, $\mathcal{R}^{ind}$, is obtained from a given fusion $\mathcal{R}$ by composing the fusions for levels $N(n)+1, ..., N(n+1)$ into one step. In this case the supertiles of $\mathcal{R}^{ind}$ are given by $\mathcal{P}_n^{ind} = \mathcal{P}_{N(n)}$, where the $N(n)$-supertiles are seen as fusions of $N(n-1)$-supertiles. The transition matrices for $\mathcal{R}^{ind}$ are given by $M_{n,n+1}^{ind} = M_{N(n),N(n+1)}$.

2.1.3. *All FLC tilings are fusion tilings.* It is possible to view *any* tiling $\mathbf{T}$ of $\mathbb{R}^d$ from a given prototile set $\mathcal{P}_0$ as a fusion tiling, as long as it has finite local complexity. Let the set $\mathcal{P}_n$ consist of all connected patches containing n tiles or less. By finite local complexity this is a finite set. Each element of $\mathcal{P}_n$ is either an element of $\mathcal{P}_{n-1}$ or is the fusion of two elements of $\mathcal{P}_{n-1}$. (The fact that these fusions typically are not unique does not matter).

2.2. **Common assumptions.** The previous section shows that the category of fusion tilings is extremely general. To prove meaningful results, we have to impose additional conditions on our fusion rules. We collect several of them into this section.

2.2.1. *Prototile- and transition-regularity.* These are the cases that are most similar to the usual definitions of symbolic and tiling substitution. When the number of supertiles at each level is constant, we can associate each n-supertile to a specific prototile, regardless of whether there is a geometric connection between the two. When we do this we call the fusion rule *prototile-regular* and rewrite it as:

$$\mathcal{R} = \{P_n(p) \mid n \in \mathbb{N} \text{ and } p \in \mathcal{P}_0\}.$$

If the number j_n of supertiles at the nth level of a fusion rule $\mathcal{R}$ has $J = \liminf j_n$ for some finite J, then the fusion rule is equivalent to a prototile-regular fusion rule by inducing on the levels for which $j_n = J$. The price we pay for taking such an induced fusion is that the transition matrices can become wildly unbounded.

In the special case where the number of supertiles at each level is a fixed constant J, if the transition matrices are all equal to a single matrix we call the fusion rule a *transition-regular* fusion rule. Being transition-regular is considerably stronger than being prototile-regular. All substitution sequences and self-affine tilings as defined in, for instance, [52, 59] are transition-regular, but not every transition-regular fusion tiling comes from a substitution.

The combinatorics and geometry of how the $(n-1)$-supertiles join to form n-supertiles can change from level to level.

Example 2.2. *A fusion that is transition-regular but not a substitution.* Consider a 1-dimensional fusion rule with transition matrix $\left(\begin{smallmatrix}2&1\\1&2\end{smallmatrix}\right)$ in which $P_n(a)$ is always given by the word $P_{n-1}(a)P_{n-1}(a)P_{n-1}(b)$, and in which $P_n(b)$ is given by $P_{n-1}(b)P_{n-1}(b)P_{n-1}(a)$ if n is prime, and is given by $P_{n-1}(a)P_{n-1}(b)P_{n-1}(b)$ if n is composite.

Remarks.

(1) Pseudo-self-similar (or self-affine) tilings, such as the Penrose tiling with kites and darts, are also transition-regular fusion tilings. In many cases these are asymptotically self-similar, and this asymptotic structure was used [27, 51] to show that such tilings are topologically equivalent to self-similar tilings with fractal boundaries.

(2) In the correspondence between one-dimensional fusion tilings with $G = \mathbb{Z}$ and Bratteli-Vershik systems, prototile-regular tilings correspond to finite Bratteli diagrams. The finite list of vertices on the nth level of the Bratteli diagram represents the finite set of n-supertiles.

(3) The one-dimensional S-adic substitution sequences of Durand [22] can be recast as fusion tilings, as can the linearly recurrent Delone sets and tower systems in [9, 3]. Example 2.2 is S-adic.

(4) The "non-constructive" combinatorial substitutions in [25] are exactly the class of prototile-regular fusion tilings.

2.2.2. *Primitivity.* A fusion rule is said to be *primitive* if, for each non-negative integer n, there exists another integer N such that every n-fusion supertile is contained in every N-supertile. When the fusion rule is transition-regular this is equivalent to some power of the transition matrix having strictly positive entries. In general it is equivalent to there existing an N for each n such that $M_{n,N}$ has all positive entries. A fusion rule is called *strongly primitive* if for every $n \geq 1$, each $(n+1)$-supertile contains at least one copy of every n-supertile. That is, all of the transition matrices $M_{n,n+1}$ have strictly positive entries. Any primitive fusion rule is equivalent to a strongly primitive one by inducing on enough levels.

Primitivity is one of the most common assumptions used in the literature on substitution sequences and tilings. It allows for Perron-Frobenius theory to be applied to the systems to determine natural frequencies, volumes, and expansion rates. We will adapt this analysis to the fusion situation in Section 3.

2.2.3. *Recognizability.* A fusion rule $\mathcal{R}$ is said to be *recognizable* if, for each n, the subdivision map σ_n from $X_{\mathcal{R}}^n$ to $X_{\mathcal{R}}^{n-1}$ is a homeomorphism. If so, then every tiling in $X_{\mathcal{R}}$ can be unambiguously expressed as a tiling with n-supertiles for every n. The uniform continuity of the inverse subdivision maps then implies that there exists a family of *recognizability radii*

r_n $(n = 1, 2, \ldots)$, such that, whenever two tilings in $X_{\mathcal{R}}$ have the same patch of radius r_n around a point $\vec{v} \in \mathbb{R}^d$, then the n-supertiles intersecting $\vec{v}$ in those two tilings are identical.

For substitution sequences and tilings, recognizability is closely related to non-periodicity [42, 60]. Recognizability implies that none of the tilings are periodic. Conversely, if G consists only of translations [60], or if G contains a set of rotation about the origin with no invariant subspaces then the absence of periodic tilings in $X_{\mathcal{R}}$ implies recognizability [33]. However, it is easy to construct fusion rules that are nonperiodic but not recognizable. For instance, the Fibonacci tiling can be generated either from the substitution $a \to ab$, $b \to a$ or from the substitution $a \to ba$, $b \to a$. By including both sets of supertiles in our fusion rule, we obtain a description of the non-periodic Fibonacci tiling space in which each tiling has at least two (actually more) decompositions into n-supertiles for $n > 0$.

We now show that fusion tiling spaces are topological factors of recognizable fusion tiling spaces using a construction inspired by the work of Robinson [54] and Mozes [43].

Example 2.3. *Constructing a recognizable extension.* Let $\mathcal{R}_0$ be a 1-dimensional fusion rule on the letters a and b, each of which is viewed as a tile of length 1. If we let the n-supertiles be all possible sequences of as and bs of length 5^n, then the space $X_{\mathcal{R}_0}$ is just the space of all bi-infinite tilings by a's and b's and $\mathcal{R}_0$ is clearly not recognizable.

Now let $\mathcal{R}$ be a 1-dimensional fusion with four letters, a_1, a_2, b_1 and b_2. We call a_1 and b_1 "type 1", and write x_1 to mean either a_1 or b_1. Likewise x_2 means either a_2 or b_2. The 1-supertiles are all 5-letter words of the general form $x_2x_1x_1x_1x_1$ (where each x_i denotes a separate choice of a_i or b_i) or $x_2x_1x_2x_2x_1$. We will use s_1^1 are shorthand for supertiles of the first type and s_2^1 for the second. Note that each supertile begins with an isolated x_2, and that isolated x_2's appear only at the beginning of supertiles. This makes the map from $X_{\mathcal{R}}^1$ to $X_{\mathcal{R}}$ invertible.

We repeat the coding at higher levels. Second-order supertiles can either take the form $s_1^2 = s_2^1s_1^1s_1^1s_1^1s_1^1$ or $s_2^2 = s_2^1s_1^1s_2^1s_2^1s_1^1$, and generally $(n+1)$-supertiles can either take the form $s_1^{n+1} = s_2^ns_1^ns_1^ns_1^ns_1^n$ or $s_2^{n+1} = s_2^ns_1^ns_2^ns_2^ns_1^n$. By the same reasoning, all decomposition maps are invertible, and $\mathcal{R}$ is recognizable. Finally, the factor map $X_{\mathcal{R}} \to X_{\mathcal{R}_0}$ just erases the subscripts on all of the letters.

The details of the construction will be different for different examples and can get complicated if the supertiles have wild shapes or combinatorics, but the basic idea is universal. Pick sufficiently many copies of your original tile set. Use some of the labels within a first-order supertile to indicate which tiles are in the supertile, and the rest to give the first-order supertiles labels. Use some of those first-order labels to define the boundaries of the second-order supertiles, and the rest to label the second-order supertiles. By continuing the process ad infinitum we obtain a recognizable fusion tiling space that factors onto the original. How close to this factor map is to being one-to-one becomes an important question.

2.2.4. *Van Hove sequences and fusion rules.* A van Hove sequence $\{A_m\}$ of subsets of $\mathbb{R}^d$ consists of sets whose boundaries are increasingly trivial relative to their interiors in a precise sense. In many cases it will be convenient to consider only fusion rules where the supertiles share this property. The use of van Hove sequences, which for $\mathbb{R}^d$ is equivalent to Følner sequences, is adopted from statistical mechanics. We follow the notation of [59] here and define, for any set $A \in \mathbb{R}^d$ and $r > 0$:

$$A^{+r} = \{x \in \mathbb{R}^d : \ \mathrm{dist}(x, A) \leq r\},$$

where "dist" denotes Euclidean distance. A sequence of sets $\{A_n\}$ of sets in $\mathbb{R}^d$ is called a *van Hove* sequence if for any $r \geq 0$

$$\lim_{n\to\infty} \frac{\mathrm{Vol}\left((\partial A_n)^{+r}\right)}{\mathrm{Vol}(A_n)} = 0,$$

where ∂A is the boundary of A and Vol is Euclidean volume.

Given a fusion rule $\mathcal{R}$, we may make a sequence of sets in $\mathbb{R}^d$ by taking one n-supertile for each n and calling its support A_n. We say $\mathcal{R}$ is a *van Hove fusion rule* if every such sequence is a van Hove sequence. Equivalently, a fusion rule is van Hove if for each $\epsilon > 0$ and each $r > 0$ there exists an integer n_0 such that each n-supertile A, with $n \geq n_0$, has $\mathrm{Vol}(\partial A)^{+r} < \epsilon \mathrm{Vol}(A)$.

2.3. **Notational conventions.** Entries of vectors and matrices are indicated as arguments, while subscripts are used to distinguish between different vectors and matrices. Thus, $M_{1,2}(3,4)$ is the (3,4) entry of the matrix $M_{1,2}$ and $v_5(2)$ is the second entry of v_5. Vectors are viewed as columns, so that the product Mv of a matrix and a vector is well-defined. Groups are denoted by capital letters, as are subsets of groups, while elements of groups are lower case. Collections of patches of tilings are given by calligraphic letters $\mathcal{P}, \mathcal{R}$, etc, and in particular our fusion rules are so denoted. Tilings are bold face. Elements of physical space $\mathbb{R}^d$ are marked with arrows, and the dot product is reserved for this setting.

3. Dynamics of fusion tilings

Let $G = G_t \rtimes G_r$, where G_t is the translation subgroup and G_r is the point group G/G_t. By assumption G contains a full rank lattice of translations, and G_r is a closed subgroup of $O(n)$. Let Vol be Haar measure on G_t, a product of Lebesgue measure in the continuous directions of G_t and counting measure in the discrete directions, and let λ_0 be normalized Haar measure on G_r. Let λ be a measure on G with $\lambda(U_t \rtimes U_r) = Vol(U_t)\lambda_0(U_r)$ for every pair of measurable sets $U_t \in G_t$, $U_r \in G_r$. We assume that we have a metric on G whose restriction to G_t is Euclidean distance and whose restriction to G_r is bounded by 1.

3.1. **Tiling metric topology and dynamical system.** Let $X_\mathcal{P}$ be the set of all possible tilings using some fixed prototile set $\mathcal{P}$ and some group G of isometries. (That is, a point in this space is an entire tiling of $\mathbb{R}^d$.) We turn $X_\mathcal{P}$ into a metric space using the so-called "big ball" metric using the metric on G as follows. Two tilings $\mathbf{T}_1$ and $\mathbf{T}_2$ are ϵ-close if there exist group elements g_1 and g_2, each of size less than or equal to ϵ, such that $g_1(\mathbf{T}_1)$ and $g_2(\mathbf{T}_2)$ exactly agree on the ball of radius $1/\epsilon$ around the origin.

This metric is not G-invariant, as it gives greatest weight to points close to the origin, but the resulting topology *is* G-invariant. A sequence of tilings $\mathbf{T}_i$ converges to a tiling $\mathbf{T}$ if there exists a sequence of group elements g_i, converging to the identity, such that for every compact subset K of Euclidean space, the tilings $g_i(\mathbf{T}_i)$ eventually agree with $\mathbf{T}$ on K.

Definition 3.1. *Let $G' \subseteq G$ contain G_t and let X be a closed, G'-invariant subset of $X_\mathcal{P}$. A* tiling dynamical system (X, G') *is the set X together with the action of G' on X.*

It is usually assumed in the tiling literature that $G' = G_t$. This can be assumed without loss of generality when G_r is a finite group simply by making extra copies of each prototile, one for each element of G_r. The situation is more complicated in cases such as the pinwheel tiling, where G_r is infinite.

3.2. **Minimality.** A topological dynamical system (X, G') is *minimal* if X is the orbit closure of any of its elements.

Proposition 3.2. *If the fusion rule $\mathcal{R}$ is primitive, then the dynamical system $(X_\mathcal{R}, G)$ is minimal.*

Proof. Let $\mathbf{T} \in X_\mathcal{R}$ be any fixed tiling. We will show that given $\mathbf{T}' \in X_\mathcal{R}$ and $\epsilon > 0$ there is a group element g such that $d(g(\mathbf{T}), \mathbf{T}') < \epsilon$. Denote by $[\mathbf{T}']_r$ the patch of tiles in $\mathbf{T}'$ that intersect the ball of radius r centered at the origin. By definition we know that any such patch is admissible by $\mathcal{R}$ and so there is an $n \in \mathbb{N}$ and a $i \in \{1, ..., j_n\}$ for which $[\mathbf{T}']_{1/\epsilon}$ is a subpatch of $P_n(i)$.

On the other hand, primitivity means that there is an N such that every N-supertile contains a copy of $P_n(i)$. Since $\mathbf{T}$ is a union of N-supertiles, it contains many copies of $P_n(i)$. Pick g to bring some particular copy of $P_n(i)$ to the origin in agreement with $[\mathbf{T}']_{1/\epsilon}$. Since $\mathbf{T}'$ and $g(\mathbf{T})$ are identical on the ball of radius $1/\epsilon$ about the origin, $d(g(\mathbf{T}), \mathbf{T}') < \epsilon$. □

Remarks.

(1) It is not necessarily true that $(X_\mathcal{R}, G_t)$, i.e. the dynamical system with only translations acting, is minimal. Consider any fusion rule having only finitely many relative orientations of the prototiles, but which for some reason we took G to be the full Euclidean group. In this case (X, G) would be minimal but $(X_\mathcal{R}, G_t)$ would not. No tiling could approximate an irrational rotation of itself.

(2) On the other hand, the pinwheel tilings [50] provide an example where G is the full Euclidean group but (X, G_t) is minimal.
(3) Primitivity is sufficient but not necessary for minimality. In particular, the Chacon transformation is not primitive, but is minimal. For each n there does not exist an N for which $P_N(b)$ contains $P_n(a)$. However, there does exists a radius r_n such that every ball of radius n contains at least one $P_n(a)$ and at least one $P_n(b)$, so the patch $[\mathbf{T}']_{1/\epsilon}$ can be found in every $\mathbf{T}$.

3.3. **Invariant measures in general tiling dynamical systems.** We begin our treatment of the ergodic theory of fusion tilings with a discussion of how invariant measures work for a general FLC tiling dynamical systems X, with a focus on patch frequency. For convenience, we assume for the remainder of Section 3 that our action is by $G = \mathbb{R}^d$ only. See section 3.7 for the modifications needed to apply this theory to other subgroups of the Euclidean group.

Let P be any patch of tiles containing the origin. Let U be a measurable subset of $\mathbb{R}^d$, let $X_{P,U}$ be the cylinder set of all tilings in X that contain $P - \vec{v}$ for some $\vec{v} \in U$, and let $\chi_{P,U}$ to be the indicator function of this cylinder set. The sets $X_{P,U}$, plus translates of these sets, generate our σ-algebra of measurable sets in X. Let μ be an invariant measure on X. If U is sufficiently small, then for every tiling $\mathbf{T} \in X$, there is at most one $\vec{v} \in U$ for which $P - \vec{v} \subset \mathbf{T}$. Since the measure is additive and translation-invariant, $\mu(X_{P,U})$ must be proportional to the volume of U and we define

$$freq_\mu(P) = \frac{1}{Vol(U)} \mu(X_{P,U}), \tag{3.1}$$

a quantity that is independent of U.

For any $A \subset \mathbb{R}^d$ we denote the number of times an equivalent copy of P appears in $\mathbf{T}$, completely contained in the set A, as $\#(P \text{ in } A \cap \mathbf{T})$. As a special case, if P' is another patch (usually some supertile), we denote by $\#(P \text{ in } P')$ the number of equivalent copies of P completely contained in P'. Next we pick a specific U_0 that is a small ball centered at the origin and define the function

$$f_P(\mathbf{T}) = \frac{1}{Vol(U_0)} \chi_{P,U_0}(\mathbf{T}). \tag{3.2}$$

This is a smeared δ-function that counts the appearances of P. $\displaystyle\int_A f_P(\mathbf{T} - \vec{v}) d\vec{v}$ is essentially $\#(P \text{ in } A \cap \mathbf{T})$, with small corrections for patches that come within the diameter of U_0 of the boundary of A. Note that $\displaystyle\int_X f_P(\mathbf{T}) d\mu = \frac{1}{Vol(U_0)} \mu(X_{P,U_0}) = freq_\mu(P)$.

We use the following version of the pointwise ergodic theorem:

Theorem 3.3. *Let $(X, \mathbb{R}^d)$ be a tiling dynamical system with invariant Borel probability measure μ. Let $\{A_m\}$ be a sequence of balls centered at the origin, with radius going to* *infinity, and let P be any finite patch. Then for μ-almost every tiling $\mathbf{T}$ the limit*

$$\lim_{m\to\infty} \frac{1}{Vol(A_m)} \int_{A_m} f_P(\mathbf{T} - \vec{v}) d\lambda(\vec{v}) = \bar{f}_P(\mathbf{T}) \tag{3.3}$$

exists. Furthermore, $\int_X \bar{f}_P(\mathbf{T}) d\mu = \int_X f_P(\mathbf{T}) d\mu = freq_\mu(P)$. If μ is ergodic, then for μ-almost every $\mathbf{T}$, $\bar{f}_P(\mathbf{T}) = freq_\mu(P)$.

The quantity $\bar{f}_P(\mathbf{T})$ corresponds to the usual notion of frequency as "number of occurrences per unit area" in $\mathbf{T}$, as computed with an expanding sequence of balls around the origin.[2] We will use the term "frequency" for this quantity, and will call $freq_\mu(P)$ an "abstract frequency". The ergodic theorem says that almost all tilings have well-defined frequencies, and that the abstract frequency $freq_\mu(P)$, while not necessarily the frequency of P in any specific tiling, is the average over all tilings of the frequency of P. Thus, any upper or lower bounds on the frequency of P that apply to μ-almost every $\mathbf{T}$ result in upper or lower bounds on the abstract frequency. If every tiling has the same frequency of P, then there is only one possible value for the abstract frequency of P, and thus for the measure of any $X_{P,U}$. Tiling spaces where all tilings have the same set of frequencies are uniquely ergodic.

For an FLC tiling, the set of all patches (up to translation) is countable, and the intersection of a countable number of sets of full measure has full measure. As a result, μ-almost every tiling $\mathbf{T}$ has well-defined frequencies for *every* patch P.

Conversely, if a tiling $\mathbf{T}$ has well-defined patch frequencies, then we can construct a probability measure on X by taking $\mu(X_{P,U}) = \bar{f}_P(\mathbf{T}) Vol(U)$ for small U and extending by additivity to larger U's. The additivity properties of measures follow from the addititivity of frequencies. For instance, if a patch P can be extended in two ways, to P' or P'', then $X_{P,U} = X_{P',U} \coprod X_{P'',U}$. The identity $\mu(X_{P,U}) = \mu(X_{P',U}) + \mu(X_{P'',U})$ follows from $\bar{f}_P(\mathbf{T}) = \bar{f}_{P'}(\mathbf{T}) + \bar{f}_{P''}(\mathbf{T})$. Countable additivity is not an issue, since the tiling space is locally the product of Euclidean space (where Lebesgue measure has all the desired properties) and a Cantor set (where the σ-algebra is based on finite partitions into clopen sets).

A measure defined in this way may or may not be ergodic. For instance, if $\mathbf{T}$ is a one-dimensional tiling with the pattern $a^\infty b^\infty$, with a tiles to the left of the origin and b tiles to the right, then the resulting measure on the orbit closure of $\mathbf{T}$ is the average of the two ergodic measures.

3.4. **Invariant measures and fusion tilings.** The possibilities for invariant measures of fusion tilings are intimately connected to the asymptotic behavior of the transition matrices

[2] Ergodic theorems are often stated not with balls, but in terms of Følner or van Hove sequences that have special properties, such as being "regular" or "tempered". That generality is useful for computing frequencies using different sampling regions, or when considering more complicated groups than $\mathbb{R}^d$. For our purposes, however, balls are sufficient.

$M_{n,N}$ as $N \to \infty$. Our analysis of these matrices takes the place of the standard Perron-Frobenius theory used so fruitfully for substitution systems. The results of this section and the next closely parallel those of [24, 12], the difference being that those papers consider discrete systems in 1 dimension, while we consider continuous systems in an arbitrary number of dimensions. We assume that our fusion rule is van Hove, recognizable, and primitive; these properties are essential.

We define the frequency $\tilde{f}_{P_n(j)}$ of a supertile $P_n(j)$ in a tiling $\mathbf{T}$ to be its frequency *as a supertile*, not as a patch. In other words, $\tilde{f}_{P_n(j)}(\mathbf{T})$ is obtained by viewing $\mathbf{T}$ as an element of $X^n_{\mathcal{R}}$, thereby excluding patches that have the same composition as $P_n(j)$, but are actually proper subsets of another n-supertile or straddle two or more n-supertiles. The abstract supertile frequency of $P_n(j)$ is obtained by averaging $\tilde{f}_{P_n(j)}$ over all tilings. By recognizability, each occurrence of a supertile $P_n(j)$ is marked by an element of a set of larger patches S_i. We then have $\tilde{f}_{P_n(j)}(\mathbf{T}) = \displaystyle\sum_i \bar{f}_{S_i}(\mathbf{T})$, and the abstract supertile frequency of $P_n(j)$ is $\displaystyle\sum_i freq_\mu(S_i)$.

Consider a sequence $\rho = \{\rho_n\}$ where each $\rho_n \in \mathbb{R}^{j_n}$ has all nonnegative entries. We say that ρ is *volume-normalized* if for all n we have $\displaystyle\sum_{i=1}^{j_n} \rho_n(i) Vol(P_n(i)) = 1$. We say that it has *transition consistency* if $\rho_n = M_{n,N}\rho_N$ whenever $n < N$. A transition-consistent sequence ρ that is normalized by volume is called a sequence of *well-defined supertile frequencies*. This terminology will be justified by the proof of Theorem 3.4.

Theorem 3.4. *Let $\mathcal{R}$ be a recognizable, primitive, van Hove fusion rule. There is a one-to-one correspondence between the set of all invariant Borel probability measures on $(X_{\mathcal{R}}, \mathbb{R}^d)$ and the set of all sequences of well-defined supertile frequencies with the correspondence that, for all patches P,*

$$freq_\mu(P) = \lim_{n\to\infty} \sum_{i=1}^{j_n} \#\left(P \ in \ P_n(i)\right) \rho_n(i) \tag{3.4}$$

Proof. Suppose μ is an invariant measure. For each $n \in \mathbb{N}$ and each $i \in \{1, 2, ..., j_n\}$, define $\rho_n(i)$ to be the abstract supertile frequency of $P_n(i)$. For a fixed n, $X_{\mathcal{R}}$ is the union of cylinder sets given by which n-supertile is at the origin. Since the measure of $X_{\mathcal{R}}$ is 1 and the measure of each of these cylinder sets is $\rho_n(i)Vol(P_n(i))$, the sequence ρ is volume-normalized.

The set of tilings where the origin lies in an n-supertile of type i is the union of disjoint sets where the origin lies in a supertile of type i, which in turn sits in an particular way in an N-supertile. There are $M_{n,N}(i,j)$ ways for $P_n(i)$ to sit in $P_N(j)$. The additivity of the measure implies that $\rho_n(i) = \displaystyle\sum_{j=1}^{j_N} M_{n,N}(i,j)\rho_N(j)$. Hence μ gives rise to a sequence of well-defined supertile frequencies.

To see that equation (3.4) applies, let P be any patch and call its diameter L_P. Since the fusion rule is van Hove, we can pick an n large enough that the fraction of each n-supertile within L_P of the boundary is so small that P patterns appearing in this region can only contribute ϵ or less to the frequency of P's in a union of n-supertiles.

To count the number of P's in a large ball A_m around the origin, we must count the number of P's in each n-supertile contained in that ball, plus the number of P's that straddle two of more n-supertiles, plus the P's in an n-supertile that is only partially in the ball. As a fraction of the whole, the third set goes to zero as $m \to \infty$ and the second goes to zero as $n \to \infty$. Thus $\#(P \text{ in } A_m \cap \mathbf{T}) = \sum_{i=1}^{j_n} \#(P \text{ in } P_n(i))\#(P_n(i) \text{ in } A_m)+$ boundary occurrences. Dividing by $Vol(A_m)$ and taking limits, first as $m \to \infty$ and then as $n \to \infty$, gives the identity

$$\bar{f}_P(\mathbf{T}) = \lim_{n\to\infty} \sum_{i=1}^{j_n} \#\left(P \text{ in } P_n(i)\right) \tilde{f}_{P_n(i)}(\mathbf{T})$$

for all tilings $\mathbf{T}$ with well-defined patch frequencies. Integrating this identity over all tilings then gives equation (3.4).

Now suppose that $\{\rho_n\}$ is a sequence of well-defined supertile frequencies. To establish the existence of an invariant measure μ for which $\{\rho_n\}$ represents the abstract supertile frequencies, we simply define $freq_\mu(P)$, and hence the measure of each cylinder set $X_{P,U}$, by equation (3.4).

To see convergence of the limit on the right hand side, note that, if $n < N$, the number of P in $P_N(j)$ is the sum of the number of P in each n-supertile in $P_N(j)$, plus a small contribution from P's that straddle two or more supertiles. That is, $\#(P \text{ in } P_N(j)) \approx \sum_i \#(P \text{ in } P_n(i))M_{n,N}(i,j)$, so

$$\sum_j \#(P \text{ in } P_N(j))\rho_N(j) \approx \sum_{i,j} \#(P \text{ in } P_n(i))M_{n,N}(i,j)\rho_N(j) = \sum_i \#(P \text{ in } P_n(i))\rho_n(i).$$

As $n \to \infty$ the contribution of P's that straddle two n-supertiles goes to zero, so the right hand side of (3.4) is a Cauchy sequence.

The non-negativity of the measure follows from the non-negativity of each ρ_n. The identity $\mu(X_{\mathcal{R}}) = 1$ follows from volume normalization. Finite additivity follows from the observation that, if a patch P can be extended to P' or P'', then $\#(P \text{ in } P_n(i)) = \#(P' \text{ in } P_n(i)) + \#(P'' \text{ in } P_n(i))$, plus a small correction for the situations where P is completely contained in $P_n(i)$ but P' or P'' is not, a correction that does not affect the limit as $n \to \infty$. As noted earlier, countable additivity is not an issue for tiling spaces. Thus μ is a well-defined measure.

$\square$

3.4.1. *Parameterization of invariant measures.* In Theorem 3.4 we showed how measures relate to well-defined sequences of supertile frequencies. We now give an explicit parametrization of the invariant measures in terms of the transition matrices $M_{n,N}$, a parametrization that we will then use to investigate unique ergodicity.

The *direction* of each column of $M_{n,N}$ is defined to be the volume-normalized vector in $\mathbb{R}^{j_n}$ collinear with it, and we define the *direction matrix* $D_{n,N}$ to be the matrix whose columns are the directions of the columns of $M_{n,N}$. That is,

$$D_{n,N}(*,k) = \frac{M_{n,N}(*,k)}{\sum_{l=1}^{j_n} M_{n,N}(l,k)Vol(P_n(l))}.$$

Let $\Delta_{n,N}$ be the convex hull of the columns of $D_{n,N}$, sitting within the hyperplane of volume-normalized vectors in $\mathbb{R}^{j_n}$. Note that the extreme points of $\Delta_{n,N}$ are columns of $D_{n,N}$, but not every column need be an extreme point. Since each column of $M_{n,N+1}$ is a sum of columns of $M_{n,N}$, each column of $D_{n,N+1}$ is a weighted average of the columns of $D_{n,N}$, so $\Delta_{n,N+1} \subset \Delta_{n,N}$. Let $\Delta_n = \cap_{N=n+1}^{\infty} \Delta_{n,N}$.

The matrix $M_{n,N}$ defines an affine map sending Δ_N to Δ_n, since if ρ_N is volume-normalized in $\mathbb{R}^{j_N}$, then so is $M_{n,N}\rho_N \in \mathbb{R}^{j_n}$. We define Δ_∞ to be the inverse limit of the polytopes Δ_n under these maps.

Corollary 3.5. *Let $(X_{\mathcal{R}}, \mathbb{R}^d)$ be the dynamical system of a recognizable, primitive, van Hove fusion rule. The set of all invariant Borel probability measures is parameterized by Δ_∞.*

Proof. By Theorem 3.4, we need only show that each element of Δ_∞ gives rise to a sequence $\{\rho_n\}$ of well-defined supertile frequencies and vice versa.

By construction, each point in Δ_∞ is a sequence of well-defined supertile frequencies, since each point in Δ_n is volume-normalized and non-negative, and since the sequence has transition consistency. For the converse, suppose that $\{\rho_n\}$ is a sequence of well-defined supertile frequencies. We must show that $\rho_n \in \Delta_n$. Since $\rho_n = M_{n,N}\rho_N$, ρ_n is a non-negative linear combination of the columns of $M_{n,N}$, and so is a weighted average of the columns of $D_{n,N}$. Thus $\rho_n \in \Delta_{n,N}$. Since this is true for every N, $\rho_n \in \Delta_n$. $\square$

3.4.2. *Measures arising from supertile sequences.* In this section we provide a concrete way of visualizing certain invariant measures, in particular the ergodic ones. The way to do it is by looking at frequencies of patches as they occur in specific sequences of nested supertiles.

Definition 3.6. *Let $\kappa = \{k_n\}$ be a sequence of supertile labels, with $k_n \in \{1, 2, \ldots, j_n\}$. For each $n < N$, we consider the frequency of each n-supertile $P_n(i)$ within $P_N(k_N)$:*

$$\rho_{n,N}(i) = M_{n,N}(i, k_N)/Vol(P_N(k_N)).$$

We say that κ has well-defined supertile frequencies *if $\rho_n(i) = \lim_{N\to\infty} \rho_{n,N}(i)$ exists for every n and every $i \in \{1, \ldots, j_n\}$.*

Note that the vectors $\rho_n(i)$, if they exist, do indeed form a sequence of well-defined supertile frequencies. For $n < n' < N$, $\rho_{n,N} = M_{n,n'}\rho_{n',N}$. Taking a limit as $N \to \infty$ gives $\rho_n = M_{n,n'}\rho_{n'}$, so the sequence has transition consistency. Likewise, it is easy to check volume normalization. We can therefore associate an invariant measure to every sequence κ that has well-defined supertile frequencies.

The purpose of using a sequence κ is to visualize a measure. Given such a sequence, one can imagine a tiling $\mathbf{T}$ where the origin sits inside a k_1 1-supertile, which sits inside a k_2 2-supertile, etc. Under mild assumptions, the supertile frequencies $\tilde{f}_{P_n(i)}(\mathbf{T})$ will then equal $\rho_n(i)$, and for any patch P, $\bar{f}_P(\mathbf{T})$ will equal $freq_\mu(P)$, where μ is constructed from the sequence $\{\rho_n\}$. The concept of using sequences κ to obtain measures applies even to non-primitive fusions, as long as the supertile frequencies are well-defined.

Example 3.7. *A minimal fusion rule with two ergodic measures.* This is a variation on an example found in [23] and illustrates the results of [21]. Consider a prototile-regular 1-dimensional fusion rule with two unit length tiles a and b and let $G = \mathbb{R}$. Let $P_n(a) = (P_{n-1}(a))^{10^n} P_{n-1}(b)$ and $P_n(b) = (P_{n-1}(b))^{10^n} P_{n-1}(a)$, so that $P_1(a) = aaaaaaaaaab$ and $P_1(b) = bbbbbbbbbba$, etc. $M_{n-1,n} = \left(\begin{smallmatrix} 10^n & 1 \\ 1 & 10^n \end{smallmatrix}\right)$ which has eigenvalues $10^n - 1$ and $10^n + 1$. Elementary linear algebra allows us to compute the frequencies as follows. Let $\alpha_n = \prod_{k=1}^{n} \dfrac{10^k - 1}{10^k + 1}$, which approaches a limit of just over 0.8 as $n \to \infty$. The fraction of a's in $P_n(a)$ is $(1 + \alpha_n)/2 \approx 0.9$, while the fraction of a's in $P_n(b)$ is $(1 - \alpha_n)/2 \approx 0.1$.

There are exactly two ergodic measures on this system. Δ_n is an interval for every value of n, with endpoints defined by the limits of the first and second columns of $D_{n,N}$. Likewise, Δ_∞ is an interval, whose endpoints μ_a and μ_b can be obtained from the supertile sequences $\kappa = (a, a, a, a, \ldots)$ and $\kappa = (b, b, b, b, \ldots)$. The first ergodic measure, μ_a, sees the frequencies of a's and b's as measured in the type-a supertiles and thus is a-heavy; the second, μ_b, reverses the roles of a and b and is b-heavy. The measure $\mu = (\mu_a + \mu_b)/2$ is invariant but not ergodic; this measure corresponds to Lebesgue measure when the system is seen as a cut-and-stack transformation.

In general, a prototile-regular fusion with j species of tiles can have at most j ergodic measures. Of course, there can be fewer, if one or more columns of $D_{n,N}$ are in the convex hull of the others for large N. The following example shows how a sequence κ may lead to a measure that is not ergodic.

Example 3.8. *A non-ergodic measure coming from a sequence κ.* Consider the following variant of the previous example. Instead of having two species of tiles or supertiles, we have three, with the fusion rules

$P_n(a) = (P_{n-1}(a))^{10^n} P_{n-1}(b) P_{n-1}(c) (P_{n-1}(a))^{10^n}$,
$P_n(b) = (P_{n-1}(b))^{10^n} P_{n-1}(a) P_{n-1}(c) (P_{n-1}(b))^{10^n}$,

$P_n(c) = (P_{n-1}(a))^{10^n} P_{n-1}(c) P_{n-1}(c) (P_{n-1}(b))^{10^n}$,
with transition matrix $M_{n-1,n} = \begin{pmatrix} 2\times 10^n & 1 & 10^n \\ 1 & 2\times 10^n & 10^n \\ 1 & 1 & 2 \end{pmatrix}$. As before, the measure μ_a coming from the sequence $a, a, \ldots$ is ergodic and describes the patterns in a high-order a supertile, which is rich in a tiles, while the measure μ_b describes the patterns in a high-order b supertile, which is similarly rich in b tiles. The measure μ_c from $c, c, \ldots$ describes a high-order c supertile, which is (essentially) half high-order a supertiles and half high-order b supertiles. In other words, $\mu_c = (\mu_a + \mu_b)/2$ is not ergodic.

3.5. **Unique ergodicity.** A system is *uniquely ergodic* if it has exactly one ergodic probability measure, in which case this measure is the only invariant probability measure whatsoever. Tiling dynamical systems are uniquely ergodic when there are uniform patch frequencies that can be computed regardless of the tiling (see e.g. Theorem 3.3 of [59]).

For each n, we say that the family of matrices $D_{n,N}$ is *asymptotically rank 1* if there is a vector $d_n \in \mathbb{R}^{j_n}$ such that the columns of $D_{n,N}$ all approach d_n as $N \to \infty$. Put another way, $D_{n,N}$ is asymptotically rank one if Δ_n consists of a single point.

Theorem 3.9. *If a primitive fusion rule $\mathcal{R}$ is van Hove and recognizable, then $D_{n,N}$ is asymptotically rank 1 for every n if and only if the tiling dynamical system $(X_{\mathcal{R}}, \mathbb{R}^d)$ is uniquely ergodic.*

Proof. By Corollary 3.5, having a unique measure is the same as Δ_∞ being a single point, which is equivalent to each Δ_n being a single point. □

Corollary 3.10. *The tiling dynamical system of a transition-regular fusion rule that is recognizable, van Hove and primitive is uniquely ergodic.*

What remains is to find checkable conditions on the transition matrices $M_{n,N}$ that imply that the direction matrices $D_{n,N}$ are asymptotically rank one. For the n-th transition matrix $M_{n-1,n}$, let

$$\delta_n = \min_k \left(\frac{\min_i M_{n-1,n}(i,k)}{\max_i M_{n-1,n}(i,k)} \right) \tag{3.5}$$

This measures the extent to which the columns of $M_{n-1,n}$ are unbalanced.

Theorem 3.11. *If $\sum_n \delta_n$ diverges, then $\mathcal{R}$ is primitive and for each n the family $D_{n,N}$ is asymptotically rank 1.*

Proof. First we show that the diameter of $\Delta_{n,N+1}$ is bounded by $(1-\delta_{N+1})$ times the diameter of $\Delta_{n,N}$. Let $v_{n,N}$ be the sum of the columns of $M_{n,N}$, and let $\hat{v}_{n,N}$ be the direction of $v_{n,N}$. Let $m_{N+1,i}$ be the smallest entry of the ith column of $M_{N,N+1}$ (which may be zero). The ith column of $M_{N,N+1}$ is then $m_{N+1,i} \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}$, plus additional terms, so the ith column of $M_{n,N+1}$ is $m_{N+1,i} v_{n,N}$, plus an additional linear combination of columns of $M_{n,N}$. This means that the direction of the ith column of $M_{n,N+1}$ is a weighted average of $\hat{v}_{n,N}$ and an unknown element of $\Delta_{n,N}$, with $\hat{v}_{n,N}$ having weight at least δ_{N+1}. Thus the direction of each column of $M_{n,N+1}$, and hence $\Delta_{n,N+1}$ lies in the convex set $\delta_{N+1}\hat{v}_{n,N} + (1-\delta_{N+1})\Delta_{n,N}$, a set whose diameter is $(1-\delta_{N+1})$ times the diameter of $\Delta_{n,N}$.

If $\sum \delta_n$ diverges, then δ_n is nonzero infinitely often, so the fusion rule is primitive. Furthermore, the infinite product $\prod_{k=n+1}^{\infty}(1-\delta_k)$ equals zero. Thus Δ_n has diameter zero, and is a single point. □

Corollary 3.12. *If $\mathcal{R}$ is a strongly primitive, van Hove and recognizable fusion rule whose transition matrices $M_{n-1,n}$ have uniformly bounded elements, then $(X_\mathcal{R}, \mathbb{R}^d)$ is uniquely ergodic.*

Proof. If the smallest matrix element of $M_{n-1,n}$ is at least 1 and the largest is at most K, then each $\delta_n \geq 1/K$. Thus $\sum_n \delta_n$ diverges and every $D_{n,N}$ is asymptotically rank 1. By Theorem 3.9, $(X_\mathcal{R}, \mathbb{R}^d)$ is uniquely ergodic. □

3.6. **Transversals, towers, and rank.** In tiling theory, especially the aperiodic order and quasicrystal branches, the concept of the transversal is an essential component to many arguments. For instance, it is used for computing the C^*-algebras and K-theory as in [37] and references therein, and it is used for gap-labelling in [9]. In ergodic theory, the concept of towers and especially the Rohlin Lemma (also called the Kakutani-Rokhlin or Halmos-Rokhlin Lemma) is a tool that has been used to great effect (see for instance [48, 46]) . One notable result [5] that uses towers and the lemma is that any aperiodic measure-preserving transformation on a standard Lebesgue space can be realized as a cutting and stacking transformation. Towers are used to define the notion of rank, which is intimately related to spectral theory.

For convenience, we will assume that $G = G_t$ and that our supertile sets $\mathcal{P}_n$ are described as follows. We position each prototile in $\mathcal{P}_0$ so that the origin is in its interior, and the place where the origin sits is called the *control point* of the prototile. In a prototile that has been translated by some element $\vec{v} \in \mathbb{R}^d$ we call $\vec{v}$ the *control point* of the new tile. Each element of $\mathcal{P}_1$ is positioned such that the origin is on the control point of one of the prototiles it contains, and this point will be considered to be the control point of the 1-supertile. Likewise, we situate the elements of $\mathcal{P}_2, \mathcal{P}_3$, etc. in such a way that the origin forms the control point of the n-supertile and lies atop the control point of the $\mathcal{P}_{n-1}$-tile at the origin.

Definition 3.13. *The* transversal *of $X_\mathcal{R}$ is the set of all tilings positioned with the origin at the control point of the tile that contains it. If $\mathcal{R}$ is recognizable, the n-*transversal *of $X_\mathcal{R}$*

is the set of all tilings positioned so that the origin is at the control point of the n-supertile containing it.

(If G_r is nontrivial, the situation is only slightly more complicated. For each prototile, we fix a preferred orientation. The transversal of a tiling space is the set of tilings with the origin at a control point, and with the tile containing the origin in the chosen orientation. The n-transversals are defined similarly.)

The transversal of $X_{\mathcal{R}}$ has a natural partition into j_0 disjoint sets, one for each type of tile. Each of these can be decomposed into pieces, one for each way that the tile containing the origin can sit in a 1-supertile, and this partitioning process can be continued indefinitely. The n-transversal of $X_{\mathcal{R}}$ can be thought of as the transversal of $X^n_{\mathcal{R}}$, and can likewise be partitioned. When the fusion rule has finite local complexity, the transversal is a totally disconnected set. The n-transversals will form the *base* for the nth tower representation.

In one-dimensional discrete dynamical systems, phase space can be visualized as a stack of Borel sets placed one above the other with the transformation taking each set to the one directly above it, except the top one, on which the action is not visualized. This representation of the system is known as a Rohlin tower. When the action is continuous, multidimensional, or by an unusual group, the "towers" no longer resemble physical towers, but the term still applies. The concept of Rohlin towers for groups other than $\mathbb{Z}$, and in particular for $\mathbb{R}^d$, is investigated in [46, 36, 18, 53] and our definitions are drawn from these. Let $(X, \mathfrak{B}, \mu)$ be a probability space acted on by some amenable group G to produce an ergodic dynamical system.

Definition 3.14.
(1) *Let $B \subset \mathfrak{B}$ and let $F \subset \mathbb{R}^d$ be relatively compact, and suppose that $g(B) \cap h(B) = \emptyset$ for any $g \neq h \in F$. In this case we call (B, F) a* Rohlin tower *with* base B*,* shape F*, and* levels $g(B)$*.*
(2) *A* tower system *is a finite list of towers $\mathcal{F} = (B_1, F_1), ..., (B_n, F_n)$ such that all levels are pairwise disjoint.*
(3) *The* support *of a tower system is the union of its levels and the* residual set *is the complement of the support in X.*
(4) *A sequence $\mathcal{F}^k$ of tower systems is said to converge to $\mathfrak{B}$ if for every Borel set $A \in \mathfrak{B}$ and every $\epsilon > 0$ there is an N such that for all $k > N$ there is a union of levels of $\mathcal{F}^k$ whose symmetric difference from A measures less than ϵ.*

Recognizable fusion tiling dynamical systems come automatically equipped with tower systems that converge to the Borel σ-algebra $\mathfrak{B}(X_{\mathcal{R}})$. The nth tower system will have one tower for each prototile $P_n(j) \in \mathcal{P}_n$, for a total of j_n towers. The base of the jth tower is the set $B_n(j)$ of all tilings in the n-transversal that have a copy of $P_n(j)$ with its control point at the origin. The shape of the jth tower, denoted $F_n(j)$, depends on whether G_r is trivial. If so, then the shape is the set of all $\vec{v} \in G_t$ such that the origin is in the interior of

$P_n(j) + \vec{v}$. This shape is, up to sign, the same as the interior of $P_n(j)$ itself, thus earning the name "shape". If instead G_r is nontrivial, the tower construction must be modified to accomodate tilings that have discrete rotational symmetry. In this case $F_n(j)$ is the set of all $g \in G$ for which the origin is in $g(P_n(j))$, provided $P_n(j)$ has no symmetry. If it does, we must restrict the rotational portion of $F_n(j)$ to keep the levels disjoint.

In ergodic theory an important idea is that of rank. A dynamical system is said to have *rank* r if for every $\epsilon > 0$ there is a tower system $(B_1, F_1), ..., (B_r, F_r)$ that approximates all elements of $\mathfrak{B}$ up to measure ϵ, where r is the smallest integer for which this is possible. It is well-known for substitution sequences and self-affine tilings that the rank is bounded by the size of the alphabet or the number of prototiles, since every tile type gets its own tower for each application of the substitution. In the case of fusion tilings, the situation is only slightly more complicated and we can say

$$\text{rank}(X_{\mathcal{R}}, G) \;\leq\; \liminf_{n\to\infty} j_n \tag{3.6}$$

In general, rank bounds spectral multiplicity.

3.7. **Groups other than $\mathbb{R}^d$.** For much of this section we have assumed that $G = \mathbb{R}^d$. However, the results can readily be adapted to tiling spaces that involve other groups. In this section we indicate what changes have to be made when G_t is a proper subgroup of $\mathbb{R}^d$, when G_r is nontrivial, or both.

In general, G_t is the product of two groups, namely a continuous translation in a subspace E of $\mathbb{R}^d$, and a discrete lattice L in the orthogonal complement of E. In place of Lebesgue measure on $\mathbb{R}^d$, the measure on G_t is the product of Lebesgue measure on E and counting measure on L. Frequencies are defined as before as occurrences per unit volume in G_t. In fact, the ergodic theorem and Rohlin towers were first developed for discrete group actions and only later extended to continuous groups.

Having G_r nontrivial is more of a complication, especially if G_r is continuous, as with the pinwheel tiling. The ergodic theorem still applies, since we can first average over G_r and then average over G_t, but the G-orbit of a tiling can no longer be identified with Euclidean $\mathbb{R}^d$. When G_r is continuous, the frequency of a patch is no longer "number per unit volume", but is "number per unit volume per unit angle", and may depend on angle.

If the group G' that defines our dynamics is the same as the group G used to construct the tiling, then invariant measures are parametrized exactly as before, by Δ_∞, or equivalently by sequences of well-defined supertile frequencies. The only difference is that $\rho_n(i)$ is the sum or integral over angle of the frequency of the supertile $P_n(i)$. That is, it counts the average number per unit area of $P_n(i)$'s appearing in *any* orientation.

If G' is different from G, then we must distinguish between the G-invariant measures, which are parametrized by Δ_∞, and the G'-invariant measures, which may not be. Determining whether every G'-invariant measure is G_r-invariant is a separate computation.

4. Spectral theory, entropy, and mixing

A vector $\vec{\alpha} \in \mathbb{R}^d$ is a *topological eigenvalue* of translation if there is a continuous map $f : X_{\mathcal{R}} \to S^1$, where S^1 is the unit circle in $\mathbb{C}$, such that, for every $\mathbf{T} \in X_{\mathcal{R}}$ and every $\vec{v} \in \mathbb{R}^d$,

$$f(\mathbf{T} - \vec{v}) = \exp(2\pi i \vec{\alpha} \cdot \vec{v}) f(\mathbf{T}). \tag{4.1}$$

The map f is called a *topological eigenfunction. Measurable* eigenvalues and eigenfunctions are defined similarly, only for f measurable rather than continuous. Of course, since continuous functions are measurable, every topological eigenvalue is a measurable eigenvalue. Given a translation-invariant measure μ on $X_{\mathcal{R}}$, one can also ask how translations act on $L^2(X_{\mathcal{R}}, \mu)$. The pure point part of the spectral measure of the translation operators is closely related to the set of measurable eigenvalues. $(X_{\mathcal{R}}, \mu)$ is said to have "pure point spectrum" if the span of the eigenfunctions is dense in $L^2(X_{\mathcal{R}}, \mu)$.

The set of topological eigenvalues, the set of measurable eigenvalues, and the spectral measure of the translation operators are all G_r-invariant. If $g \in G_r$ and f is an eigenfunction with eigenvalue $\vec{\alpha}$, then $f \circ g$ is an eigenfunction with eigenvalue $g^{-1}(\vec{\alpha})$. If G_r is continuous, then this means that the number of nontrivial eigenvalues is either uncountable or zero. The first is impossible, as $L^2(X_{\mathcal{R}}, \mu)$ is separable, and as every topological eigenvalue is a measurable eigenvalue. The search for (topological or measurable) eigenvalues only has meaning, then, when G_r is discrete, in which case we can increase our prototile set by counting each orientation separately and take G_r trivial.

Standing assumption for Section 4: For the remainder of this section, we assume $G = \mathbb{R}^d$.

A measurable dynamical system is said to be *weakly mixing* if there are no nontrivial measurable eigenvalues. A topological dynamical system is *topologically weakly mixing* if there are no nontrivial topological eigenvalues. For primitive substitution tiling spaces there is no distinction between the two sorts of weak mixing, as it has been proven [34, 15] that every measurable eigenfunction (with respect to the unique invariant measure) can be represented by a continuous function. For fusion tilings, the situation is more subtle. In Theorem 4.1 we develop necessary and sufficient conditions for a vector to be a topological eigenvalue. This theorem is similar to Theorem 3.1 of [13], and of earlier 1-dimensional results of [8]. The key differences are that we work with $\mathbb{R}^d$ actions rather than $\mathbb{Z}^d$ actions and that we do not assume linear repetitivity.

Unlike the substitution situation it is possible for a fusion tiling space to have a measurable eigenvalue that is not a topological eigenvalue. In example 4.4 we exhibit a fusion tiling space that has pure point measurable spectrum but that is topologically weakly mixing. After this example was announced, it was noted [38] that the vertices of this tiling form a diffractive point pattern that is not Meyer.

4.1. **Topological eigenvalues.** For self-affine substitution tilings there are well-established criteria for a vector being an (topological or measurable) eigenvalue [59]. Given a substitution with stretching map L, there is a finite list of vectors $\vec{v}_i$ such that $\vec{\alpha}$ is an eigenvalue if and only if

$$\vec{\alpha} \cdot L^n(\vec{v}_i) \to 0 \pmod 1 \tag{4.2}$$

for each i.

Our first task is to construct an analogous criterion for topological eigenvalues of fusion tilings. Assuming strong primitivity, each $(n+2)$-supertile contains multiple copies of each n-supertile (at least one per $(n+1)$ supertile). Let $\mathcal{V}^n$ be the set of relative positions of two n-supertiles, of the same type, within an $(n+2)$-supertile. This is a finite set, since there are only finitely many kinds of $(n+2)$-supertiles and each $(n+2)$-supertile contains only finitely many n-supertiles. We call the elements of $\mathcal{V}^n$ *return vectors*. For each $\vec{\alpha} \in \mathbb{R}^d$, let $\eta_n(\vec{\alpha}) = \max_{\vec{v} \in \mathcal{V}^n} |\exp(2\pi i \vec{\alpha} \cdot \vec{v}) - 1|$.

Theorem 4.1. *Let $\mathcal{R}$ be a strongly primitive and recognizable van Hove fusion rule with finite local complexity. A vector $\vec{\alpha} \in \mathbb{R}^d$ is a topological eigenvalue if and only if $\sum_n \eta_n(\vec{\alpha})$ converges.*

For primitive substitution tilings, $\mathcal{V}^{n+1} = L\mathcal{V}^n$ and $\eta_n(\vec{\alpha})$ either goes to zero exponentially fast, or does not go to zero at all [59]. In such cases, the convergence of $\sum_n \eta_n(\vec{\alpha})$ is equivalent to $\eta_n(\vec{\alpha}) \to 0$, which is equivalent to the criterion (4.2), where the vectors $\vec{v}_i$ range over $\mathcal{V}^0$.

Proof. Since $X_{\mathcal{R}}$ is minimal, a continuous eigenfunction with a given eigenvalue is determined by its value on a single tiling $\mathbf{T}$. Fix $\mathbf{T} \in X_{\mathcal{R}}$ and $\vec{\alpha} \in \mathbb{R}^d$ and define $f(\mathbf{T}) = 1$. For each $\vec{x} \in \mathbb{R}^d$ let $f(\mathbf{T} - \vec{x}) = \exp(2\pi i \vec{\alpha} \cdot \vec{x})$. If this function is continuous on the orbit of $\mathbf{T}$, then it extends to an eigenfunction on all of $X_{\mathcal{R}}$. If it is not continuous, then $\vec{\alpha}$ cannot be a topological eigenvalue.

Suppose that $\sum_n \eta_n(\vec{\alpha})$ converges. We will show that f is continuous on the orbit of $\mathbf{T}$. Choose $\epsilon > 0$ and pick n large enough that $\sum_{k=n}^{\infty} \eta_n(\vec{\alpha}) < \epsilon$. We will show that if $\mathbf{T} - \vec{x}$ and $\mathbf{T} - \vec{y}$ agree to the nth recognizability radius ρ_n, then $f(\mathbf{T} - \vec{x})$ and $f(\mathbf{T} - \vec{y})$ are within ϵ. The following lemma states that $\vec{y} - \vec{x}$ can be expressed as a sum of return vectors.

Lemma 4.2. *Suppose that $\vec{x}$ and $\vec{y}$ are corresponding points in n-supertiles of the same type within the same N-supertile, with $N \geq n+2$. Then $\vec{y} - \vec{x}$ can be written as $\sum_{k=n}^{N-2} \vec{v}_k$, where $\vec{v}_k \in \mathcal{V}^k$.*

Proof of lemma. For each n, we work by induction on N. The base case $N = n+2$ follows from the definition of $\mathcal{V}^n$. Now suppose the lemma is true for $N = N_0$, and suppose that $\vec{x}$ and $\vec{y}$ sit in the same (N_0+1)-supertile. The point $\vec{x}$ sits in an (N_0-1) supertile S_x, say of type i, and $\vec{y}$ sits in an N_0-supertile, say of type j. By strong primitivity, there is an (N_0-1) supertile S_y of type i in the N_0-supertile that contains $\vec{y}$. Let $\vec{z}$ be the point in S_y corresponding to where $\vec{x}$ sits in S_x. (See Figure 1.) Then $\vec{z}-\vec{x}$ is a return vector

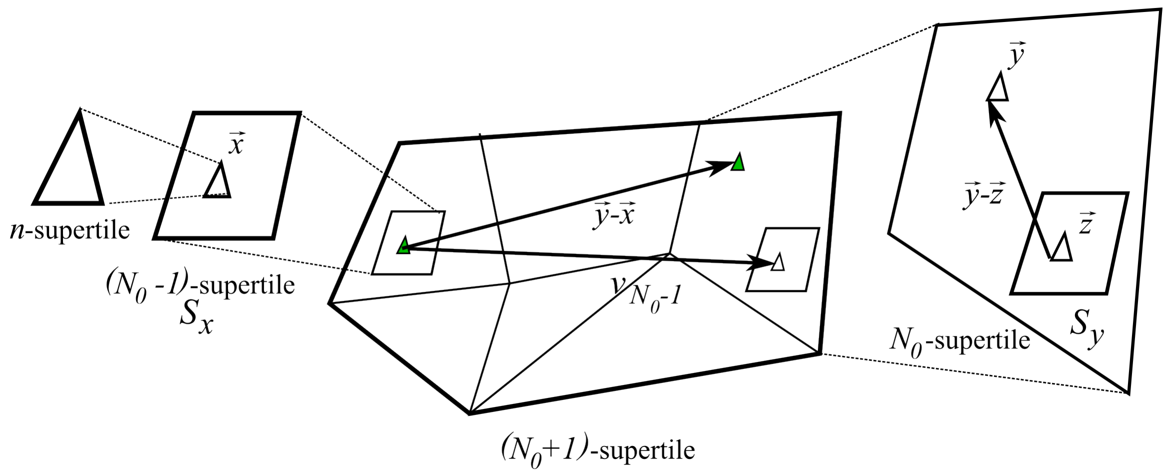


FIGURE 1. The induction step identifies a return between the n-supertiles (shown as shaded triangles) inside their (N_0+1)-supertile.

from S_x to S_y which we denote $\vec{v}_{N_0-1} \in \mathcal{V}^{N_0-1}$. Meanwhile, $\vec{y}$ and $\vec{z}$ sit in the same kind of n-supertile within the same N_0-supertile, so $\vec{y}-\vec{z} = \sum_{k=n}^{N_0-2} \vec{v}_k$. This means that $\vec{y}-\vec{x} = \sum_{k=1}^{N_0-1} \vec{v}_k$, as desired. $\square$

If $\vec{x}$ and $\vec{y}$ lie in the same N-th order supertile, the lemma implies that $\vec{y}-\vec{x} = \sum_{k=n}^{N-2} \vec{v}_k$, so

$$\begin{aligned} |f(\mathbf{T}-\vec{y}) - f(\mathbf{T}-\vec{x})| &= |\exp(2\pi i \vec{\alpha}\cdot(\vec{y}-\vec{x})) - 1| \\ \leq \sum_{k=n}^{N-2} |\exp(2\pi i\vec{\alpha}\cdot\vec{v}_k) - 1| &\leq \sum_{k=n}^{N-2} \eta_n(\vec{\alpha}) < \epsilon. \end{aligned}$$

Even if $\vec{x}$ and $\vec{y}$ do not lie in the same N-supertile of T for any N, it is still true that any patch containing $\vec{x}$ and $\vec{y}$ is congruent to a patch that lies within an N-supertile, so $\vec{y}-\vec{x}$ still takes the form $\sum_{k=n}^{N-2} \vec{v}_k$ for some N and we still obtain that $|f(\mathbf{T}-\vec{y}) - f(\mathbf{T}-\vec{x})| < \epsilon$. This estimate proves that f is continuous on the orbit of $\mathbf{T}$. By minimality, it extends to a continuous eigenfunction on all of $X_{\mathcal{R}}$. This proves half of Theorem 4.1.

For the converse, suppose that $\sum_n \eta_n(\vec{\alpha})$ diverges. Then there exists a subsequence $\sum_k \eta_{n+3k}(\vec{\alpha})$ that also diverges. We have the following lemma, that states that any finite sum of return vectors separated by three levels appears in $X_{\mathcal{R}}$ as the return of some n-supertile to itself.

Lemma 4.3. *For a given n, pick N such that $N+1-n$ is divisible by 3. For $k = n, n+3, \ldots, N-2$ pick $\vec{v}_k \in \mathcal{V}^k$ and let $\vec{v} = \vec{v}_n + \vec{v}_{n+3} + \cdots + \vec{v}_{N-2}$. For every such set of choices, there exists an N-supertile containing two n-supertiles of the same type, such that the relative position of the two n-supertiles is $\vec{v}$.*

Proof. Again we work by induction on N. If $N = 2$, then this follows from the definition of $\mathcal{V}^n$. Now suppose it is true for $N = N_0$, and we shall attempt to prove it for $N = N_0+3$. By the inductive hypothesis, there exist points $\vec{x}_0$ and $\vec{y}_0$ in corresponding n-supertiles within the same N_0-supertile S_1 such that $\vec{y}_0 - \vec{x}_0 = \vec{v} = \vec{v}_n + \vec{v}_{n+3} + \cdots + \vec{v}_{N_0-2}$, and there exist two (N_0+1)-supertiles S_2 and S_3, of the same type and with relative position $\vec{v}_{N_0+1}$, within an (N_0+3) supertile. (See Figure 2). By primitivity, S_2 and S_3 each contain copies of S_1 (in

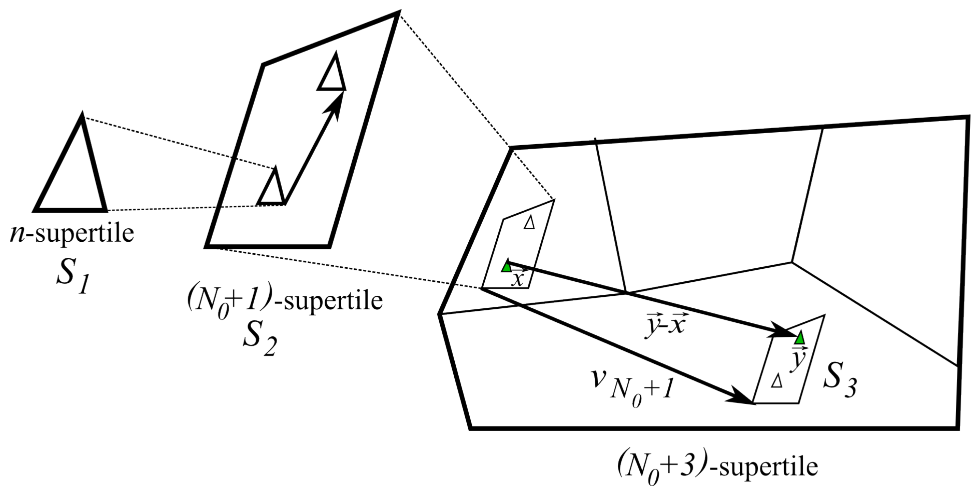


FIGURE 2. The induction step gives a return vector $\vec{y} - \vec{x}$ between two copies of S_1, shown shaded inside an (N_0+3)-supertile.

corresponding positions). Let $\vec{x}$ be the point corresponding to $\vec{x}_0$ in the copy of S_1 inside S_2, and let $\vec{y}$ be the point corresponding to $\vec{y}_0$ in the copy of S_1 inside S_3. Then $\vec{y} - \vec{x} = \vec{v}$. $\square$

Thus for any ϵ and for infinitely many values of n, one can find vectors $\vec{v}_n, \vec{v}_{n+3}, \cdots, \vec{v}_{N-2}$ with $\vec{v}_k \in \mathcal{V}^k$, such that $|\exp(2\pi i\vec{\alpha} \cdot \sum \vec{v}_k) - 1| > 2\epsilon$. By restricting to a subsequence we can assume that the complex numbers $\exp(2\pi i\vec{\alpha} \cdot \vec{v}_k)$ are either all in the first quadrant or all in the fourth quadrant, and that $|\exp(2\pi i\vec{\alpha} \cdot \sum \vec{v}_k) - 1| > \epsilon$. By Lemma 4.3 there then exist, for n arbitrarily large, two n-supertiles of the same type with relative position $\sum \vec{v}_k$. Pick $\vec{x}$ and $\vec{y}$ to be corresponding points of these supertiles in $\mathbf{T}$, such that a big ball around $\vec{x}$ and $\vec{y}$ lies entirely in the supertile. (This is possible since the supertiles form a van Hove sequence.) Then $f(\mathbf{T} - \vec{x})$ and $f(\mathbf{T} - \vec{y})$ differ in phase by at least ϵ, so our purported eigenfunction is not continuous. $\square$

4.2. **Measurable eigenvalues.** In this section we provide an example that has pure discrete spectrum from a measurable standpoint while being weakly mixing from a topological one.

Example 4.4. *The scrambled Fibonacci tiling.* We consider four fusions, denoted by the letters $\mathcal{F}$, $\mathcal{A}$, $\mathcal{E}$ and $\mathcal{S}$, with the last being the "scrambled Fibonacci" fusion whose tiling space has the desired properties. All use the prototile set $\{a, b\}$ where the length of a is the golden mean ϕ and the length of b is 1.

The first fusion rule is the usual Fibonacci rule $\mathcal{F}$, which is prototile- and transition-regular with $(n+1)$-supertiles given by $F_{n+1}(a) = F_n(a)F_n(b), F_{n+1}(b) = F_n(a)$. This fusion rule generates the self-similar Fibonacci tiling space $X_{\mathcal{F}}$, which is known to have measurable and topological pure point spectrum with eigenvalue set $\dfrac{1}{\sqrt{5}}\mathbb{Z}[\phi]$. Importantly for our calculations, the Euclidean length of $F_{n-1}(a)$ and $F_n(b)$ is ϕ^n, which deviates from an integer by $\pm\phi^{-n}$, and differs from an integer multiple of $1/\phi$ by $\pm\phi^{-(n+2)}$. The transition matrix is $M_0 = \left(\begin{smallmatrix} 1 & 1 \\ 1 & 0 \end{smallmatrix}\right)$. To make the second fusion rule, called "accelerated" Fibonacci, we first fix some increasing sequence of positive integers $\{N(n)\}_{n=1}^{\infty}$, where we assume $N(n) - N(n-1) > 2$. We define $\mathcal{A}$ to be the induced fusion rule on $N(n)$ levels so that $A_n(a) = F_{N(n)}(a)$ and $A_n(b) = F_{N(n)}(b)$. The lengths of the a and b n-supertiles for $\mathcal{A}$ are $\phi^{N(n)+1}$ and $\phi^{N(n)}$, respectively. This fusion rule is prototile-regular but not necessarily transition-regular, since now the transition matrices M_n are given by $M_0^{N(n)-N(n-1)}$.

The third fusion, which we call "exceptional", is derived from the accelerated rule by introducing a third supertile type on all odd levels and using it to introduce a relatively small defect in the next (even) level. On both the odd and even levels of $\mathcal{E}$, the supertiles $E_n(a)$ and $E_n(b)$ are constructed with the same populations of prototiles and $(n-1)$ supertiles as $A_n(a)$ and $A_n(b)$. When n is odd, and only when n is odd, there is an additional tile $E_n(e)$ with the same population of $(n-1)$-supertiles as $E_n(b)$. The fusion rules for $\mathcal{E}_n$ are as follows:

When n is odd, $E_n(a)$ and $E_n(b)$ are built from $\mathcal{E}_{n-1}$ in exactly the same way that the $\mathcal{A}_n$-supertiles are built from $\mathcal{A}_{n-1}$. The exceptional supertile $E_n(e)$ is obtained from $E_n(b)$ by permuting the $(n-1)$-supertiles so that all of the $E_{n-1}(a)$-supertiles come before any of the $E_{n-1}(b)$-supertiles. When n is even, $E_n(a)$ and $E_n(b)$ are built from $\mathcal{E}_{n-1}$ in exactly the same way that the $\mathcal{A}_n$-supertiles are built from $\mathcal{A}_{n-1}$, except that one copy of $E_{n-1}(b)$ is replaced with $E_{n-1}(e)$.

We can make the fusion rule prototile-regular by taking the induced fusion of $\mathcal{E}$ on even levels. We call this last fusion rule the *scrambled Fibonacci* $\mathcal{S}$, but most of our proofs center on the equivalent space $X_{\mathcal{E}}$. By controlling the sequence $N(n)$ we can change spectral properties of the scrambled Fibonacci fusion.

The space $X_{\mathcal{F}}$ is well known to be recognizable, and the recognizability of $X_{\mathcal{A}}$ is similar. The same patterns that allow us to recognize supertiles in $X_{\mathcal{F}}$ also work (with small modifications) in $X_{\mathcal{E}}$ and $X_{\mathcal{S}}$. We can thus freely speak of the (unique) n-supertile containing a particular tile.

Proposition 4.5. *If $N(2n)-N(2n-1)$ goes to infinity fast enough that $\sum_n \phi^{N(2n-1)-N(2n)}$ converges, then all four fusion spaces have pure point measurable spectrum with eigenvalues $\frac{1}{\sqrt{5}}(\mathbb{Z}+\phi\mathbb{Z}) = \frac{1}{\sqrt{5}}\mathbb{Z}[\phi]$.*

Proof. We will show that the four spaces $X_{\mathcal{F}}$, $X_{\mathcal{A}}$, $X_{\mathcal{E}}$ and $X_{\mathcal{S}}$ are all measurably conjugate. Then, since $X_{\mathcal{F}}$ is well known to have pure point spectrum with eigenvalue set $\frac{1}{\sqrt{5}}\mathbb{Z}[\phi]$, the others must as well. Since $X_{\mathcal{F}}$ and $X_{\mathcal{A}}$ are manifestly the same, and since $X_{\mathcal{E}} = X_{\mathcal{S}}$, we need only show that $X_{\mathcal{A}}$ and $X_{\mathcal{E}}$ are measurably conjugate.

In the tilings in $X_{\mathcal{E}}$, we call a supertile of any level *exceptional* if it lies in an $E_m(e)$-supertile of some level. Note that $E_{2m+2}(a)$ and $E_{2m+2}(b)$ each contain only one $E_{2m+1}(e)$ supertile and a large number (of order $\phi^{N(2m+2)-N(2m+1)}$) of supertiles of type $E_{2m+1}(a)$ and $E_{2m+1}(b)$. The fraction of $(2n)$-supertiles that are exceptional in $X_{\mathcal{E}}^{2n}$ is thus bounded by a constant times $\epsilon_n = \sum_{m=n}^{\infty} \phi^{-(N(2m+2)-N(2m+1))}$, which by assumption goes to zero as $n \to \infty$.

Suppose T is a tiling in $X_{\mathcal{E}}$. If the origin lies in an unexceptional supertile of some level n, and hence also at levels $n+1$, $n+2$, etc., and if the union of these supertiles is the entire line,[3] then we can convert this to a tiling in $X_{\mathcal{A}}$ by replacing each unexceptional $\mathcal{E}$-supertile containing the origin with the corresponding $\mathcal{A}$-supertile. From the definitions of the supertiles, this operation on $n+1$-supertiles is consistent with the operation on n-supertiles.

The measure of the tilings for which the origin is in an exceptional n-supertile is bounded by a constant times ϵ_n, and so goes to zero as $n \to \infty$. Thus, with probability 1, the origin lies in an unexceptional supertile of some level. Likewise, with probability 1, the union of the supertiles containing the origin is all of $\mathbb{R}$. Thus we have a map from $X_{\mathcal{E}}$ to $X_{\mathcal{A}}$ that is defined except on a set of measure zero. This map is readily seen to preserve measure and to commute with translation. □

Proposition 4.6. *If $\lim_{n\to\infty} N(2n+1)-2N(2n) = +\infty$, then $X_{\mathcal{S}}$ is topologically weakly mixing.*

Proof. First we show that elements of $\frac{1}{\sqrt{5}}\mathbb{Z}[\phi]$ cannot be topological eigenvalues. Then we show that real numbers that are not of this form cannot be topological eigenvalues.

[3]Note that this condition is translation-invariant, as every point in T would then lie in a sequence of unexceptional supertiles whose union is the entire line.

The supertiles $S_n(a) = E_{2n}(a)$ and $S_n(b) = E_{2n}(b)$ have length $\phi^{N(2n)+1}$ and $\phi^{N(2n)}$, respectively. For $\alpha \in \dfrac{1}{\sqrt{5}}\mathbb{Z}[\phi]$, $|\exp(2\pi i\alpha|S_n(a)|)-1|$ and $|\exp(2\pi i\alpha|S_n(b)|)-1|$ are bounded above and below by constants (depending on α) times $\phi^{-N(2n)}$. Each supertile $E_{2n+1}(e)$ contains the Fibonacci number $f_{N(2n+1)-N(2n)}$ consecutive copies of $E_{2n}(b)$, since that is how many $N(2n)$-supertiles of type b there are in the $N(2n+1)$-supertile $F_{N(2n+1)}(b)$. We thus find at least that many consecutive copies of $E_{2n}(b)$ in $S_{n+2}(a)$ and $S_{n+2}(b)$, so there exist vectors $v_k = k|S_n(b)|$ between n-supertiles of the same type in the same $(n+2)$-supertile, where k is any positive integer up to $f_{N(2n+1)-N(2n)}$. Since $|\exp(2\pi i\alpha|S_n(b)|)-1|$ is bounded below by a constant times $\phi^{-N(2n)}$, and since $f_{N(2n+1)-N(2n)}$ is of order $\phi^{N(2n+1)-N(2n)}$, and since $\phi^{N(2n+1)-2N(2n)}$ grows without bound, there are k for which for which $\exp(2\pi i\alpha v_k)$ is not close to 1. In fact, by taking n sufficiently large and picking k appropriately, we can get $\exp(2\pi i\alpha v_k)$ to be as close as we want to any number on the unit circle.

On the other hand, if α is not in $\dfrac{1}{\sqrt{5}}\mathbb{Z}[\phi]$, then by Pisot's theorem, $\exp(2\pi i\phi^n\alpha)$ does not approach 1 as $n \to \infty$. Since for arbitrarily large patches P there exist return vectors of length ϕ^n for n sufficiently large, α cannot be a topological eigenvalue. $\square$

It is simple to construct sequences $N(n)$ that meet the conditions of both Propositions (4.5) and (4.6). For instance, we could take $N(n) = 3^n$. Thus there exist fusion tilings that are topologically weakly mixing but are measurably pure point.

4.3. **Pure point spectrum.** An important and widely studied problem in substitution sequences and substitution tilings is determining when the (measure-theoretic) tiling dynamics have pure point spectrum. A key tool is Dekking's coincidence criterion [19], first developed for 1-dimensional substitutions of constant length and later extended to arbitrary substitutions, with generalizations in higher dimensions such as Solomyak's overlap algorithm [59]. In this section we explore the extent to which the analog of Dekking's criterion determines spectral type for fusions. The differences between substitutions and fusions are already apparent in the simplest category, namely one dimensional fusions of constant length.

We say a 1-dimensional fusion (or substitution) has *constant length* if, for each n, all of the n-supertiles $P_n(i)$ have the same size. This implies that tiles all have the same length and that, for fixed n, each n-supertile contains the same number L_n of $(n-1)$-supertiles. The fusion is *coincident* if, for each n, there exists an N such that any two N-supertiles agree on at least one n-supertile. The fusion is *coincident with finite waiting* if there exists a fixed integer k such that $N = n + k$ works for every n. For substitution tilings, coincidence is equivalent to coincidence with finite waiting, but for fusions it is not.

To each fusion of constant length we associate a solenoid $S_{\mathcal{R}}$, obtained as the inverse limit of the circle $\mathbb{R}/\mathbb{Z}$ under a series of maps, with the n-th map being multiplication by L_n. $S_{\mathcal{R}}$ is a topological factor of $X_{\mathcal{R}}$, with a point in $S_{\mathcal{R}}$ describing where the origin lies in a tile, a 1-supertile, a 2-supertile, etc, but not generally determining which type of n-supertile the origin sits in. There is a natural translational action on $S_{\mathcal{R}}$, and the span of the eigenfunctions of this action is dense in $L^2(S_{\mathcal{R}})$. If the factor map from $X_{\mathcal{R}}$ to $S_{\mathcal{R}}$ is a measurable conjugacy, or equivalently if there is a set of full measure on $X_{\mathcal{R}}$ where the factor map is 1:1, then $X_{\mathcal{R}}$ has pure point spectrum. If the factor map is not a conjugacy, and if every eigenfunction on $X_{\mathcal{R}}$ is obtained from an eigenfunction on $S_{\mathcal{R}}$,[4] then $X_{\mathcal{R}}$ does not have pure point spectrum.

For substitutions of constant length, the situation is clear-cut:

Theorem 4.7 ([19])**.** *A one dimensional tiling space obtained from a primitive and recognizable substitution of constant length and height one has pure point measurable spectrum if and only if it is coincident.*

There are two reasons why this theorem does not apply to general fusions. First, a coincident fusion may not be uniquely ergodic. For each ergodic measure, the question isn't whether a generic point in $S_{\mathcal{R}}$ corresponds to a single tiling, but whether it corresponds to a single tiling *in a suitably chosen set of full measure*. Second, a coincidence, or even a coincidence with finite waiting, only implies that supertiles agree somewhere. Unless we have some control over the transition matrices, we cannot conclude that high-order supertiles agree on a fraction approaching 1 of their length, which is what is needed to obtain a measurable conjugacy between $X_{\mathcal{R}}$ and $S_{\mathcal{R}}$.

In Example 3.7, the fusion is not coincident, as $P_n(a)$ and $P_n(b)$ disagree at every site. The map from $X_{\mathcal{R}}$ to $S_{\mathcal{R}}$ is nowhere 1:1, being 4:1 over the orbit where there exist two infinite-order supertiles, and 2:1 over all other orbits.

However, for each ergodic measure, $X_{\mathcal{R}}$ *does* have pure point spectrum. For instance, for the ergodic measure that comes from the supertile sequence $\{a, a, \ldots\}$, the measure of the tilings where the origin sits in a supertile $P_n(a)$ is exponentially close to 1. With probability 1, for all sufficiently large n the n-th order supertile containing the origin is of type a. Also with probability 1, the infinite-order supertile containing the origin covers the entire line. The set of tilings with both these properties has full measure, and the factor map to $S_{\mathcal{R}}$ is 1:1 on this set.

Example 4.8. To see how coincidence with finite waiting is insufficient to prove pure point spectrum we make a fusion based on the substitution $\sigma(b) = bc^5b^4$, $\sigma(c) = cb^5c^4$. Repeated substitution produces words $\sigma^n(b)$ and $\sigma^n(c)$; by abusing notation we write $\sigma^n(P_{n-1}(b))$ and $\sigma^n(P_{n-1}(c))$ to mean the fusion of $(n-1)$-supertiles of types b and c in the order given by the letters of $\sigma^n(b)$ and $\sigma^n(c)$. We introduce coincidence with finite waiting by defining the

[4]This is connected to the *height* of a substitution or fusion. If a substitution has height one, then all eigenvalues of $X_{\mathcal{R}}$ are eigenvalues of $S_{\mathcal{R}}$ [47]. One can similarly define a notion of height for fusions.

fusion rule to be

$$P_n(b) = P_{n-1}(b)\sigma^n(P_{n-1}(b))P_{n-1}(c), \qquad P_n(c) = P_{n-1}(b)\sigma^n(P_{n-1}(c))P_{n-1}(c).$$

The transition matrix $M_{n-1,n} = \left(\begin{smallmatrix} 5 & 5 \\ 5 & 5 \end{smallmatrix}\right)^n + \left(\begin{smallmatrix} 1 & 1 \\ 1 & 1 \end{smallmatrix}\right) = (5 \times 10^{n-1} + 1)\left(\begin{smallmatrix} 1 & 1 \\ 1 & 1 \end{smallmatrix}\right)$ has rank 1, so the system is uniquely ergodic. The length of an n-supertile is $\prod_{j=1}^{n}(10^j + 2)$, and $P_n(b)$ and $P_n(c)$ differ on 10^j $(n-1)$-supertiles, implying that they differ on $\prod_{j=1}^{n} 10^j$ tiles. Thus $P_n(b)$ and $P_n(c)$ agree on a positive fraction of their tiles, namely $1 - \prod_{j=1}^{n} \frac{10^j}{10^j+2}$. As $n \to \infty$, this fraction increases but does not approach 1. This implies that one can find disjoint measurable sets of positive measure that map to the same set on the solenoid. Any function that distinguishes between these sets cannot be approximated by a function on the solenoid, so the span of the eigenfunctions is not dense and the spectrum is not pure point. For an example of such a function, let $f(\mathbf{T})$ equal 1 if the origin is in a b tile and 0 if the origin is in a c tile.

Example 4.9. It is also possible for different ergodic measures for the same fusion to have different spectral types. Consider the 1-dimensional non-primitive substitution

$$\sigma(a) = a^{10}, \qquad \sigma(b) = bc^5b^4, \qquad \sigma(c) = cb^5c^4$$

of constant length 10. Next consider a 1-dimensional fusion tilings with three prototiles a, b, c, each of unit length. Using the same abuse of notation as in the previous example we define the fusion rule as

$$\begin{aligned} P_n(a) &= P_{n-1}(a)\sigma^n(P_{n-1}(a))P_{n-1}(b)P_{n-1}(c) \\ P_n(b) &= P_{n-1}(a)\sigma^n(P_{n-1}(b))P_{n-1}(b)P_{n-1}(c), \\ P_n(c) &= P_{n-1}(a)\sigma^n(P_{n-1}(c))P_{n-1}(b)P_{n-1}(c), \end{aligned}$$

This fusion is coincident with waiting time 1, in that all n-supertiles begin with $P_{n-1}(a)$ and end with $P_{n-1}(b)P_{n-1}(c)$. However, that is only 3 out of $L_n = 10^n + 3$ $(n-1)$-supertiles, and the n-supertiles disagree on the rest! For large n, the supertiles $P_n(a)$, $P_n(b)$ and $P_n(c)$ disagree at roughly 70% of their tiles, 97% of their 1-supertiles, 99.7% of their 2-supertiles, etc.

The transition matrices

$$M_{n-1,n} = \begin{pmatrix} 10^n + 1 & 1 & 1 \\ 1 & 5 \cdot 10^{n-1} + 1 & 5 \cdot 10^{n-1} + 1 \\ 1 & 5 \cdot 10^{n-1} + 1 & 5 \cdot 10^{n-1} + 1 \end{pmatrix}$$

have rank 2. There are two ergodic measures, one coming from the supertile sequence $\{a, a, \ldots\}$ and the other coming from an arbitrary sequence of b's and c's.

When we take the ergodic measure from the sequence $\{a, a, \ldots\}$, $X_{\mathcal{R}}$ is measurably conjugate to the solenoid $S_{\mathcal{R}}$ and has pure point spectrum. When we take the other ergodic measure, however, there is a set of full measure where, for all sufficiently large n, the origin is either in $P_n(b)$ or $P_n(c)$, but is not in the two right-most $n-1$ supertiles within $P_n(b$ or $c)$. This set admits a measure-preserving involution where, for all sufficiently large n, the supertiles $P_n(b)$ containing the origin are replaced by $P_n(c)$ and vice-versa. On any set of full measure, the factor map is (almost everywhere) at least 2:1, and the tiling dynamical system does not have pure point spectrum.

In other words, almost every point of the solenoid corresponds to three tilings. One set of preimages has full measure with respect to the $\{a, a, \ldots\}$ ergodic measure, while the other two preimage sets have full measure with respect to the other ergodic measure. Since the first ergodic measure only "sees" one preimage, it has pure point spectrum. Since the other measure "sees" two preimages, it does not have pure point spectrum.

To get pure point spectrum from coincidence, we must control the transition matrices.

Theorem 4.10. *Let $\mathcal{R}$ be a primitive, recognizable, prototile-regular, 1-dimensional fusion of constant length. If the fusion is coincident with finite waiting, and if the transition matrices $M_{n-1,n}$ are uniformly bounded, then $X_{\mathcal{R}}$ is uniquely ergodic and has pure point spectrum.*

Proof. Suppose that there are J species of prototiles, that the fusion is coincident with waiting k, and that $M_{n-1,n}(i,j) \leq C$ for all n, i, j. Then $L_n \leq CJ$. Any two nk-supertiles agree on at least one $(n-1)k$-supertile, at least one $(n-2)k$-supertile in each remaining $(n-1)k$-supertile, at least one $(n-3)k$-supertile in each remaining $(n-2)k$-supertile, etc. This means that any two nk-supertiles agree on at least a fraction $1 - \left(\frac{C^k J^k - 1}{C^k J^k}\right)^n$ of their tiles, a fraction that approaches 1 as $n \to \infty$. In particular, the density of tiles (and likewise, of n-supertiles for any fixed n) is asymptotically the same in all N-supertiles as $N \to \infty$, implying unique ergodicity.

A point in $S_{\mathcal{R}}$ thus determines all but a set of density zero of the tiles in the infinite-order supertile containing the origin. The probability of there being an undetermined tile in any fixed bounded region is thus zero. Since the real line is a countable union of bounded regions, and since the probability of having two infinite-order supertiles in a single tiling is also zero, almost every point in $S_{\mathcal{R}}$ corresponds to a tiling with *no* undetermined tiles. Thus the factor map $X_{\mathcal{R}} \to S_{\mathcal{R}}$ is a measurable conjugacy, and $X_{\mathcal{R}}$ has pure point spectrum. $\square$

Theorem 4.10, while modest in scope, is typical of the theorems that can be proven about fusions that are not of constant length, or that are not 1-dimensional. Given any concidence-based test for pure point spectrum in the category of substitution tilings (e.g., the balanced pair algorithm or the overlap algorithm), one can construct an analogous test for fusions. However, a positive result from such a test will only demonstrate pure point spectrum if one can also show that the coincidences happen frequently enough. This requires estimates both on how long one must wait for a coincidence, and on how much the system has grown in the process.

4.4. **Entropy.** Standard results in symbolic and tiling substitution dynamics say that such systems cannot have positive entropy. The obstruction is that the transitions from level to level do not contain much 'new' information. This continues to be the case for fusion tilings when one assumes that both the number and shapes of supertiles remain fairly well-controlled. This section contains a simple example of a minimal and uniquely ergodic fusion rule with positive entropy and a sufficient condition for a fusion space to have zero entropy.

Configurational entropy is based on counting configurations, and for this we need G to be discrete. We therefore assume that $G = \mathbb{Z}^d$, so we are essentially dealing with subshifts. Let $\#_n$ be the number of configurations that can appear in a d-dimensional cube of side n (this is the complexity function). The configurational entropy is

$$\lim_{n\to\infty} \frac{\log \#_n}{n^d}$$

For subshifts, configurational entropy is known to be the same as topological entropy.

Positive entropy implies that there is a lot of randomness in the system, while unique ergodicity means that all patterns appear with well-defined frequencies. These ideas might seem to be in conflict, but Jewett [35] and Krieger [39] showed that uniquely ergodic dynamical systems can exhibit a very wide range of dynamical properties, and in particular can have positive topological entropy. The following example is in the spirit of their construction.

Example 4.11. *A strictly ergodic fusion rule with positive entropy.* We construct a fusion rule $\mathcal{R}$ with $\mathcal{P}_0 = \{a, b\}$ recursively. Let $\mathcal{P}_1$ be all words of length 3 in which each letter appears at least 1 time but no more than 2 times; we have $j_1 = 6$ distinct 1-supertiles. Now let $\mathcal{P}_2$ be all fusions of $\frac{3j_1^2}{2} = 54$ 1-supertiles in which each supertile appears between j_1 and $2j_1$ times. The expected number of 1-supertiles in any fusion of $\frac{3j_1^2}{2}$ of them is $\frac{3j_1}{2}$, so we are including the highest-probability fusions in our set $\mathcal{P}_2$.

In general, let j_n be the number of n-supertiles and let $\mathcal{P}_{n+1}$ be all fusions of $\frac{3j_n^2}{2}$ n-supertiles in which each n-supertile appears between j_n and $2j_n$ times. Since having more than $2j_n$ or fewer than j_n occurrences in a span of size $\frac{3j_n^2}{2}$ is already highly improbable, restricting to these $(n+1)$-supertiles only reduces the number of configurations slightly, and the system so constructed has positive entropy. The transition matrices are enormous and grow super-exponentially but always have all nonzero entries, making the system strongly primitive and hence minimal. Moreover, the constant δ_n used in equation (3.5) (to measure how balanced the columns of the transition matrices are) is always $\frac{j_n}{2j_n} = 1/2$, so the tiling space is uniquely ergodic.

The fusion rule $\mathcal{R}$ is not recognizable, but we can build a recognizable fusion rule $\mathcal{R}'$ from $\mathcal{R}$ as in Example 2.3. Since the entropy of the factor $X_{\mathcal{R}}$ is bounded by the entropy of $X_{\mathcal{R}'}$, $X_{\mathcal{R}'}$ has positive entropy. It is easy to check that the addition of subscripts does not affect unique ergodicity.

This example involved the number j_n of n-supertile types growing exponentially with the size of the supertiles. If the growth is slower than exponential, and if the shapes of the supertiles are not too distorted, then the system will have zero entropy.

Proposition 4.12. *Let d_n be the diameter of the largest n-supertile, let j_n be the number of n-supertile types and suppose that there exist constants β, K such that each cube of side βd_n touches at most K n-supertiles. If $\lim_{n\to\infty} \frac{\log j_n}{d_n^d} = 0$, then the configurational entropy of $X_{\mathcal{R}}$ is zero.*

Proof. To determine the configuration in a cube of side βd_n, one must specify the kinds of n-supertiles that intersect that cube, and also specify the locations of those supertiles. There are at most j_n^K choices for the first, and at most V^K choices for the second, where V is the volume of the largest n-supertile, which is bounded by d_n^d. Thus the log of the number of configurations, divided by the volume of the cube, is bounded by $\frac{K\log(j_n) + Kd\log d_n}{\beta^d d_n^d}$, which goes to zero as $n \to \infty$. □

The upshot of Proposition 4.12 is that positive entropy either requires the number j_n of n-supertiles to grow exponentially with volume, or for the shapes and relative sizes of supertiles to be so wild, and for the ways that supertiles fit together be so varied, that there are many ways for supertiles to fit together. The geometric issues do not apply to dimension 1, where supertiles simply concatenate, but could in principle apply in dimensions 2 or more. However, we know of no examples where positive entropy is achieved without j_n growing exponentially fast.

4.5. **Strong mixing.** A (measurable) dynamical system is *strongly mixing* if for any pair of measurable sets A, B, and for any sequence of vectors $\vec{v}_n$ tending to infinity, $\lim \mu(A \cap (B - \vec{v}_n)) = \mu(A)\mu(B)$. The dynamical systems of primitive substitution sequences and self-similar tilings are never strongly mixing [20, 59]. Because of the rigidity of the substitution process, knowing the location of one copy of a patch gives a higher probability that it will be seen again at certain intervals. However, there are "staircase" cut-and-stack transformations in one and several dimensions that have been shown to be strongly mixing [1, 2], thus it is possible to have strongly mixing fusion tiling systems. As in the case of entropy, this is only possible when the system has increasing complexity at higher levels of the hierarchy.

In this section we establish sufficient criteria for fusion tilings not to be strongly mixing. These criteria involve both uniform bounds on the number of supertiles and on the transition matrices, and are not necessary criteria. For instance, the accelerated Fibonacci fusion discussed in Example 4.4 does not have bounded matrices, but is essentially the same as ordinary Fibonacci and is not strongly mixing.

Theorem 4.13. *The dynamical system of a strongly primitive van Hove fusion rule with a constant number of supertiles at each level and bounded transition matrices, and with group $G = \mathbb{R}^d$, cannot be strongly mixing.*

Proof. Our proof is an adaptation of Solomyak's [59], which in turn is an adaptation of Dekking and Keane's [20]. By Corollary 3.12, $X_{\mathcal{R}}$ is uniquely ergodic, so for any patch P, $freq_\mu(P)$ can be computed from the actual frequency of P in any increasing sequence of supertiles. We will find a patch P and a sequence of vectors $\vec{v}_n$, tending to infinity, such that the frequency of $P \cup (P + \vec{v}_n)$ is bounded away from zero. Then, supposing that $freq_\mu(P) = \delta$ and $freq_\mu(P \cup (P + \vec{v}_n)) > \epsilon$, we pick a set $U \subset \mathbb{R}^d$ whose volume is less than $\dfrac{\epsilon}{2\delta^2}$, and which is small enough that $\mu(X_{P,U}) = freq_\mu(P)Vol(U)$. Let $A = B = X_{P,U}$. Since $X_{P\cup(P+\vec{v}_n),U} \subset A \cap (B - \vec{v}_n)$, we have

$$\mu(A \cap (B - \vec{v}_n)) \geq \mu(X_{P\cup P+\vec{v}_n,U}) \geq \epsilon Vol(U) > 2\delta^2 Vol(U)^2 = 2\mu(A)\mu(B), \tag{4.3}$$

so $\mu(A \cap B - \vec{v}_n)$ cannot approach $\mu(A)\mu(B)$ as $n \to \infty$.

To find the vectors $\vec{v}$, we suppose the number of supertiles at each level is the constant J. Since $X^n_{\mathcal{R}}$ can be expressed as the union of J cylinder sets defined by which n-supertile is at the origin, it must be that at least one of those cylinder sets has measure at least $1/J$. For each n, choose $l_n \in \{1, 2, ...J\}$ corresponding to an n-supertile with this property, so that $Vol(P_n(l_n))\rho_n(l_n) \geq 1/J$, where ρ_n is the supertile frequency vector. Now choose $\vec{v}_n \in \mathcal{V}^n$ to be a return vector for $P_n(l_n)$ inside $P_{n+2}(l_{n+2})$, as in Theorem 4.1. Because of strong primitivity and the fact that our transition matrices are uniformly bounded, we can find a $\delta' > 0$ for which $\dfrac{Vol(P_n(l_n))}{Vol(P_{n+2}(l_{n+2}))} \geq \delta'$ for all n. (Specifically, if each n-supertile contains at most K $(n-1)$-supertiles, then the ratio of volume between the largest and smallest n-supertile is at most K, and $Vol(P_n(l_n))/Vol(P_{n+2}(l_{n+2})) \geq 1/K^3$.)

The patch P is arbitrary. By choosing n large we can make $\dfrac{\#(P \text{ in } P_n(l_n))}{Vol(P_n(l_n))}$ arbitrarily close to $freq_\mu(P)$, and hence greater than a fixed constant $freq_\mu(P)/2$ for all n. The reader can refer to Figure 3 to see that $\#(P \cup P + \vec{v}_n \text{ in } P_{n+2}(l_{n+2})) \geq \#(P \text{ in } P_n(l_n))$. Since the fraction of volume from the supertiles $P_n(l_n)$ is at least $1/J$, this implies that the frequency of $P \cup (P + \vec{v}_n)$ is at least $\dfrac{freq_\mu(P)Vol(P_n(l_n))}{2JVol(P_{n+2}(l_{n+2}))}$, hence at least $\dfrac{\delta' freq_\mu(P)}{2J}$. □

Proposition 4.14. *The dynamical system of a strongly primitive van Hove fusion rule with a constant number of supertiles at each level and bounded transition matrices, and with group $G = \mathbb{Z}^d$, cannot be strongly mixing.*

Proof. The previous proof does not apply to $\mathbb{Z}^d$ actions because we cannot choose U arbitrarily small. However, we have already shown that for *any* patch P and any sufficiently

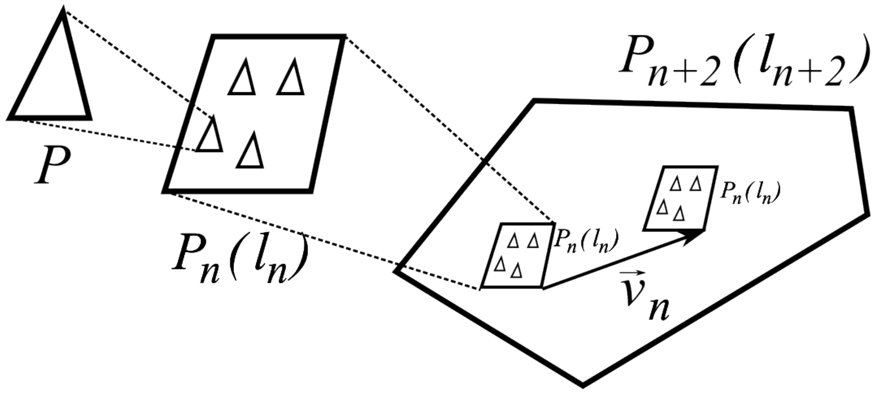


FIGURE 3. Each copy of P in $P_n(l_n)$ makes a copy of $P \cup (P + \vec{v}_n)$ in $P_{n+2}(l_{n+2})$.

large n, there exist large $\vec{v}$ with $freq_\mu(P \cup (P + \vec{v})) \geq \delta' freq_\mu(P)/2J$. We then find a patch P whose frequency is less than $\delta'/4J$, so that $freq_\mu((P \cup (P + \vec{v}_n)) \geq 2 freq_\mu(P)^2$. Taking U to consist of one point, this implies that $\mu(A \cap (A - \vec{v}_n)) \geq 2\mu(A)^2$, where $A = X_{P,U}$. □

5. Inverse limit structures, collaring, and cohomology

In this section we consider topological properties of spaces $X_\mathcal{R}$ of fusion tilings, including their structure as inverse limit spaces, their Čech cohomology groups, and the significance of these groups.

Standing assumptions for Section 5: We assume that $G = \mathbb{R}^d$. Unlike in Section 4, this is more for convenience than from necessity. Modifications for other groups can be done exactly as for substitution tilings [45, 57]. We also assume that our fusion rules are recognizable and, as always, have finite local complexity.

Tiling spaces can always be represented as inverse limits of CW complexes [9, 56]. The challenge is finding a representation that allows for efficient calculation and for the proving of theorems. To this end we present generalizations of the Anderson-Putnam complex [4] and of the partial collaring scheme of Barge, Diamond, and their collaborators [6, 7]. (See also [30] for another method of computing the Čech cohomology of transition-regular 1-dimensional fusion tiling spaces that meet additional assumptions.)

In all cases, we construct a sequence of spaces and maps

$$\Gamma_0 \xleftarrow{f_0} \Gamma_1 \xleftarrow{f_1} \Gamma_2 \xleftarrow{f_2} \Gamma_3 \xleftarrow{f_3} \cdots,$$

where each *approximant* Γ_i describes a region of the tiling, each Γ_{i+1} describes a larger region of the tiling, and $f_i : \Gamma_{i+1} \to \Gamma_i$ is the *forgetful map* that loses the additional information carried in Γ_{i+1}. The *inverse limit* $\varprojlim(\Gamma, f)$ is the set of infinite sequences $(x_0, x_1, \ldots)$ such that each $x_i \in \Gamma_i$ and each $x_i = f_i(x_{i+1})$. Such a sequence is a set of consistent instructions for tiling larger and larger regions of $\mathbb{R}^d$. If the union of these regions is all of $\mathbb{R}^d$ for all sequences in the inverse limit, then there is a natural homeomorphism between $\varprojlim(\Gamma, f)$ and

$X_{\mathcal{R}}$. The key is to make sure that every tiling in $X_{\mathcal{R}}$ can be built up from the approximants in a unique way. A common obstruction is when the approximants can build an infinite tiling that covers only a portion of $\mathbb{R}^d$. "Border-forcing" fusions, discussed next, do not have this obstruction. Later we will describe the technique of "collaring" to make fusion rules become border-forcing.

5.1. **Forcing the border.** A fusion rule always tells us how n-supertiles make up the interiors of larger N-supertiles. But sometimes the N-supertiles also determine which n-supertiles are on their exterior as well. When this happens we say the fusion rule forces the border, and we have a natural way to see the space as an inverse limit.

Definition 5.1. *A fusion rule* forces the border *if for each integer n there exists an N with the following property: If S_1 and S_2 are two N-supertiles of the same type appearing in tilings T_1 and T_2 in $X_{\mathcal{R}}$, then the patch of n-supertiles that touch S_1 in T_1 is equivalent to the patch of n-supertiles touching S_2 in T_2.*

Example 5.2. *Compare and contrast: border forcing.* The 1-dimensional substitution $a \to abb$, $b \to abbb$ forces the border in that every $n+1$-supertile of type a is preceded by an n-supertile of type b and followed by an n-supertile of type a, and likewise every $n+1$-supertile of type b is also preceded by an n-supertile of type b and followed by an n-supertile of type a. By contrast, the substitution $a \to ab$, $b \to aa$ does not force the border, since an N-supertile of type a can be preceded either by an n supertile of type a or b.

5.2. **The Anderson-Putnam complex.** To build Γ_0, we start out with one copy of each prototile from $\mathcal{P}_0$. Then, if somewhere in some tiling two prototiles meet, we identify the corresponding points on their boundaries. The resulting branched manifold is compact [4]. (If we take the periodic tiling of unit squares lined up edge-to-edge, it is easy to see that Γ_0 is the torus.) We build Γ_1 by taking one copy of each supertile from $\mathcal{P}_1$ and identifying the boundaries whenever they meet as above, and continue making each approximant Γ_n similarly. Put another way, Γ_n for the space $X_{\mathcal{R}}$ is Γ_0 for $X^n_{\mathcal{R}}$.

There is a natural map from $X_{\mathcal{R}}$ to Γ_n that maps a tiling to the location of the origin within its n-supertile. Thus, a point in Γ_n can be viewed as a set of instructions for placing an n-supertile containing the origin. Or course, if we know the $(n+1)$-supertile containing the origin, then we necessarily know the n-supertile containing the origin, so the forgetful map f_n is well-defined.

Theorem 5.3. *If the recognizable fusion rule $\mathcal{R}$ forces the border, then $X_{\mathcal{R}}$ is homeomorphic to the inverse limit $\varprojlim(\Gamma_n, f_n)$ of Anderson-Putnam complexes.*

Proof. We will construct a homeomorphism from the inverse limit to $X_{\mathcal{R}}$ by constructing maps from each approximant to partial tilings of $\mathbb{R}^d$. Pick an increasing sequence of integers $n_1, n_2, \ldots$ such that all the n_i-supertiles bounding an n_{i+1}-supertile are determined by the type of the n_{i+1}-supertile. Our map takes a point x_N in Γ_N to the N-supertile with the origin where x_N is, together with all of the lower order supertiles that are determined by that N-supertile. If $N \geq n_i$, then this includes all the n_{i-1}-supertiles touching the N-supertile, all the n_{i-2} supertiles touching the n_{i-1} supertiles, all the n_{i-3} supertiles touching the n_{i-2} supertiles, and so on. If x_N is on the boundary of an N-supertile, then there are multiple tilings that can come from this process, but they all agree on the n_{i-1}-supertiles in all directions around the origin, as well as the lower-order supertiles determined by the n_{i-1}-supertiles. In particular, x_N determines at least $i-1$ layers of supertiles of various sizes around the origin, and so determines at least $i-1$ layers of ordinary tiles around the origin. By choosing N large enough, we can get i to be arbitrarily large. Thus as $N \to \infty$ the points in the approximants determine tilings of larger and larger balls around the origin, so a point in the inverse limit gives a set of consistent directions for tiling increasing regions of $\mathbb{R}^d$ whose union is all of $\mathbb{R}^d$. Such instructions are clearly in 1:1 correspondence with tilings of $\mathbb{R}^d$. Checking that the topology of $X_{\mathcal{R}}$ corresponds to the topology of the inverse limit (as a subset of the infinite product $\prod \Gamma_n$) is a straightforward exercise that is left to the reader. □

Example 5.4. *A short Čech cohomology computation.* Consider a transition-regular fusion rule in one dimension, with two tile types a and b. Let $P_n(a) = P_{n-1}(a)P_{n-1}(b)P_{n-1}(b)$ and let $P_n(b)$ be a permutation of two $P_{n-1}(a)$'s and three $P_{n-1}(b)$'s, with the permutation depending on the level. As long as a permutation beginning in $P_{n-1}(a)$ occurs infinitely often and a permutation ending in $P_{n-1}(b)$ occurs infinitely often, this fusion rule forces the border. The approximant Γ_n consists of two intervals, one representing $P_n(a)$ and one representing $P_n(b)$, with all four endpoints identified to form a figure-8. The map f_n wraps the $P_{n+1}(a)$ circle around the $P_n(a)$ circle once and then around the $P_n(b)$ circle twice. It also wraps the $P_{n+1}(b)$ circle around the $P_n(a)$ circle twice and around the $P_n(b)$ circle three times, in an order determined by the fusion rule at level $n+1$. By Theorem 5.3, $X_{\mathcal{R}}$ is the inverse limit of these figure-8's under these maps.

From a Čech cohomology standpoint we can see the figure-8 as the chain complex of each approximant, so that both $\check{H}_1(\Gamma_n)$ and $\check{H}^1(\Gamma_n)$ are isomorphic to $\mathbb{Z}^2$. The first Čech cohomology of the inverse limit (and of $X_{\mathcal{R}}$) is the direct limit of $\mathbb{Z}^2$ under the pullback of the maps f_n, which always come out to be $\left(\begin{smallmatrix} 1 & 2 \\ 2 & 3 \end{smallmatrix}\right)$ even though the order for the $P_n(b)$'s varies. Since that matrix is invertible over $\mathbb{Z}$, we see that $\check{H}^1(X_{\mathcal{R}}) = \mathbb{Z}^2$.

5.3. **Anderson-Putnam collaring.** Most fusion rules, like most substitution rules, do not force the border. However there is a simple trick, due to Anderson and Putnam in the setting of substitutions, for replacing an arbitrary fusion rule $\mathcal{R}_0$ with a hierarchical rule $\mathcal{R}$ that forces the border, such that $X_{\mathcal{R}_0}$ is homeomorphic (and topologically conjugate) to $X_{\mathcal{R}}$. We can then express our original tiling space $X_{\mathcal{R}_0}$ as the inverse limit of the Anderson-Putnam complexes of $\mathcal{R}$.

A *collared tile to distance* r, or an r-collared tile, is a tile together with a label that describes the types and relative positions of all of that tile's neighbors out to distance r. For instance, in a 1-dimensional tiling with patch $abbaba$, the three b's are all different as collared tiles to distance 1, as one is preceded by an a and followed by a b, one is preceded by a b and followed by an a, and one is both preceded and followed by a's. Likewise, a collared n-supertile to distance r is an n-supertile, together with a label indicating the pattern of nearby n-supertiles out to distance r.

Take an infinite increasing sequence of radii $r_0 < r_1 < \cdots$, tending to infinity. We take $\mathcal{P}_n$ to be the set of collared n-supertiles to distance r_n. Clearly, any complete tiling can be equally well-described in terms of (super)tiles or collared (super)tiles. However, by construction, $\mathcal{R}$ forces the border, since if r_N is greater than r_n plus the diameter of the largest n-supertile, then a collared N-supertile determines its surrounding r_n-collared n-supertiles.

Note that the label of a collared n-supertile contains information about all the neighboring n-supertiles out to distance r_n, and in particular determines all of the neighboring $(n-1)$-supertiles out to distance r_{n-1}. This means that each collared n-supertile is uniquely decomposed as a union of collared $(n-1)$-supertiles, and gives a well-defined map from $X^n_{\mathcal{R}}$ to $X^{n-1}_{\mathcal{R}}$. The hierarchical rule $\mathcal{R}$ is a generalized fusion in the sense of Footnote 1, since the $(n-1)$-supertiles contained in an n-supertile do not determine the n-supertile. The collared n-supertiles have strictly more information than the collared $(n-1)$-supertiles, which is the whole point of collaring!

If the shapes of the supertiles are not too wild, one can pick the r_n's to grow slowly compared to the size of the supertiles, so that collaring to distance r_n just means specifying the nearest neighbors of the n-supertile, as is usually done for substitution tilings. However, for some fusion rules it is possible that knowing the n-supertiles containing the origin and the ones touching this supertile, for all n, will not determine the tiling of all of $\mathbb{R}^d$.

The process of collaring does have its drawbacks, as $\mathcal{R}$ may not have the same transition-regularity or even prototile-regularity properties as $\mathcal{R}_0$. Collaring increases the number of tile types, and there is no reason to expect the increase to be the same at all levels. Indeed, even if the number of uncollared supertiles is uniformly bounded it is entirely possible that the number of collared n-supertiles will grow without bound as $n \to \infty$. This happens, for instance, in Example 6.4.

5.4. **Barge-Diamond collaring.** The idea behind Barge-Diamond collaring [6, 7] is to collar points rather than tiles. As before, pick an increasing sequence of radii $r_0 < r_1 < \cdots$ tending to infinity. Take a tiling $\mathbf{T}$, and identify points $\vec{x}$ and $\vec{y}$ if $[B_{r_0}]^{\mathbf{T}-\vec{x}} = [B_{r_0}]^{\mathbf{T}-\vec{y}}$. That is, if the tiling looks the same around $\vec{x}$ and $\vec{y}$ to distance r_0 (with $\vec{x}$ and $\vec{y}$ playing corresponding roles). Let Γ_0 be the quotient space. To get Γ_1, identify points for which the corresponding tiling in $X^1_{\mathcal{R}}$ agrees to distance r_1. That is, points $\vec{x}$ and $\vec{y}$ for which all of the 1-supertiles that exist within a distance r_1 of $\vec{x}$ and $\vec{y}$ agree. Likewise, Γ_n is $\mathbb{R}^d$ modulo identification of points $\vec{x}$ and $\vec{y}$ for which all of the n-supertiles within distance r_n of $\vec{x}$ and $\vec{y}$ agree.

As before, we have a map from $X_{\mathcal{R}}$ to Γ_n, taking a tiling to a description of how a ball of radius r_n around the origin sits in one or more n-supertiles. Since $r_n \to \infty$ as $n \to \infty$, a point in the inverse limit is a consistent set of instructions for tiling all of $\mathbb{R}^d$, so the inverse limit is homeomorphic to $X_{\mathcal{R}}$ as long as the orbit closure of $\mathbf{T}$ is $X_{\mathcal{R}}$ (which is always the case when $X_{\mathcal{R}}$ is minimal).

When the fusion is asymptotically self-similar or self-affine, one can take r_n to be much smaller than the size of an n-supertile, but still to go to infinity. For 2-dimensional tilings, this means that there are three kinds of points. Most points are farther than r_n from the nearest n-supertile boundary. These points are identified with corresponding points of other n-supertiles, without regard for the supertile's neighbors. Some points are within r_n of one of the supertile's edges. These points are identified with corresponding points of other n-supertiles that have the same n-supertile neighbor across the specific edge. Finally, some points are within r_n of two or more edges, and hence are close to a vertex. There is a stratification of Γ into points-near-vertices, points-near-edges, and interior points, and this stratification makes for much easier computations of tiling cohomology than Anderson-Putnam collaring.

6. Direct product variations

An easy way to make higher-dimensional substitution sequences is to take the direct product of two or more one-dimensional substitutions. To break the direct product structure, one can rearrange the substitution carefully so that at each stage the blocks still fit, creating what is called a *direct product variation* or *DPV*. Introduced as examples of combinatorial substitutions in [25], DPVs are quite flexible when viewed as examples of fusion rules.

Example 6.1. *The Fibonacci DPV.* This simple example of a prototile- and transition-regular fusion rule in two dimensions is based on the Fibonacci substitution $0 \to 01, 1 \to 0$. We use it to illustrate almost all of the ideas and computations discussed for fusion rules.

The prototile set consists of four unit-square tiles with label set $\{a, b, c, d\}$ and so $\mathcal{P}_0 = \left\{\boxed{a}, \boxed{b}, \boxed{c}, \boxed{d}\right\}$.[5]

[5]There is some flexibility with the geometry of the prototiles. They could be parallelograms or rectangles, and there are two vertical and two horizontal degrees of freedom for the lengths of the sides.

For the 1-supertiles we choose $\mathcal{P}_1 = \left\{ \begin{array}{|c|c|}\hline c & a \\ \hline d & b \\ \hline \end{array}, \begin{array}{|c|c|}\hline a & c \\ \hline \end{array}, \begin{array}{|c|}\hline b \\ \hline a \\ \hline \end{array}, \begin{array}{|c|}\hline a \\ \hline \end{array} \right\}$, where we list the supertiles in the obvious order $\{P_1(a), P_1(b), P_1(c), P_1(d)\}$. To make the 2-supertiles we concatenate the 1-supertiles in combinatorially the same way:

$$\mathcal{P}_2 = \left\{ \begin{array}{|c|c|c|}\hline b & c & a \\ \hline a & d & b \\ \hline a & a & c \\ \hline \end{array}, \begin{array}{|c|c|c|}\hline c & a & b \\ \hline d & b & a \\ \hline \end{array}, \begin{array}{|c|c|}\hline a & c \\ \hline c & a \\ \hline d & b \\ \hline \end{array}, \begin{array}{|c|c|}\hline c & a \\ \hline d & b \\ \hline \end{array} \right\} = \left\{ \begin{array}{|c|c|}\hline P_1(c) & P_1(a) \\ \hline P_1(d) & P_1(b) \\ \hline \end{array}, \begin{array}{|c|c|}\hline P_1(a) & P_1(c) \\ \hline \end{array}, \begin{array}{|c|}\hline P_1(b) \\ \hline P_1(a) \\ \hline \end{array}, \begin{array}{|c|}\hline P_1(a) \\ \hline \end{array} \right\}$$

In general we construct $\mathcal{P}_{n+1}$ from $\mathcal{P}_n$ with exactly the same combinatorics as the rightmost version of the 2-supertiles shown above. It is not difficult to show that the sides of $P_n(a)$ and the long sides of $P_n(b)$ and $P_n(c)$ are the Fibonacci numbers f_{n+2}, while the sides of the n-supertile of type d and the short sides of the b and c supertiles are the Fibonacci numbers f_{n+1} (using the convention that $f_0 = 0$ and $f_1 = 1$). This means that at each stage, the supertiles fit together to form squares and rectangles with Fibonacci side lengths.

Recognizability is straightforward and proceeds by induction. The $P_{n+1}(a)$ supertiles are determined by the presence of a $P_n(d)$, each $P_{n+1}(b)$ is determined by a $P_n(c)$ that is not in a $P_{n+1}(a)$, each $P_{n+1}(c)$ is determined by a $P_n(b)$ that is not in a $P_{n+1}(a)$, and each remaining $P_n(a)$ is a $P_{n+1}(d)$.

The Fibonacci DPV is transition-regular with $M = M_{n-1,n} = \begin{pmatrix} 1&1&1&1 \\ 1&0&1&0 \\ 1&1&0&0 \\ 1&0&0&0 \end{pmatrix}$. This is a primitive matrix so $M_{n,N} = M^{N-n}$ is asymptotically rank 1; the dynamical system is uniquely ergodic. The Perron-Frobenius eigenvalue is ϕ^2, where ϕ is the golden mean; this number represents the asymptotic volume expansion of the supertiles from level to level.

To compute the ergodic measure, it suffices to compute the frequencies of the n-supertiles and then use equation (3.4) of Theorem 3.4 to get the frequencies of arbitrary patches. The vectors ρ_n are the volume-normalized directions of the asymptotic columns of $M_{n,N} = M^{N-n}$, and thus they are collinear with the right Perron-Frobenius eigenvector of M, which is $\begin{pmatrix} \phi^2 \\ \phi \\ \phi \\ 1 \end{pmatrix}$. Since we have chosen unit square prototiles, the volumes of the n-supertiles are $f_{n+2}^2, (f_{n+2}f_{n+1}), (f_{n+2}f_{n+1})$, and f_{n+1}^2 respectively. We compute $\rho_n = \phi^{-(2n+4)} \begin{pmatrix} \phi^2 \\ \phi \\ \phi \\ 1 \end{pmatrix}$.

Next we turn to computing the topological and measure-theoretic spectrum. Technically we should take the induced fusion rule that composes two levels at once to get strong primitivity, then go up two more levels at a time to find all of the return vectors in $\mathcal{V}^n$. Fortunately, in this example there are always return vectors of the form $(f_n, 0)$ and $(0, f_n)$ (see Figure 4). Any eigenvalue $\vec{\alpha} = (\alpha_1, \alpha_2)$, topological or measure-theoretic, must have the property that

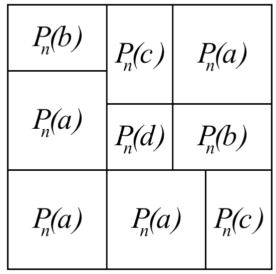


FIGURE 4. The induced fusion for $P_{n+2}(a)$.

$\lim_{n\to\infty} \vec{\alpha}\cdot\vec{v}_n = 0 \pmod 1$, and this means that $\lim_{n\to\infty} \alpha_k f_n = 0 \pmod 1$ for $k = 1, 2$. Pisot's theorem then implies that $\vec{\alpha} \in \mathbb{Z}[\phi] \times \mathbb{Z}[\phi]$[6]. For such $\vec{\alpha}$, the criterion for topological eigenvalues in Theorem 4.1 is satisfied because the convergence of $\eta_n(\vec{\alpha})$ to 0 is exponential. In short, the computations for both topological and measure-theoretic eigenvalues are exactly the same as for the product of two Fibonacci tilings (built from unit length prototiles), and we have that both eigenvalue sets are $\mathbb{Z}[\phi] \times \mathbb{Z}[\phi]$. In particular, any measurable eigenfunction can be chosen to be continuous.

Next we compute the Čech cohomology of $X_{\mathcal{R}}$. We use Barge-Diamond collaring [6, 7], picking the collaring radius r_n to grow slowly with n. We stratify our tiling space into three pieces $\Xi_0 \subset \Xi_1 \subset \Xi_2 = X_{\mathcal{R}}$. Ξ_2 is the entire tiling space, Ξ_1 is the set of tilings where the origin is within r_n of the boundary of an n-supertile for every n, and Ξ_0 is the set of tilings where the origin is within a distance r_n of two supertile edges, and hence is near a supertile corner. The cohomology of Ξ_0 is the cohomology of a CW complex with one cell for each possible pattern by which three or more high-order supertiles can meet at a vertex. There are 78 such patterns. The relative cohomology of the pair (Ξ_1, Ξ_0) is computed from a CW complex containing 52 cells that describe the ways that two supertiles can meet along a common edge. The relative cohomology of the pair (Ξ_2, Ξ_1) is computed from the matrix M. The techniques for generating these cells and computing the cohomology are similar to those for substitution tilings, and yield

$$\begin{array}{rcc}
\check{H}^0(\Xi_0) = \mathbb{Z}; & \check{H}^1(\Xi_0) = 0; & \check{H}^2(\Xi_0) = \mathbb{Z}^{42} \\
\check{H}^0(\Xi_1, \Xi_0) = 0; & \check{H}^1(\Xi_1, \Xi_0) = \mathbb{Z}^4; & \check{H}^2(\Xi_1, \Xi_0) = \mathbb{Z}^{18} \\
\check{H}^0(\Xi_1) = \mathbb{Z}; & \check{H}^1(\Xi_1) = \mathbb{Z}^4; & \check{H}^2(\Xi_1) = \mathbb{Z}^{60} \\
\check{H}^0(\Xi_2, \Xi_1) = 0; & \check{H}^1(\Xi_2, \Xi_1) = 0; & \check{H}^2(\Xi_2, \Xi_1) = \mathbb{Z}^4 \\
\check{H}^0(X_{\mathcal{R}}) = \mathbb{Z}; & \check{H}^1(X_{\mathcal{R}}) = \mathbb{Z}^4; & \check{H}^2(X_{\mathcal{R}}) = \mathbb{Z}^{64}.
\end{array}$$

The generators of $\check{H}^1(X_{\mathcal{R}}) = \mathbb{Z}^4$ are easily described. Pick a value of $n \geq 4$. Each edge of an n-supertile either has length f_{n+1} or f_{n+2}. The first generator counts the horizontal edges of the first type, the second generator counts the horizontal edges of the second type,

[6]The absence of the $\sqrt{5}$ that is present in Example 4.4 is due to the integer size of the prototiles.

and the third and fourth generators similarly count vertical edges. The boundaries of two supertiles may overlap on intervals of size $f_n = f_{n+2} - f_{n+1}$ or $f_{n-1} = 2f_{n+1} - f_{n+2}$. The first (or third) generator assigns the numbers -1 and 2 to these partial edges, while the second (or fourth) assigns the numbers 1 and -1, as these are the coefficients of f_{n+1} and f_{n+2}. Picking different values of n gives different generators, but the group they generate is the same.

Deformations of a tiling, by changing the shape and size of (possibly collared) tiles, are parametrized up to mutual local derivability by $\check{H}^1(X_{\mathcal{R}}, \mathbb{R}^d)$ [16]. For the Fibonacci DPV, $\check{H}^1$ is the same as for the product of two 1-dimensional Fibonacci tiling spaces, and the deformations are the same. Thus, *any* deformation of the sizes and shapes yields a tiling space that is topologically conjugate to a linear transformation of $\mathbb{R}^2$ applied to the original tiling space. In particular, a self-similar version of the DPV, in which the a, b, c, and d tiles have dimensions $\phi \times \phi$, $\phi \times 1$, $1 \times \phi$ and 1×1, is topologically conjugate to a DPV where all tiles are congruent squares (of side $\sqrt{5}/\phi$).

The difference between the Fibonacci DPV and the product of two 1-dimensional Fibonacci tilings is seen in the second Čech cohomology, where that of the DPV has rank 64 and that of the product has rank 4. The rank of the top cohomology is closely related to the independent appearance of patterns in the tiling, via the following theorem:

Theorem 6.2 ([58]). *If the rank of $\check{H}^d$ of a d-dimensional tiling space is k, then there exist k patterns $P_1, \ldots, P_k$, such that for any patch P there exist rational numbers $c_1, \ldots, c_k$ and c_P such that, for any region R in any tiling T,*

$$\#(P \text{ in } R) = \sum_{i=1}^{k} c_i \#(P_i \text{ in } R) + e(P, R),$$

where the error term $e(P, R)$ is computable from the patterns on the boundary of R, and is bounded by c_P times the $(d-1)$-volume of the boundary of R.

We call $P_1, \ldots, P_k$ *control patches.* For the product of two 1-dimensional Fibonacci tilings, we can take our control patches to be the four basic tiles. For the Fibonacci DPV, however, there are 60 additional control patches. They can be chosen from the generators of $\check{H}^2(\Xi_0)$ and $\check{H}^2(\Xi_1, \Xi_0)$. That is, we have 9 control patches that involve supertiles meeting along horizontal edges, 9 that involve supertiles meeting along vertical edges, and 42 that involve three or four supertiles meeting at a vertex.

Example 6.3. *A scrambled Fibonacci DPV.* We can construct a scrambled version of the Fibonacci DPV in much the same way as the 1-dimensional scrambled Fibonacci tiling of Example 4.4. We pick an increasing sequence $N(n)$ and induce on this sequence to get an accelerated scrambled Fibonacci rule $\mathcal{A}$. We then introduce an exceptional supertile $\mathcal{E}_n(e)$ at each odd level, whose population in terms of $(n-1)$-supertiles is the same as $\mathcal{A}_n(d)$, but rearranged so that all of the $\mathcal{E}_{n-1}(a)$ tiles appear in the lower left corner, all the $\mathcal{E}_{n-1}(b)$ appear in the lower right, all the $\mathcal{E}_{n-1}(c)$ appear in the upper left, and all the $\mathcal{E}_{n-1}(d)$ appear in the upper right. On even levels, the n-supertiles are built from the $(n-1)$-supertiles exactly as for the accelerated DPV, only with one $\mathcal{E}_{n-1}(d)$ in each n-supertile replaced by an $\mathcal{E}_{n-1}(e)$. Finally, we induce on even levels to obtain a prototile-regular fusion $\mathcal{S}$.

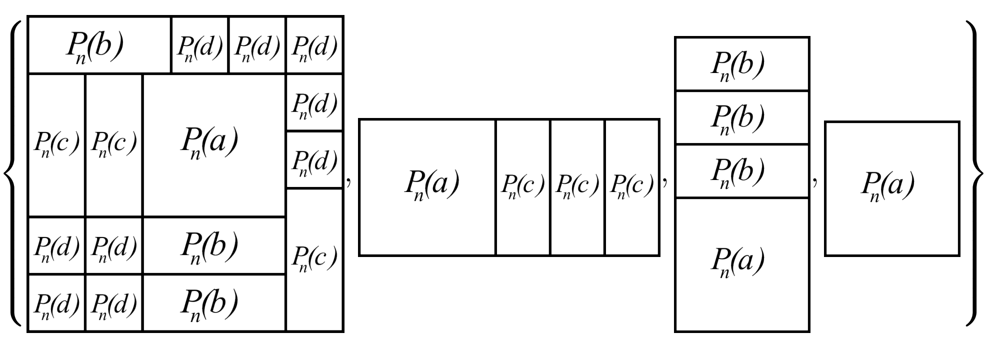
As before, if we choose the sequence $N(n)$ to grow sufficiently fast, and if we give the prototiles the same shape as the asymptotic supertiles, with the a, b, c, d prototiles having dimensions $\phi \times \phi$, $\phi \times 1$, $1 \times \phi$ and 1×1, then the scrambled Fibonacci DPV space is topologically weakly mixing but has pure point measurable spectrum, being measurably conjugate to the unscrambled Fibonacci DPV. However, if we choose the prototiles to be unit squares, then every $\alpha \in \mathbb{Z} \times \mathbb{Z}$ is manifestly a topological eigenvalue.

This discrepancy means that the deformation theory for the scrambled DPV is *not* the same as for the unscrambled DPV. Either the first cohomologies are different, or, more likely, the cohomologies are isomorphic but the two tilings have different "asymptotically negligible" [16] subspaces of $\check{H}^1(X, \mathbb{R}^2)$ that describe deformations that are topological conjugacies but that are not mutually locally derivable from the original. As for $\check{H}^2$, the rank must be at least 64, since the control patches for the Fibonacci DPV are still present in the scrambled DPV.

This example suggests two directions for future work. One is to understand deformation theory better, and in particular the role of the asymptotically negligible classes. These are well-understood for substitution tilings, but not for fusions. Another is to develop new techniques for computing tiling cohomology for spaces that do not come from substitutions. The Anderson-Putnam and Barge-Diamond complexes were defined in Section 5 for all tilings, but almost every existing method for studying these complexes relies on an underlying substitution.

Example 6.4. *A non-Pisot DPV.* We base this DPV on the one-dimensional substitution $a \to abbb, b \to a$, which despite its apparent similarity to the Fibonacci DPV exhibits significantly different dynamical behavior. The prototile set is the same as for the Fibonacci DPV, and again we choose the fusion to be both prototile- and transition-regular. This time we choose our fusion rule at each stage to be given by

$\mathcal{P}_{n+1} =$



Recognizability is easily established, almost exactly as with the Fibonacci DPV.

The transition matrix has $((1+\sqrt{13})/2)^2$ as its largest eigenvalue, which is not a Pisot number. The side lengths of the supertiles grow as solutions to the recursion $l_{n+1} = l_n + 3s_n, s_{n+1} = l_n$, which are nontrivial linear combinations of $((1 \pm \sqrt{13})/2)^n$. Since both of those numbers are greater than one in modulus, the side lengths are not well-approximated by powers of the positive eigenvalue. The effect this has on the system is profound. It means that the combinatorics of the fusion tilings are exceptionally complicated, in that the number of ways that n-supertiles can be adjacent to one another grows without bound as $n \to \infty$.

This increasing complexity with scale shows up in the topology of $X_{\mathcal{R}}$. Both $\check{H}^1$ and $\check{H}^2$ are infinitely generated, the first indicating that there are infinitely many "interesting" deformations of size and shape, and the second indicating that there are infinitely many control patterns. Unlike the Fibonacci DPV, changes in tile sizes, while preserving the fusion rule, can change the dynamics and in fact the topology of the tiling space. If we were to choose irrationally related side lengths for our prototiles, then the resulting tiling would not have finite local complexity [26].

Examples 6.1 and 6.4 lead us to a discussion of the combinatorial and geometric behavior of supertiles as $n \to \infty$. In some cases one or the other will approach a limit as $n \to \infty$. Consider a prototile-regular fusion rule, and suppose that there is some invertible linear map $L : \mathbb{R}^d \to \mathbb{R}^d$ such that $\lim_{n\to\infty} L^{-n} P_n(p)$ exists for each prototile type. If in addition the combinatorics of how the $(n-1)$-supertiles lie inside their n-supertiles stabilizes for large values of n, then we call the fusion rule *asymptotically self-affine* (or -similar if L is a similarity). This means that there is a self-affine tiling that is related to the fusion tiling. The precise nature of the relationship varies, and no general theorems about it are known to the authors at this time. Both of the previous examples are asymptotically self-similar, with the limiting prototile sets having edge lengths in the ratios $\phi : 1$ in the Fibonacci case and $(1+\sqrt{13}) : 2$ in the non-Pisot case.

A fusion tiling may have finite local complexity in the usual sense while failing to be locally finite in an asymptotic sense. We call a fusion rule *asymptotically FLC* if there is a constant B such that each pair of n-supertiles can form at most B connected two-supertile patches. Example 6.1 is asymptotically FLC, but Example 6.4 is not. If an asymptotically self-affine tiling is not asymptotically FLC, then the self-affine tiling obtained from the limiting shapes will have infinite local complexity.

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