TOWARDS A CHARACTERIZATION OF SELF-SIMILAR TILINGS IN TERMS OF DERIVED VORONOÏ TESSELLATIONS

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Abstract. In this paper, a technique for analyzing levels of hierarchy in a tiling $\mathcal{T}$ of Euclidean space is presented. Fixing a central configuration P of tiles in $\mathcal{T}$, a "derived Voronoï" tessellation $\mathcal{T}_P$ is constructed based on the locations of copies of P in $\mathcal{T}$. A family of derived Voronoï tilings $\mathcal{F}(\mathcal{T})$ is formed by allowing the central configurations to vary through an infinite number of possibilities. The family $\mathcal{F}(\mathcal{T})$ will normally be an infinite one, but we show that for a *self-similar* tiling $\mathcal{T}$ it is finite up to similarity. In addition, we show that if the family $\mathcal{F}(\mathcal{T})$ is finite up to similarity, then $\mathcal{T}$ is *pseudo-self-similar*. The relationship between self-similarity and pseudo-self-similarity is not well understood, and this is the obstruction to a complete characterization of self-similarity via our method. A discussion and conjecture on the connection between the two forms of hierarchy for tilings is provided.

1. Introduction

Our study of tilings takes a dynamical viewpoint, where tilings can be seen as higher-dimensional analogues of points in symbolic dynamical systems. Any such generalization is complicated due to the presence of nontrivial geometry in two or more dimensions. Important work has been done on tilings and their associated "tiling dynamical systems" by a variety of authors, including C. Radin [10, 11], E. A. Robinson Jr. [14], and B. Solomyak [19, 20].

In this work we study tilings with hierarchical properties. These tilings are intended to be generalizations of limit points of substitutions on sequences (many results on such sequences are presented in M. Queffelec's book [9]). An exposition on hierarchy in tilings was given by C. Radin in [12]. A generalization of fixed points of constant-length substitutions is "self-similar" tilings, defined by W. Thurston [21]. We will also present a definition of "pseudo-self-similar" tilings, a slightly less restrictive condition which may ultimately be seen to be equivalent. There are many examples of and dynamical results about self-similar tilings and pseudo-self-similar tilings in the literature, some of which will be mentioned in Section 3.1.

The main results to be presented here were inspired by the work of F. Durand [2] in the context of substitutive sequences. He defined "derived sequences" and used them to characterize the limit points of primitive substitutions on sequences. Given a minimal sequence X and a finite block u from X, the sequence $D_u(X)$ is derived based on a recoding of X in terms of the occurrences of u in X. Durand's result is as follows.

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Theorem (Durand)**.** *X is a primitive substitutive sequence if and only if the number of derived sequences $D_u(X)$ is finite, as u ranges throughout all possible finite initial words of X.*

In order to generalize this result to the tilings case we define "derived Voronoï (DV) tilings", the analogue of Durand's derived sequences. Given a tiling $\mathcal{T}$ and a patch $P \subset \mathcal{T}$, we describe how to derive a new tiling $\mathcal{T}_P$ using a Voronoï construction on the locations of copies of the patch P in $\mathcal{T}$. The resulting DV tiling contains information about the layout of translates of P in $\mathcal{T}$ and provides a notion of which translates of P are "first returns" of one another.

We can use DV tilings to study hierarchy of various kinds. In addition to self-similar tilings, combinatorially substitutive tilings as well as pseudo-self-similar tilings were studied by the author in terms of DV tilings in [8]. The results concerning self-similar and pseudo-self-similar tilings are presented in this work. In order to use DV tilings to study hierarchy, we construct the DV tiling $\mathcal{T}_{P_r} = \mathcal{T}_r$ of a central patch P_r determined by the ball of radius r centered at 0. Letting r vary, we obtain an (infinite) family $\mathcal{F}(\mathcal{T})$ of DV tilings. One way to classify elements of $\mathcal{F}(\mathcal{T})$ is via similarity of $\mathbb{R}^d$; we give a precise definition later. Our two main results can be summarized as follows:

Theorem. *If $\mathcal{T}$ is a nonperiodic self-similar tiling of $\mathbb{R}^d$, then the DV family $\mathcal{F}(\mathcal{T})$ is finite up to similarity.*

Theorem. *If for a nonperiodic tiling $\mathcal{T}$ of $\mathbb{R}^d$ the DV family $\mathcal{F}(\mathcal{T})$ is finite up to similarity, then it is pseudo-self-similar.*

Taken together, these results nearly classify the set of all self-similar tilings. Attempts to make a complete classification are under way—possible ways to form a characterization are listed in Section 5. There may be a way to use these results to prove that any pseudo-self-similar tiling is a self-similar tiling with "deformed" tiles (the tilings are "mutually locally derivable"). Currently, this is just a conjecture.

The second result is quite important as it provides a system for checking an arbitrary tiling for hierarchy. Upon examination of the proof, we find that the DV tiling is a useful tool for detecting hierarchy on a very practical level. Given a tiling $\mathcal{T}$, we need not necessarily construct a DV tiling for every initial configuration and then classify them all in order to deduce that there is hierarchical structure in $\mathcal{T}$. The proof shows that it is enough to find two similar DV tilings constructed from initial patches of sufficiently different sizes to deduce that there is hierarchy in the tiling. How different the sizes of the central patches need to be will depend on estimates on the sizes of the return tiles in each DV tiling. This could be a very usable method for determining hierarchy in specific examples.

The results presented here are based on the author's Ph.D. dissertation [8] written at the University of North Carolina at Chapel Hill under the direction of Karl Petersen.

2. Tilings and Tiling Dynamical Systems

2.1. **Prototiles and tilings.** Prototiles are the basic tile shapes which can be used to form a tiling. A tiling will be constructed from some finite set of prototiles by covering $\mathbb{R}^d$ with translates of the prototiles, allowing them to overlap only on their boundaries.

Definition 2.1. Given a set $A \subset \mathbb{R}^d$ homeomorphic to the closed unit disk $\{x \in \mathbb{R}^d : \|x\| \leq 1\}$ and an integer $l \in \{1, 2, ..., L\}$, define a *prototile* t as the pair (A, l).

The *support* of t is $\operatorname{supp}(t) = A$; the *label* (or *tile type*) of t is $l(t) = l$. A *prototile set* is a set τ of prototiles so that if $t \in \tau$, then $l(t) \in \{1, 2, ..., L\}$, and if $l(t_1) = l(t_2)$, then $\operatorname{supp}(t_1) = \operatorname{supp}(t_2)$.

The labels in this definition make available a high level of control to distinguish (or not distinguish) the various tile shapes as we choose. We could use the labelling to distinguish prototiles having congruent supports which are not translates of one another, but are instead rotations or reflections of one another. Or we could use the labelling to color tiles which have translationally congruent supports so that they appear distinct. If the requirement that labels uniquely determine a prototile's shape and orientation was dropped, then all of the prototiles in a prototile set could be given the same label. From a combinatorial standpoint, the study of such a tiling would lose all consideration for the geometry of the tiles, since any homeomorphism of $\mathbb{R}^d$ produces a tiling which carries the same combinatorial structure. We avoid this situation, preferring to allow labellings on graph elements to preserve their geometric connection to the tiling.

In order to tile the plane with copies of the prototiles, a subgroup G of $\mathbb{R}^d$ is selected which must at least contain $\mathbb{Z}^d$. These are the *allowable translations* of the prototiles. A tiling is created out of copies of the prototile set under allowable translations.

Definition 2.2. Given a prototile set τ and a group of allowable translations G, a *tile* T is a pair $(\operatorname{supp}(t) - g, l(t))$ for some $g \in G$ and $t \in \tau$ (making $\operatorname{supp}(T) = \operatorname{supp}(t) - g$ and $l(T) = l(t)$). We say

$$\mathcal{T} = \{T_j = (\operatorname{supp}(t_{i_j} - g_j), l(t_{i_j})) \text{ for } j \in \mathbb{N}, t_{i_j} \in \tau, \text{ and } g_j \in G\}, \tag{1}$$

is a (τ, G)-*tiling* if $\mathbb{R}^d = \bigcup_j \operatorname{supp}(T_j)$ and $\operatorname{int}(T_i) \cap \operatorname{int}(T_j) = \emptyset$ for $i \neq j$.

When the prototile set and allowable group of translations are unambiguous we will refer to $\mathcal{T}$ simply as a tiling. For convenience of notation we will suppress subscripts and refer to any $\mathcal{T}$-tile as $T \in \mathcal{T}$.

Example 1. Suppose we let $t_1 = ([0, 1] \times [0, 1], 1)$ and $t_2 = ([0, 1] \times [0, 1], 2)$ be two prototiles forming the prototile set τ. We can let the subgroup G of allowable translations be either $\mathbb{Z}^2$ or $\mathbb{R}^2$; in either case an example of a (τ, G)-tiling is shown in Figure 1. Since the tiles have congruent supports, the labelling plays an important role in the appearance of the tiling. One can see this tiling as a configuration in $\mathbb{Z}^2$; sometimes it is more useful to consider it as part of a dynamical system whose action in $\mathbb{R}^d$ instead.

Remark 2.1. Note that a tiling can be seen as a higher-dimensional analogue of a sequence on a finite number of letters (prototile types). The introduction of nontrivial geometry can cause complex patterns of adjacency between tiles in a tiling. Unlike the situation for sequence spaces, arbitrary concatenation of the tiles may not result in a tiling at all. Whether or not a given prototile set can actually form a tiling is an undecidable question [15, 22].

Remark 2.2. At this point we should also note that a more general framework is used by many authors, (C. Radin has contributed to the theory of tiling dynamical systems in this framework), in which rotations of prototiles can be used along with translations to tile $\mathbb{R}^d$. Although the format used here can allow rotations of prototiles, it restricts us to a finite group of rotations. However, most of the

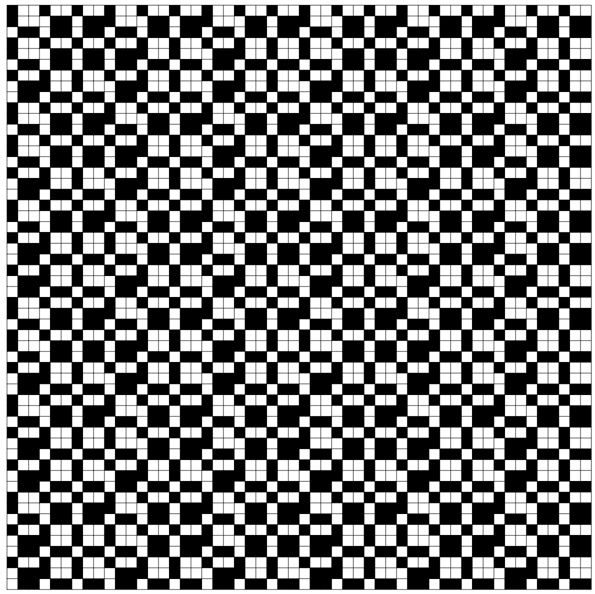

FIGURE 1. A tiling of $\mathbb{R}^2$ with colored unit square tiles.

results in this work could be generalized to the case in which an infinite number of rotations of prototiles is allowed. This would require alterations of several key estimates used in the proofs.

Definition 2.3. A $\mathcal{T}$-*patch* P is given by

$$P = \{T_{i_1}, ..., T_{i_n}\} \subset \mathcal{T}.$$

Patches in tilings are analogous to words in sequences. To a subset U of $\mathbb{R}^d$ we can associate two $\mathcal{T}$-patches in a natural fashion: one has support contained in U and the other has support containing U. Using the notation of [19], call the *inner patch of U*

$$]U[^{\mathcal{T}} = \{T \in \mathcal{T} \text{ such that } \operatorname{supp}(T) \subseteq U\} \tag{2}$$

and the *outer patch of U*

$$[U]^{\mathcal{T}} = \{T \in \mathcal{T} \text{ such that } \operatorname{supp}(T) \cap U \neq \emptyset\}. \tag{3}$$

An important outer patch is the one associated to a point $y \in \mathbb{R}^d$ given by the *elementary patch* $[y]^{\mathcal{T}}$. The elementary patches can be used to form a sort of atlas of local tile configurations, since they show every possible way (up to translation) any point in $\mathbb{R}^d$ is covered by the tiles in a tiling. Elementary patches can be (1) a single tile, (2) two tiles meeting along an edge, or (3) several tiles which share a common vertex and whose collective support contains a neighborhood of that vertex.

Tiles and patches can be acted upon by translation. This action will be of primary importance since it is the action that will produce the dynamical system that we intend to study.

Definition 2.4. Given a patch P, and a translation $g \in G$ to which it corresponds, define the patch

$$P - g = \{(\mathrm{supp}(T) - g, l(T)) : T \in P\}.$$

The $\mathcal{T}$-patches P_1 and P_2 are said to be *equivalent* and we write

$$P_1 \sim P_2 \text{ if and only if there exists } g \in G \text{ with } P_1 = P_2 - g. \tag{4}$$

Definition 2.5. A tiling $\mathcal{T}$ is *normal* if all of its tiles are uniformly bounded topological disks which intersect in connected sets. A tiling $\mathcal{T}$ has a *finite number of local patterns* if for any $R > 0$ there is an integer n and $\mathcal{T}$-patches $P_1, ..., P_n$ such that for any $x \in \mathbb{R}^d$, $[B_R(x)]^{\mathcal{T}}$ is equivalent to P_i for some $i \in \{1, ..., n\}$.

Tilings in this work are always assumed to satisfy Definition 2.5.

Definition 2.6. A tiling $\mathcal{T}$ of $\mathbb{R}^d$ is said to be *periodic* if there exists a basis $g_1, g_2, ..., g_d$ of $\mathbb{R}^d$ so that $\mathcal{T} - g_i = \mathcal{T}$ for $i = 1, 2, ..., d$. It will be considered *nonperiodic* if there is no nonzero $g \in \mathbb{R}^d$ with $\mathcal{T} - g = \mathcal{T}$. A set τ of prototiles is called *aperiodic* if it can tile the plane, but only nonperiodically.

For quite some time the existence of aperiodic prototile sets was an open question, and was the key to the solution of H. Wang's Tiling Problem [22]. Once it was shown that there are aperiodic sets of prototiles, Wang was able to show that the question of whether an arbitrary set of prototiles can form a tiling is not decidable.

If a tiling is not periodic it can still have strong repetitive properties. The following is analogous to minimality for sequences.

Definition 2.7. A tiling $\mathcal{T}$ is called *almost periodic* if for any patch $P \in \mathcal{T}$ there is a real number R such that for any $x \in \mathbb{R}^d$ there is a $\mathcal{T}$-patch P' such that $\mathrm{supp}(P') \subset B_R(x)$ and $P' \sim P$. The minimum such R, denoted $R(P)$, is called the *almost periodicity radius* of P.

Using the notation of Solomyak [19] we can associate several positive constants to any almost periodic tiling with the local finiteness property. These constants will be useful for a variety of estimates and computations throughout this work.

Definition 2.8. Let $\mathcal{T}$ be an almost periodic tiling with a finite number of local patterns. Define positive constants C_1, C_2, and C_3 such that

$$C_1 = \max\{\mathrm{diam}\, T : T \in \mathcal{T}\}; \tag{5}$$

$$\text{if } \|x - y\| < C_2 \text{ , then there exists } z \in \mathbb{R}^d \text{ such that } x, y \in \mathrm{supp}[z]^{\mathcal{T}}; \tag{6}$$

$$\text{Any ball of radius } C_3 \text{ contains copies of all elementary } \mathcal{T}\text{-patches.} \tag{7}$$

It is clear that C_1 exists due to our assumption that the prototile set is finite. The existence of C_2 can be seen from the local finiteness property along with the fact that there are a finite number of prototiles. It is clear that C_3 can be constructed by taking the maximum of all return radii of elementary $\mathcal{T}$-patches. These return radii exist by almost periodicity, and since by local finiteness there are only a finite number of elementary patches, the maximum exists and is also finite.

2.2. **Tiling systems.** A tiling space X will be defined to be made up of some or all of the tilings which can be created from a given set of prototiles using a given allowable translation group. If it contains all possible such tilings, then it can be seen as analogous to the full shift.

Definition 2.9. Given a prototile set τ and an allowable translation group G define the *full tiling space*

$$X(\tau, G) = \{\mathcal{T} \text{ such that } \mathcal{T} \text{ is a } (\tau, G)\text{-tiling } \}.$$

We define basic open sets which form a topology on the tiling space $X(\tau, G)$ and are akin to cylinder sets in shift spaces. The version of a basic open set which will be used in this work is given by B. Solomyak in [20]. Let P be a $\mathcal{T}$-patch and $U \subset \mathbb{R}^d$ a Borel set. Define a basic open set to be

$$X_{P,U} = \{\mathcal{S} \in X : P + u \subset \mathcal{S} \text{ for some } u \in U\}. \tag{8}$$

The topology generated by the basic open sets generates in turn the Borel σ-algebra $\mathcal{B}$ on X.

Radin, Robinson, and Solomyak have proposed metrics for tiling spaces based on the notion of near-agreement on large balls about the origin. The various metrics are equivalent on tiling spaces for which they are mutually valid; we present Solomyak's version here [20]. Let

$$\begin{aligned} \tilde{\rho}(\mathcal{T}, \mathcal{S}) = \inf\{\epsilon : & \text{ there exist } P \subset \mathcal{T} \text{ and } Q \subset \mathcal{S} \text{ with } B(0, \frac{1}{\epsilon}) \subset P, \\ & B(0, \frac{1}{\epsilon}) \subset Q \text{ and } P = Q + g \text{ for some } g \text{ with } \|g\| < \epsilon\}. \end{aligned} \tag{9}$$

The metric on X is given by

$$\rho(\mathcal{T}, \mathcal{S}) = \min(1, \tilde{\rho}(\mathcal{T}, \mathcal{S})). \tag{10}$$

It has been shown [13] that the full tiling space $X(\tau, G)$ is compact in this metric. Since translation can be defined on entire tilings by letting $\mathcal{T} - g = \{T_j - g \text{ for } j \in \mathbb{N}\}$, we can define closed, translation-invariant subspaces which are analogous to subshifts in symbolic dynamics. It is clear that each $g \in G$ is a bijection of the tiling space $X(\tau, G)$ and that the action $G \times X(\tau, G) \to X(\tau, G)$ is jointly continuous.

Definition 2.10. Let τ be a prototile set, let G be a group of allowable translations, and let $X(\tau, G)$ be the corresponding full tiling space. A *tiling space* is a subset $X \subset X(\tau, G)$ which is closed in the metric topology and is translation invariant. Fixing a tiling $\mathcal{T}$, the *tiling space of* $\mathcal{T}$, denoted $X_{\mathcal{T}}$, is given by $X_{\mathcal{T}} = \overline{\{\mathcal{T} - g : g \in G\}}$.

Definition 2.11. Let $X \subset X(\tau, G)$ be a tiling space. A *tiling dynamical system* (X, G) is given by the tiling space X acted upon by the group of translations G.

If $\mathcal{T}$ is almost periodic, then all of its translates are as well, forcing all of the limit points of the orbit to be almost periodic. In this case the tiling dynamical system $(X_{\mathcal{T}}, G)$ is a minimal system.

2.3. **Local derivability.** In symbolic dynamics, there is a notion of a "sliding-block coding": given a sequence one can construct another sequence by keeping track of the order in which the blocks of size $(2N+1)$ appear in the original sequence. Given a code f that replaces each $(2N+1)$-block on an alphabet A by a symbol in the alphabet B, f extends to a shift-commuting map from any subshift of $A^{\mathbb{Z}}$ into $B^{\mathbb{Z}}$ by defining $(f(x))_k = f(x_{k-n}...x_{k+n})$ for each k in $\mathbb{Z}$. If two sequences have sliding-block codes between them which are inverse to one another, then they are the same for some purposes—the dynamical systems which they generate are topologically conjugate. We define a similar notion for tilings.

Fix an $R > 0$ and let $\mathcal{P} = \{P_1, P_2, ..., P_m\}$ be a set of patches in $\mathcal{T}$ so that for every $x \in \mathbb{R}^d$ there exists a unique $i \in 1, 2, ..., m$ so that $]B_R(x)[^{\mathcal{T}} \sim P_i$. Then $\mathcal{T}$ can be seen as an infinite concatenation of the P_i's: it is covered by overlapping translates of elements of $\mathcal{P}$. Let τ' be a finite set of prototiles, and let $\mathcal{P}'$ be a finite set of (τ', G)-patches. Let $\mathcal{C} : \mathcal{P} \to \mathcal{P}'$ be a map which associates to each patch in $\mathcal{P}$ a patch in $\mathcal{P}'$. (It is not necessary to assume that the locations of the patches and their associated images coincide, but it is not hard to show that this assumption can be added without loss of generality.) We can extend $\mathcal{C}$ to a map of the tiling $\mathcal{T}$ "by concatenation":

$$\mathcal{C}_\infty(\mathcal{T}) = \{\mathcal{C}(P_{i_j}) - g_j : P_{i_j} - g_j \subset \mathcal{T}\}. \tag{11}$$

This set may or may not form a (τ, G)-tiling. If it does, then we will consider $\mathcal{C}$ to be a local code which extends to a coding of one tiling onto another.

Definition 2.12. Let $\mathcal{P}, \tau', \mathcal{P}', \mathcal{C}$, and $\mathcal{C}_\infty$ be as above. If $\mathcal{C}_\infty(\mathcal{T})$ is a (τ', G)-tiling then $\mathcal{C}$ is an *local code* from $\mathcal{T}$ to $\mathcal{C}_\infty(\mathcal{T})$, and we say that the tiling $\mathcal{C}_\infty(\mathcal{T})$ is *locally derivable from* $\mathcal{T}$. If, in addition, there exists an local code taking $\mathcal{C}_\infty(\mathcal{T})$ onto $\mathcal{T}$, then $\mathcal{C}_\infty(\mathcal{T})$ and $\mathcal{T}$ are *mutually locally derivable.*

Notice that the full tiling space $X(\tau', G)$ may be empty. In this case, any tiling using the prototiles in τ' must have been formed with a different translation group G'. Such a tiling is not locally derivable from a (τ, G) tiling for any prototile set τ.

Once a local code has been established between a tiling $\mathcal{T}$ and a tiling $\mathcal{S}$, $\mathcal{C}_\infty$ can be extended to a mapping from $X_{\mathcal{T}}$ to $X_{\mathcal{S}}$. In fact, the tiling dynamical system $(X_{\mathcal{S}}, G)$ is a topological factor of $(X_{\mathcal{T}}, G)$:

Proposition 2.1. *Let $\mathcal{S}$ be locally derivable from $\mathcal{T}$ with local code $\mathcal{C}$ and let G be the group of allowable translations of $\mathcal{T}$. Then $\mathcal{C}_\infty$ uniquely determines a factor map from $X_{\mathcal{T}}$ to $X_{\mathcal{S}}$, and $(X_{\mathcal{S}}, G)$ is a topological-dynamical factor of $(X_{\mathcal{T}}, G)$.*

Proof. Let $\mathcal{P}, \tau'$, and $\mathcal{P}'$, be defined for $\mathcal{C}$ as above. First we will show that $\mathcal{C}_\infty$ is well-defined and continuous by demonstrating that it is uniformly continuous on the orbit $\mathcal{O}(\mathcal{T}) = \{\mathcal{T} - g : g \in G\}$.

For $g \in G$, $\mathcal{C}_\infty(\mathcal{T} - g) = \mathcal{C}_\infty(\mathcal{T}) - g$:

$$\begin{aligned}
\mathcal{C}_\infty(\mathcal{T} - g) &= \{\mathcal{C}(P_{i_j} - g_j) : P_{i_j} - g_j \in \mathcal{T} - g\} \\
&= \{\mathcal{C}(P_{i_j} - h_j - g) : P_{i_j} - h_j - g \in \mathcal{T} - g \text{ where } g_j = h_j + g\} \\
&= \{\mathcal{C}(P_{i_j} - h_j - g) : P_{i_j} - h_j \in \mathcal{T}\} \\
&= \{\mathcal{C}(P_{i_j} - h_j) - g : P_{i_j} - h_j \in \mathcal{T}\} \\
&= \{\mathcal{C}(P_{i_j} - h_j) : P_{i_j} - h_j \in \mathcal{T}\} - g \\
&= \mathcal{C}_\infty(\mathcal{T}) - g \\
&= \mathcal{S} - g.
\end{aligned}$$

Now fix $\epsilon > 0$. We must show that there exists a $\delta > 0$ such that for any $\mathcal{T}_1, \mathcal{T}_2 \in \mathcal{O}(\mathcal{T})$ with $d(\mathcal{T}_1, \mathcal{T}_2) < \delta$, we have that $d(\mathcal{C}_\infty(\mathcal{T}_1), \mathcal{C}_\infty(\mathcal{T}_2)) < \epsilon$. Let D be the minimum number such that P_i and $\mathcal{C}(P_i)$ are contained in a closed ball of diameter D for all $i = 1, ..., m$. Let $1/\delta = D + 1/\epsilon$. Let $\mathcal{T}_1, \mathcal{T}_2 \in \mathcal{O}(\mathcal{T})$ with $d(\mathcal{T}_1, \mathcal{T}_2) < \delta$. Then there exists $g \in G$ with $\|g\| < \delta$ and

$$[B_{1/\delta}(0)]^{\mathcal{T}_1} = [B_{1/\delta}(0)]^{\mathcal{T}_2} - g.$$

Every tile in $[B_{1/\epsilon}(0)]^{\mathcal{C}_\infty(\mathcal{T}_i)}$ is given by an R-patch in $[B_{1/\delta}(0)]^{\mathcal{T}_i}$, since any R-patch is within D of its image under $\mathcal{C}$. Since $[B_{1/\delta}(0)]^{\mathcal{T}_1} = [B_{1/\delta}(0)]^{\mathcal{T}_2} - g$ for $\|g\| < \delta < \epsilon$, we obtain the result

$$[B_{1/\epsilon}(0)]^{\mathcal{C}_\infty(\mathcal{T}_1)} = [B_{1/\epsilon}(0)]^{\mathcal{C}_\infty(\mathcal{T}_2)} - g,$$

and so we conclude that $d(\mathcal{C}_\infty(\mathcal{T}_1), \mathcal{C}_\infty(\mathcal{T}_2)) < \epsilon$. Hence $\mathcal{C}_\infty$ is uniformly continuous on $\mathcal{O}(\mathcal{T})$. By basic topology, since $X_\mathcal{T}$ is a compact space there is a unique continuous extension of $\mathcal{C}_\infty$ to the whole space $X_\mathcal{T}$.

It is clear that $\mathcal{C}_\infty$ commutes with translation, so it remains to show that $\mathcal{C}_\infty$ is onto. Note that any translate of $\mathcal{S}$ is the translate of the image under $\mathcal{C}_\infty$ of $\mathcal{T}$ by the same vector. Suppose we have a limit point $\mathcal{S}' = \lim_{n\to\infty} \mathcal{S} - g_n$. Since $X_\mathcal{T}$ is a compact space there is a limit point $\mathcal{T}'$ of the sequence $\mathcal{T} - g_n$. Since $\mathcal{C}_\infty$ is continuous,

$$\mathcal{S}' = \lim_{n\to\infty} \mathcal{C}_\infty(\mathcal{T} - g_n) = \mathcal{C}_\infty(\lim_{n\to\infty} \mathcal{T} - g_n),$$

proving that $\mathcal{S}' = \mathcal{C}_\infty \mathcal{T}'$. This implies that $\mathcal{C}_\infty$ is onto and finishes the proof. □

Corollary 2.2. *If the (τ, G)-tiling $\mathcal{T}$ and the (τ', G)-tiling $\mathcal{S}$ are mutually locally derivable, then the dynamical systems $(X_\mathcal{T}, G)$ and $(X_\mathcal{S}, G)$ are topologically conjugate.*

Proof. Suppose $\mathcal{T}$ and $\mathcal{S}$ are mutually locally derivable. By Proposition 2.1 there are translation-commuting factor maps $\mathcal{C}_\infty : X_\mathcal{T} \to X_\mathcal{S}$ and $\mathcal{D} : X_\mathcal{S} \to X_\mathcal{T}$ such that $\mathcal{C}_\infty(\mathcal{T}) = \mathcal{S}$ and $\mathcal{D}(\mathcal{S}) = \mathcal{T}$. The factor maps are inverses on the orbits of $\mathcal{T}$ and $\mathcal{S}$: for any $g \in G$,

$$\mathcal{D} \circ \mathcal{C}_\infty(\mathcal{T} - g) = \mathcal{D}(\mathcal{S} - g) = \mathcal{T} - g, \text{ and } \mathcal{C}_\infty \circ \mathcal{D}(\mathcal{S} - g) = \mathcal{C}_\infty(\mathcal{T} - g) = \mathcal{S} - g.$$

Since the compositions $\mathcal{D} \circ \mathcal{C}_\infty$ and $\mathcal{C}_\infty \circ \mathcal{D}$ are continuous, they are identity maps on $X_\mathcal{T}$ and $X_\mathcal{S}$. So $\mathcal{C}_\infty$ is a translation-commuting homeomorphism from $X_\mathcal{T}$ to $X_\mathcal{S}$, establishing that $(X_\mathcal{T}, G)$ and $(X_\mathcal{S}, G)$ are topologically conjugate. □

3. Hierarchical Tiling Dynamical Systems

A very general definition is given by M. Senechal [17]: "A tiling is said to be *hierarchical* if its tiles can be merged (composed) to form a tiling on a larger scale with a finite protoset, and these tiles can then be composed to form a tiling on a still larger scale with a finite protoset, and so on *ad infinitum*." We will consider two related types of hierarchy which are based on substitution for sequences. A substitution for a sequence takes each letter in the sequence and replaces it with a word; a sequence is a limit point of the substitution if it is invariant under these replacements. If a tiling is invariant under an analogous replacement process, then it will be viewed as hierarchical. The two specific variations we consider are self-similar tilings and pseudo-self-similar tilings.

Self-similarity is a strict geometric condition: each tile in a self-similar tiling $\mathcal{T}$ is replaced by a configuration of tiles which is geometrically similar to itself. When every tile has been replaced, the resulting tiling is the same as the original. There is a large body of dynamical information piling up from sources like [6, 14, 20, 21] on these tilings.

A tiling $\mathcal{T}$ will be defined to be pseudo-self-similar if there is an expanding linear map ϕ so that when each patch P belonging to a fixed set of patches is expanded by ϕ and replaced by a $\mathcal{T}$-patch which fits 'almost exactly' onto $\phi(P)$, the resulting tiling is $\mathcal{T}$. Pseudo-self-similarity is not quite as strict a condition as self-similarity.

3.1. **Self-similar tilings.** We begin by defining what it will mean to expand a tiling with a linear map.

Definition 3.1. Let ϕ be a linear map of $\mathbb{R}^d$ and let $\mathcal{T}$ be a (τ, G)-tiling. Let $\tau' = \{(\phi(\text{supp}(t)), l(t)) : t \in \tau\}$ and let G' be the translation group given by $\{\phi(g) : g \in G\}$. Define $\phi(\mathcal{T})$ to be the (τ', G')-tiling given by

$$\phi(\mathcal{T}) = \bigcup_{T \in \mathcal{T}} (\phi(\text{supp}(T)), l(T)).$$

Definition 3.2. A tiling $\mathcal{T}$ is *ϕ-subdividing* if there exists an expansive, linear, diagonalizable map $\phi : \mathbb{R}^d \to \mathbb{R}^d$ such that:

1. For all $T \in \mathcal{T}, \phi(T)$ is a union of tiles in $\mathcal{T}$, and
2. If T and T' have the same tile type, then the union of $\mathcal{T}$-tiles composing $\phi(T)$ and $\phi(T')$ are translates of one another.

If $\mathcal{T}$ is almost periodic, then it is a *self-affine tiling.* If it is also true that ϕ is a similarity of $\mathbb{R}^d$, (i.e. all of its eigenvalues have equal modulus) then the tiling will be called a *self-similar tiling.* If $\mathcal{T}$ is a self-similar tiling of $\mathbb{R}^d$ we define the *expansion factor* $\lambda \in \mathbb{R}$ to be the real number $|\det \phi|^{1/d}$.

While it is true that a ϕ-subdividing tiling displays a high degree of self-similar structure, it can fail to be almost periodic, causing the dynamical system associated to it to have no nontrivial invariant probability measure. There may be interesting translation-invariant infinite measures, (see [3] for a measure on a non-primitive substitution sequence space induced by the first-return map on a cylinder), but we will not discuss that here.

A tiling which is self-similar with map ϕ given by $\phi(z) = 3z$ for $z \in \mathbb{R}^2$ is shown in Figure 2. In Figure 3, the substitution $\phi(T)$ on the tile T is given for several of the prototile types.

For self-similar tilings of $\mathbb{R}$ and $\mathbb{R}^2$ we can define the *expansion constant* λ of the tiling, which represents the action of the similarity ϕ: the linear map ϕ is just multiplication by λ. The expansion constant of a self-similar tiling has important implications for the dynamics of the tiling system. Thurston and Kenyon showed [7] that a complex number can be the expansion constant of a self-similar tiling of the plane if and only if that number is a Perron number.

Self-similar tilings are uniquely ergodic, i.e. there is only one translation-invariant probability measure. This result is proved by B. Solomyak [20] using ideas which mimic the proof for substitution dynamical systems. The unique ergodic measure is based on the frequency of occurrence of patches in the tiling space.

It is known [18] that substitution and self-similar tiling systems have entropy 0 and are not mixing, mimicing the symbolic substitution system case yet again (in

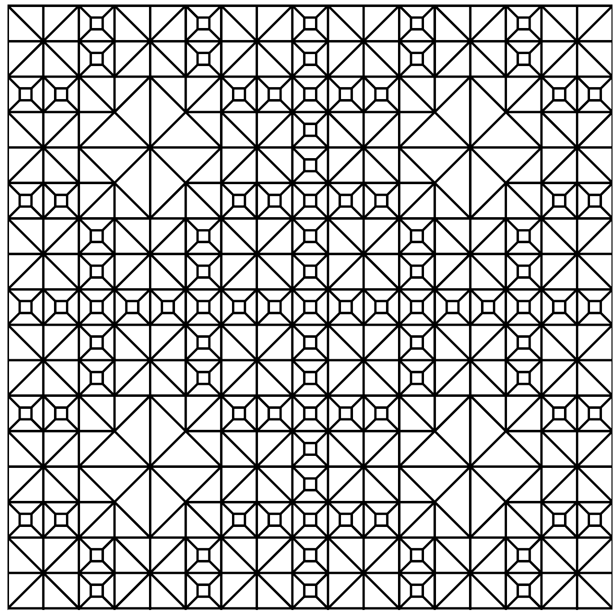

FIGURE 2. A self-similar tiling of $\mathbb{R}^2$ with polygonal tiles.

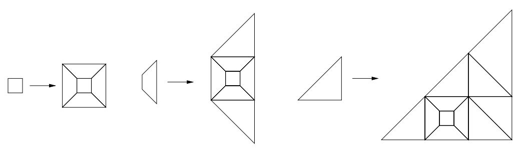

FIGURE 3. The substitution on some of the prototiles used to make Figure 2.

fact, in this paper J. Shieh shows that uniquely ergodic tiling systems have entropy 0). Using a constructive proof, a condition for the existence of eigenvalues is given by B. Solomyak in [20]. For self-similar tilings of $\mathbb{R}$ and $\mathbb{R}^2$, another result of B. Solomyak [20] specifies when the tiling systems $(X_{\mathcal{T}}, G)$ are weakly mixing in terms of the expansion constant λ. Part of the spectrum of these systems is identified.

An important property for self-similar tilings is *recognizability*—the ability to compose a tiling $\mathcal{T}$ into $\phi\mathcal{T}$-tiles in a unique fashion. This is an extension of the same idea for symbolic substitution systems.

Definition 3.3. A self-similar tiling $\mathcal{T}$ is *recognizable* if there exists a real number ρ such that

$$[B_\rho(y)]^{\mathcal{T}} = [B_\rho(x)]^{\mathcal{T}} + (y - x) \qquad \text{implies} \qquad [y]^{\phi\mathcal{T}} = [x]^{\phi\mathcal{T}} + (y - x). \tag{12}$$

The number ρ is called the *recognizability radius* of $\mathcal{T}$.

B. Solomyak proved [19] that nonperiodic self-similar tilings are recognizable. Recognizability implies that we can construct $\phi\mathcal{T}$ from $\mathcal{T}$ locally; of course the reverse is also true, so the two tilings are mutually locally derivable and their dynamical systems are topologically conjugate.

3.2. **Pseudo-self-similar tilings.** The definition proposed here was suggested by E. A. Robinson, Jr.

Definition 3.4. Let $\mathcal{T}$ be a tiling. We call $\mathcal{T}$ *pseudo-self-affine* if there exists an expansive linear map ϕ of $\mathbb{R}^d$ so that the tiling $\phi(\mathcal{T}) = \bigcup_{T\in\mathcal{T}}(\phi(\text{supp}(T)), l(T))$ is mutually locally derivable from $\mathcal{T}$. If the map ϕ is a similarity then we will say that $\mathcal{T}$ is a *pseudo-self-similar tiling*.

The Penrose tiling with darts and kites (introduced to the public in [4] and analyzed dynamically in [14]) is pseudo-self-similar. Another example of a pseudo-self-similar tiling is the one discovered by Godrèche and Lançon [5]. Both of these tilings are mutually locally derivable from self-similar tilings.

A pseudo-self-similar tiling $\mathcal{T}$ with expansion map $\phi(z) = 3z$ is given in Section 4 as Figure 8. The local code from $\mathcal{T}$ to $\phi(\mathcal{T})$ is not at all obvious. It is because $\mathcal{T}$ is a derived Voronoï tiling of a self-similar tiling that we know that it is pseudo-self-similar. Using the same expansion map as that for the self-similar tiling along with the knowledge that the origin is in the center of the portion of the tiling shown, we can figure out the local code between $\mathcal{T}$ and $\phi(\mathcal{T})$.

Self-similar tilings are pseudo-self-similar by recognizablilty. That is, given a large enough patch in the tiling $\mathcal{T}$ around a point $x \in \mathbb{R}^d$ (the recognizability radius ρ determines this size) we can determine which $\phi(\mathcal{T})$-tile contains x. Conversely, the property (1) in Definition 3.2 establishes a local code from $\phi(\mathcal{T})$ back to $\mathcal{T}$. (Replace a $\phi(\mathcal{T})$-tile by the appropriate configuration of $\mathcal{T}$-tiles.) This establishes the mutual local derivability of $\mathcal{T}$ and $\phi(\mathcal{T})$, showing that $\mathcal{T}$ is pseudo-self-similar if it is self-similar.

E. A. Robinson, Jr. has conjectured that all tilings satisfying the pseudo-self-similar property are mutually locally derivable from self-similar tilings. In the end of this paper we discuss a possibility for solving this problem which involves the use of derived Voronoï tilings.

4. First Returns in Tilings

In the introduction, a result of F. Durand [2] characterizing limit sequences of symbolic substitutions was discussed. In the characterization, sequences are recoded in terms of "return words" of a fixed block u—words beginning and ending in u and containing no other copy of u. There are a variety of ways to interpret the term "first return" when continuous, multidimensional time is the parameter of the dynamical system. Given a tiling $\mathcal{T}$ and a fixed, finite, central patch P in $\mathcal{T}$, we will derive a tiling (the *DV tiling*) which carries all of the information about where the other copies of P are. If we change the central patch we will see a potentially different DV tiling; how does it compare?

4.1. **Derived Voronoï tilings.** Fixing a nonempty patch P in $\mathcal{T}$, we will define the *locator set* $\mathcal{L}_P(\mathcal{T})$ to be

$$\mathcal{L}_P(\mathcal{T}) = \{q \in \mathbb{R}^d : \text{ there exists } P' \subset \mathcal{T} \text{ with } P = P' - q\}. \tag{13}$$

The elements of this set pinpoint the locations of all equivalent copies of P in the tiling $\mathcal{T}$. When the tiling $\mathcal{T}$ is understood we will suppress it and write $\mathcal{L}_P$. A tiling from unit square tiles with two labels on the tiles (black and white) is seen in Figure 1 of Section 2. This tiling is a self-similar tiling with substitution shown in

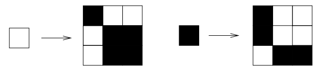

FIGURE 4. The substitution on the tiling from Figure 1.

Figure 4. We will illustrate the construction of a Derived Voronoï tiling using this tiling as an example.

The locator set derived from the two-by-two patch of tiles shown in Figure 5 can be seen in Figure 6.

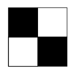

FIGURE 5. A central patch of tiles for which we make a Derived Voronoï tiling.

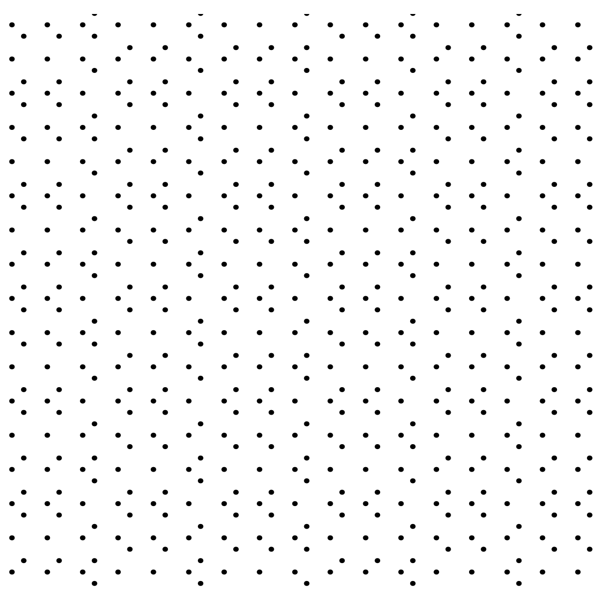

FIGURE 6. A locator set.

Our tilings are assumed to have a finite number of local patterns and to be almost periodic, therefore $\mathcal{L}_P$ forms a *Delaunay set* [17]: a relatively dense set whose elements are uniformly bounded away from each other. This is exactly the type of set for which it is possible to form a (normal) Voronoï tessellation [17], a tessellation which clumps together points which are "closest" to an element of the set.

Definition 4.1. The *Voronoï cell* for q, which forms the support of the return tile t_q, is given by

$$\text{supp}(t_q) = \{x \in \mathbb{R}^d |\ d(x,q) \leq d(x,q') \text{ for all } q' \in \mathcal{L}_P\}. \tag{14}$$

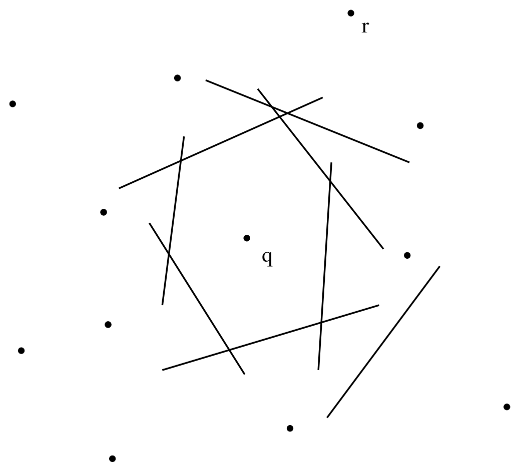

FIGURE 7. The Voronoï cell for q is independent of points as distant as r.

Since there is the possibility that geometrically congruent return tiles arise from non-equivalent $\mathcal{T}$-patches, it will be necessary to label them by noting from which $\mathcal{T}$-patches they originated. To define the label set $\mathcal{H}_P$, we must decide on a radius for the $\mathcal{H}_P$-patches which provides the required information. Since the copies of P in T occur with almost periodicity radius $R(P)$ (recall Definition 2.7), we can ensure that all locator points neighboring q in $\mathcal{L}_P$ appear in a ball of radius $2R(P)$ (this is Lemma 4.3). So to figure out the shape of the return tile t_q, we need only search in a ball of radius $2R(P)$ in the tiling to find all of the neighboring copies of P (see Figure 7). In certain situations it may be useful to consider labellings arising from $\mathcal{H}_P$-patches which are given a larger radius than $2R(P)$. Such a choice of radius may produce a larger label set than the one provided by the (minimal) radius $2R(P)$.

Definition 4.2. Fix an $R \geq 2R(P)$. The set of $\mathcal{H}_P(\mathcal{T}, R)$*-patches* is defined to be

$$\mathcal{H}_P(\mathcal{T}, R) = \{[B_R(q)]^{\mathcal{T}} : q \in \mathcal{L}_P\}. \tag{15}$$

Considering both $\mathcal{T}$ and R fixed, we refer only to the set $\mathcal{H}_P$. Since $\mathcal{T}$ has a finite number of local patterns, $\mathcal{H}_P$ has a finite number of patches up to (translation) equivalence. We will use the translation equivalence ($\sim$) classes of elements of $\mathcal{H}_P$ to provide a finite number of labels on our DV tiles.

Definition 4.3. Let $\mathcal{T}, P$, and R be fixed as above. Let $H_1, H_2, ...H_{N(P)}$ denote representatives of the equivalence classes of patches in $\mathcal{H}_P$. For any $q \in \mathcal{L}_P$, the *label of the return tile* t_q is given by

$$l(t_q) = i, \text{ where } [B_R(q)]^{\mathcal{T}} \sim H_i. \tag{16}$$

Definition 4.4. Let $\mathcal{T}, P$, and R be fixed as above, and let $q \in \mathcal{L}_P$. The *return tile* t_q is defined to be $t_q = (\text{supp}(t_q), l(t_q))$. A *DV tiling for the patch* P is given by

$$\mathcal{T}_P(R) = \bigcup_{q \in \mathcal{L}_P} t_q. \tag{17}$$

If $R(P)$ is the almost periodicity radius (Definition 2.7), denote the DV tiling $\mathcal{T}_P(2R(P))$ as $\mathcal{T}_P$.

The Voronoï tiling for the locator set shown in Figure 6 is shown in Figure 8. When labelled by $\mathcal{H}_P$-patches, it is the DV tiling $\mathcal{T}_P$. It should be noted that this is an interesting example because it is a DV tiling of a self-similar tiling, and therefore is pseudo-self-similar.

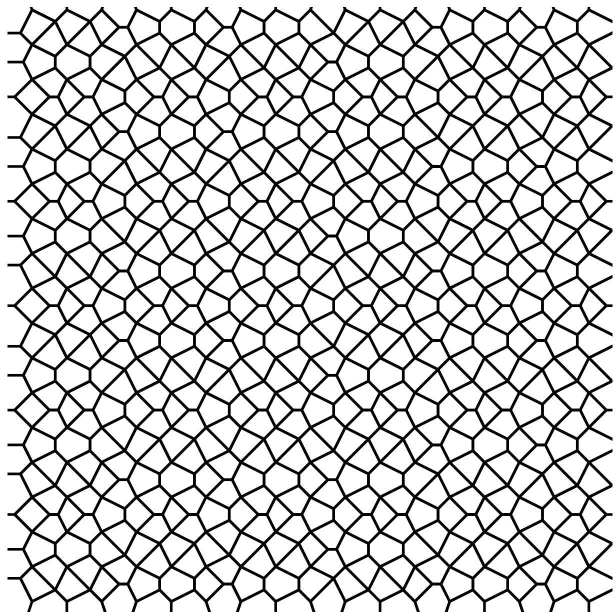

FIGURE 8. Part of a Voronoï tiling.

We refer to *the* DV tiling $\mathcal{T}_P$ because it has the minimal possible label set. As we will see below, all DV tilings from a tiling $\mathcal{T}$ are mutually locally derivable, so most salient features of the tilings are preserved under relabelling by differently-sized $\mathcal{H}_P$-patches.

4.2. **Properties of DV tilings.** By construction, a DV tiling is made from a finite set of prototiles. The tiles of $\mathcal{T}_P(R)$ can be constructed locally from the $\mathcal{H}_P$-patches in $\mathcal{T}$; conversely, patches in $\mathcal{T}$ are uniquely determined by tiles in $\mathcal{T}_P$ (Figure 9). There is a one-to-one map from $\mathcal{L}_P$ to $\mathcal{H}_P$ given by $q \mapsto H_P(q) = [B_R(q)]^{\mathcal{T}}$. This map is easily converted into a local code from $\mathcal{T}_P$ to $\mathcal{T}$ as described in Section 2.3. It is clear that there is a local code from $\mathcal{T}$ to $\mathcal{T}_P$ given by the reverse of this map. This implies that $\mathcal{T}$ and $\mathcal{T}_P$ are mutually locally derivable. Using Corollary 2.2, we conclude that

Proposition 4.1. *The dynamical systems* $(X_{\mathcal{T}}, G)$ *and* $(X_{\mathcal{T}_P}, G)$ *are topologically conjugate.*

Note that the relationship between patches in $\mathcal{T}$ and patches in $\mathcal{T}_P$ require that $\mathcal{T}_P$ inherit the properties of almost periodicity and finite number of local patterns. It follows that DV tilings have all of the properties we require for tilings. DV tilings lend themselves to computations as the tiles and their adjacents have the following known properties (see [17]).

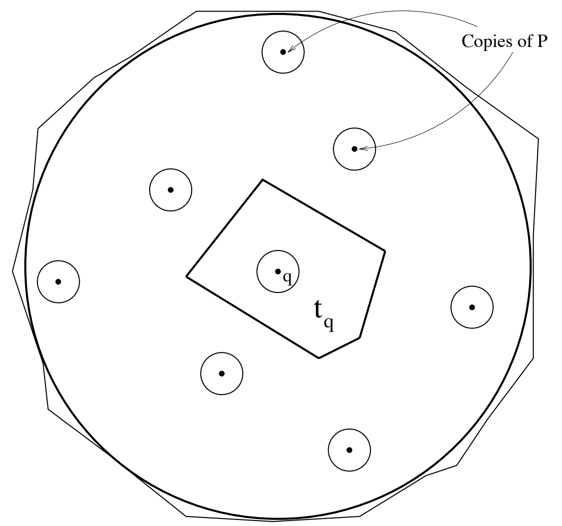

FIGURE 9. An $\mathcal{H}_P$-patch and the associated return tile t_q.

Proposition 4.2. *Let $\mathcal{L}$ be a Delaunay set in $\mathbb{R}^d$, and let $\mathcal{T}(\mathcal{L})$ be the Voronoï tiling of $\mathcal{L}$. Then*

1. *The tiles of $\mathcal{T}(\mathcal{L})$ are convex polytopes which intersect along whole faces; no two tiles have a common interior point;*
2. *The points of $\mathcal{L}$ whose Voronoï tiles share a vertex v lie on a sphere, centered at v, that has no points of $\mathcal{L}$ in its interior.*

The field of computational geometry has provided a variety of algorithms for constructing the Voronoï tessellations of point sets in several dimensions. A convenient algorithm for local construction is to construct the perpendicular bisectors of the line segments qq', for $q, q' \in \mathcal{L}_P$. The smallest open convex region containing q and bounded by the bisectors is the interior of the tile t_q.

A bit of experimentation provides convincing evidence that the appearance of a copy of P in any ball of radius R implies that the return tiles for P have diameter no bigger than $2R$. We record this observation in the following Lemma.

Lemma 4.3. *Let $\mathcal{T}$ be a tiling of $\mathbb{R}^d$ which is almost periodic and has a finite number of local patterns, and let P be a $\mathcal{T}$-patch. If there is a translate of P in $\mathcal{T}$ in any ball of radius R, then for any $q \in \mathcal{L}_P$ and return tile $t_q \in \mathcal{T}_P$, we have that*

$$\mathrm{supp}(t_q) \subset B_R(q).$$

Thus all points which are neighbors of q (in that their return tiles share edges with t_q) are contained in $B_{2R}(q)$.

Proof. Let $w \in t_q$ so that $d(w, q) \le d(w, q')$ for all $q' \in \mathcal{L}_P$. If $d(w, q) > R$, then $d(w, q') > R$, so there are no copies of P in $B_R(w)$. This contradiction shows that for all $w \in \mathrm{supp}(t_q)$, $d(w, q) \le R$. □

4.3. **The derived Voronoï family $\mathcal{F}(\mathcal{T})$.** Given a fixed tiling $\mathcal{T}$, we consider the family of DV tilings of central patches of the form $P_r = [B_r(0)]^{\mathcal{T}}, r \geq 0$, where $B_r(0)$ is the closed ball of radius r about 0. We truncate the notation so that the derived Voronoï tiling $\mathcal{T}_{P_r}$ is simply $\mathcal{T}_r$, $\mathcal{H}_{P_r}$ is simply $\mathcal{H}_r$, and so on. Let

$$\mathcal{F}(\mathcal{T}) = \{\mathcal{T}_r \text{ such that } r \in [0,\infty)\} \tag{18}$$

For a nonperiodic tiling, there will be an infinite number of tilings in the family $\mathcal{F}(\mathcal{T})$; this is a consequence of the following Lemma. Let $R(\mathcal{T}_r) = \sup\{R \in \mathbb{R} : B_R(q) \subset \text{supp}(t_q) \text{ for all } q \in \mathcal{L}_r\}$, the maximum size of a ball contained in any return tile in $\mathcal{T}_r$. (Note that $B_{R(\mathcal{T}_r)}(q) \subset \text{supp}(t_q)$, since $\text{supp}(t_q)$ is a closed set.)

Lemma 4.4. *Let $\mathcal{T}$ be a nonperiodic, almost periodic tiling of $\mathbb{R}^d$. Then*

$$R(\mathcal{T}_r) \to \infty \quad \text{as} \quad r \to \infty.$$

Proof. In search of a contradiction, suppose that there is an $R \in \mathbb{R}$ such that for all $r \in \mathbb{R}$ there is a $q \in \mathcal{L}_r$ with $B_R(q) \not\subset \text{supp}(t_q)$. Fixing any such r and q, we see that there must exist a $q' \in \mathcal{L}_r$ with $\|q - q'\| \leq 2R$.

By local finiteness of the tiling $\mathcal{T}_0$, there are only a finite number of distances $q - q'$ with $\|q - q'\| \leq 2R$ in $\mathcal{L}_0$. For every $s \geq 0$, the locator set $\mathcal{L}_s$ is contained in $\mathcal{L}_0$. Thus there are only a finite number of distances $q - q'$ of modulus not exceeding $2R$ with q, q' in any locator set $\mathcal{L}_s$. This implies that there is an $z \in \mathbb{R}^d$ with $\|z\| \leq 2R$ such that there are $q, q' \in \mathcal{L}_r$ with $z = q - q'$ for infinitely many r.

We will show that for all $T \in \mathcal{T}$, $T + z \in \mathcal{T}$, establishing that $\mathcal{T} + z = \mathcal{T}$ and contradicting the nonperiodicity of $\mathcal{T}$. Choose $r \in \mathbb{R}$ such that $\text{supp}(T)$ and $\text{supp}(T + z)$ are contained in $B_r(0)$. We have that $T \in P_r$ and must show that $T + z \in P_r$. Choose q and $q' \in \mathcal{L}_r$ such that $q - q' = z$. Then $P_r - q \subset \mathcal{T}$ and $P_r - q' \subset \mathcal{T}$; in particular $T - q \in \mathcal{T}$ and $T - q' \in \mathcal{T}$. But $T - q' = T - (q - z)$, so $(T + z) - q \in P_r - q$ by choice of r. Therefore $T + z \in P_r$, and hence in $\mathcal{T}$, as desired. □

So we see that in general $\mathcal{F}(\mathcal{T})$ is likely to have an infinite number of similarity classes. If it does not, we will see that this is an indication of hierarchy in the original tiling $\mathcal{T}$.

5. Hierarchy and Derived Voronoï Tilings

In this section we prove the main results about DV tilings and their connection to hierarchical tilings. Inspired by Durand's work on substitution sequences [2], we will show that the number of similarity isomorphism classes of DV tilings in $\mathcal{F}(\mathcal{T})$ is linked to the presence or absence of hierarchy in $\mathcal{T}$. We begin by making a rigorous definition of what it means for a set of tilings to have a finite number of similarity classes.

Definition 5.1. Let $\mathcal{G}$ be a set of tilings of $\mathbb{R}^d$, and suppose $\phi : \mathbb{R}^d \to \mathbb{R}^d$ is an expanding similarity. We say that $\mathcal{G}$ is *ϕ-finite* if there exist tilings $\mathcal{S}_1, ..., \mathcal{S}_m \in \mathcal{G}$ such that for any tiling $\mathcal{T} \in \mathcal{G}$, there exist $i \in \{1, ..., m\}$ and $k \in \{0, 1, 2, ...\}$ with $\mathcal{T} = \phi^k(\mathcal{S}_i)$.

Theorem 5.1. *Let $\mathcal{T}$ be a nonperiodic self-similar tiling of $\mathbb{R}^d$ with expansion map ϕ. Then the family $\mathcal{F}(\mathcal{T})$ is ϕ-finite.*

The proof is based on the "core argument": the locations of a large initial $\mathcal{T}$-patch in $\mathcal{T}$ are given by the locations of $\phi^k\mathcal{T}$-patches ("cores") with many fewer ($\phi^k\mathcal{T}$-) tiles in $\phi^k\mathcal{T}$. This proof has little in common with the proof used by Durand for limit points of primitive substitutions. Although some steps of the proof hold for self-affine tilings, the proof style holds little promise for generalization beyond the self-similar case. Problems arise in the self-affine case because an arbitrary expanding linear map acting on an arbitrary Delaunay set may cause complicated differences in the Voronoï tilings produced from the original Delaunay set and the expanded Delaunay set. When the linear map is a similarity, the Voronoï tessellations are simply rescalings of one another. It may be that alterations to either the Voronoï construction (such as the "Laguerre tilings" cited in [16]) or to the shapes of the initial patches will result in some other type of derived tiling which would better suit the self-affine case.

An analogue of the argument proving Theorem 5.1 can be used to prove the analogous part of Durand's theorem in the case of substitutions of constant length. In the case of substitutions of non-constant length, it may be possible to use this argument in conjunction with Perron-Frobenius theory to prove the "necessary" part of Durand's result.

In an attempt to classify self-similar tilings and those locally derived from them, the following theorem was obtained.

Theorem 5.2. *Let $\mathcal{T}$ be a nonperiodic, almost periodic tiling of $\mathbb{R}^d$ such that there exists a similarity ϕ under which $\mathcal{F}(\mathcal{T})$ is ϕ-finite. Then $\mathcal{T}$ is pseudo-self-similar and there exists an integer I such that ϕ^I is the expansion map of $\mathcal{T}$.*

In Section 5.2, we prove Theorem 5.2 in detail. Notice that this theorem is very close to being a converse to Theorem 5.1, except that the relationship between pseudo-self-similar tilings and self-similar tilings is not understood. It is not clear whether a core argument and estimates can be applied to pseudo-self-similar tilings which prove Theorem 5.1 for self-similar tilings, and attempts to make such adjustments have broken down in the last step of the proof. Still, we conjecture the following is true: *A tiling $\mathcal{T}$ is a pseudo-self-similar tiling if and only if it has a finite number of DV tilings up to similarity.* An alternative suggestion would be to prove the conjecture: *Any pseudo-self-similar tiling is mutually locally derivable from a self-similar tiling.*

5.1. **The proof of Theorem 5.1.** The proof of this theorem proceeds in the following manner. First it is shown that the locator sets of initial patches for any self-similar tiling fall into a finite number of similarity classes. This implies that there are a finite number of (unlabelled) Voronoï tilings from these locator sets up to similarity. Finally, it is shown that there are only a finite number of ways the $\mathcal{H}_r$-patches can produce labellings for DV tilings, and the result follows.

It will be necessary to have the following technical result showing that given a large enough patch of tiles in a self-similar tiling $\mathcal{T}$, one can uniquely compose a $\phi^k\mathcal{T}$-tile.

Lemma 5.3. *Let ϕ be the expansion map for the self-similar tiling $\mathcal{T}$, let λ be the expansion factor $|\det\phi|^{1/d}$ of $\mathcal{T}$, let l be the minimal integer for which $\lambda^l(\lambda-1) > 1$,*

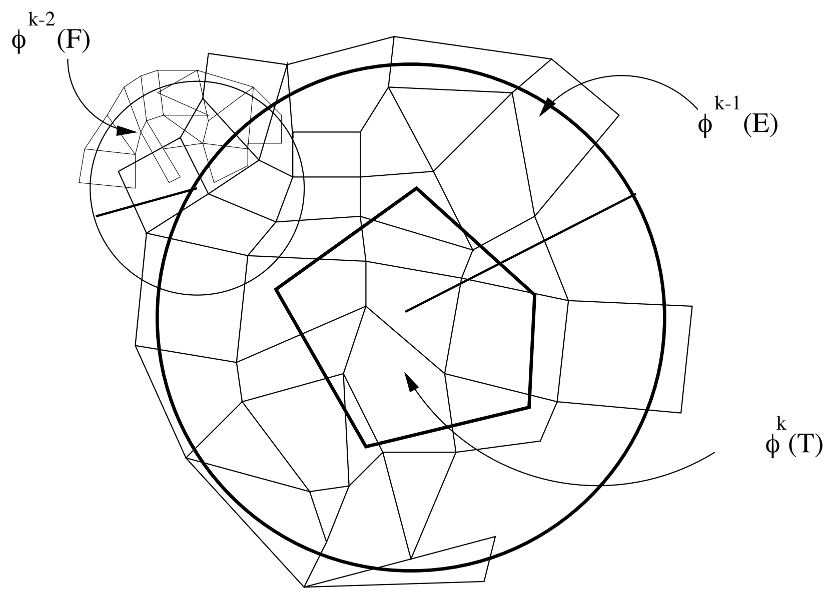

FIGURE 10. A patch of $\phi^{k-2}\mathcal{T}$-tiles reveals how it composes a unique $\phi^k\mathcal{T}$-tile.

and let $k \geq 0$. Let ρ be the recognizability radius of $\mathcal{T}$ (recall Definition 3.3). Then

(19)

$$[B_{\lambda^{k+l}\rho}(y)]^{\mathcal{T}} = [B_{\lambda^{k+l}\rho}(x)]^{\mathcal{T}} + (y-x) \quad \textit{implies} \quad [y]^{\phi^k\mathcal{T}} = [x]^{\phi^k\mathcal{T}} + (y-x).$$

Proof. By recognizability of $\phi^k\mathcal{T}$ we know that

$$[B_{\lambda^{k-1}\rho}(y)]^{\phi^{k-1}\mathcal{T}} = [B_{\lambda^{k-1}\rho}(x)]^{\phi^{k-1}\mathcal{T}} + (y-x)$$

implies $[y]^{\phi^k\mathcal{T}} = [x]^{\phi^k\mathcal{T}} + (y-x)$. For any $z \in B_{\lambda^{k-1}\rho}(y)$ we can uniquely write $[z]^{\phi^{k-1}\mathcal{T}}$ as a composition of $\phi^{k-2}\mathcal{T}$-tiles by looking in $[B_{\lambda^{k-2}\rho}(z)]^{\phi^{k-2}\mathcal{T}}$.

Thus we can express $[B_{\lambda^k\rho}(y)]^{\phi^k\mathcal{T}}$ as a unique composition of $\phi^{k-2}\mathcal{T}$-tiles by looking at $[B_{\lambda^{k-1}\rho+\lambda^{k-2}\rho}(y)]^{\phi^{k-2}\mathcal{T}}$. (See Figure 5.1.) Continuing in this fashion, we determine the size of the $\phi^{k-3}\mathcal{T}$ patch uniquely composing $[B_{\lambda^{k-1}\rho+\lambda^{k-2}\rho}(y)]^{\phi^{k-2}\mathcal{T}}$, and so on until we finally conclude that $\mathcal{T}$-patches of radius $(\Sigma_{j=0}^{k-1}\lambda^j)\rho$ uniquely determine $\phi^k\mathcal{T}$-tiles. The first few stages of the composition process are depicted in Figure 5.1. Formally, if

$$[B_{(\Sigma_{j=0}^{k-1}\lambda^j)\rho}(y)]^{\mathcal{T}} = [B_{(\Sigma_{j=0}^{k-1}\lambda^j)\rho}(x)]^{\mathcal{T}} + (y-x) \quad \text{then} \quad [y]^{\phi^k\mathcal{T}} = [x]^{\phi^k\mathcal{T}} + (y-x).$$

To obtain the final form of the result note that

$$\Sigma_{j=0}^{k-1}\lambda^j = \frac{\lambda^k - 1}{\lambda - 1} < \frac{\lambda^k}{\lambda - 1} = \frac{\lambda^k\lambda^l}{(\lambda-1)\lambda^l}.$$

By choice of l the denominator is greater than 1, and so we can conclude that

$$\Sigma_{j=0}^{k-1}\lambda^j < \lambda^{k+l}.$$

□

The following Proposition shows that a self-similar tiling has a finite number of locator sets up to similarity using the core argument.

Proposition 5.4. *Let $\mathcal{T}$ be a self-similar tiling of $\mathbb{R}^d$ with expansion map ϕ and expansion factor $\lambda = |\det \phi|^{1/d}$. There exist $\mathcal{T}$-patches $F_1, F_2, ..., F_N$ and translations $g_1, g_2, ..., g_N$ in $\mathbb{R}^d$ such that for any r there exist $i_1, i_2, ..., i_m \in \{1, 2, ..., N\}$ and $k \in \mathbb{Z}$ such that $\mathcal{L}_r = \phi^k((\mathcal{L}_{F_{i_1}} + g_{i_1}) \cup (\mathcal{L}_{F_{i_2}} + g_{i_2}) \cup ... \cup (\mathcal{L}_{F_{i_m}} + g_{i_m}))$.*

Proof. We begin the proof by defining the $\mathcal{T}$-patch E which will be used to form the cores of initial patches. The core will have a finite list of *extensions* $F_1, ..., F_N$ which can be used to pinpoint the locations of copies of the initial patches. The core E is defined to be $E = [0]^{\mathcal{T}}$, the elementary patch consisting of all tiles in $\mathcal{T}$ whose supports contain 0. We extend the patch E as follows. Let l be as in Lemma 5.3 and let $\mathcal{L}_E$ be the locator set for E in $\mathcal{T}$; that is, $\mathcal{L}_E = \{q \in \mathbb{R}^d :$ there exists $E' \subset \mathcal{T}$ with $E' - q = E\}$. Consider the set of $\mathcal{T}$-patches given by $\{[B_{\rho\lambda^{l+1}}(q)]^{\mathcal{T}} : q \in \mathcal{L}_E\}$. By local finiteness, this set has a finite number of equivalence classes up to translation. Let $F_1, ..., F_N$ be representatives of these equivalence classes, and let $g_1, ..., g_N$ be the elements of $\mathcal{L}_E$ such that $F_i = [B_{\rho\lambda^{l+1}}(g_i)]^{\mathcal{T}}$ for $i = 1, ..., N$. We have defined $\mathcal{L}_{F_i} = \{q \in \mathbb{R}^d :$ there exists $F \subset \mathcal{T}$ with $F - q = F_i\}$.

We claim that

$$\mathcal{L}_E = (\mathcal{L}_{F_1} + g_1) \cup ... \cup (\mathcal{L}_{F_N} + g_N). \tag{20}$$

To show $\mathcal{L}_E \subset (\mathcal{L}_{F_1} + g_1) \cup ... \cup (\mathcal{L}_{F_N} + g_N)$, let $q \in \mathcal{L}_E$. Then $[B_{\rho\lambda^{l+1}}(q)]^{\mathcal{T}} = F_i + q - g_i$ for some $i \in 1, ..., N$, since $q - g_i$ is the translation taking the appropriate F_i onto $[B_{\rho\lambda^{l+1}}(q)]^{\mathcal{T}}$. This implies that $q - g_i \in \mathcal{L}_{F_i}$, so $q \in \mathcal{L}_{F_i} + g_i$.

Now let $q \in \mathcal{L}_{F_i} + g_i$ for some $i \in \{1, ..., N\}$, that is, $q - g_i \in \mathcal{L}_{F_i}$. By definition of $\mathcal{L}_{F_i}$, there exists $F' \subset \mathcal{T}$ with $F' - (q - g_i) = F_i$. We know that $E \subset F_i - g_i$ by the definition of g_i (since $g_i \in \mathcal{L}_E$), so $E \subset F' - q$. This implies that $E + q \subset \mathcal{T}$ and hence that $q \in \mathcal{L}_E$, finishing the proof of Equation 20. We have established that E appears at a certain spot in $\mathcal{T}$ if and only if one of its extensions appears there.

We now show that any initial patch P_r has locator set which depends only on the locator sets of certain of the extensions. Fix r and let k be the integer with $\rho\lambda^{k+l} < r \leq \rho\lambda^{k+l+1}$. If $r \leq \rho\lambda^l$, then $P_r \subset F_i$ for some i, and the following proof is valid for these r when $k = 0$.

By Lemma 5.3 on the recognizability of $\phi^k\mathcal{T}$ (given sufficiently large $\mathcal{T}$-patches), since $r > \rho\lambda^{k+l}$, for any $q \in \mathcal{L}_r$, since $P_r = [B_r(0)]^{\mathcal{T}} = [B_r(q)]^{\mathcal{T}} - q$, we have that $[0]^{\phi^k\mathcal{T}} = [q]^{\phi^k\mathcal{T}} - q$. (If k is fixed as 0, we have a priori that $[0]^{\mathcal{T}} = [q]^{\mathcal{T}} - q$.) But $[0]^{\phi^k\mathcal{T}} = \phi^k E$, which shows that $q \in \phi^k(\mathcal{L}_E)$. So $q \in \phi^k(\mathcal{L}_{F_i} + g_i)$ for some $i \in \{1, ..., N\}$. Choose $i_1, ..., i_m$ to be the set of all integers in $1, ..., N$ so that $\mathcal{L}_r \cap \phi^k(\mathcal{L}_{F_{i_j}} + g_{i_j}) \neq \emptyset$.

Thus, $\mathcal{L}_r \subset \phi^k((\mathcal{L}_{F_{i_1}} + g_{i_1}) \cup (\mathcal{L}_{F_{i_2}} + g_{i_2}) \cup ... \cup (\mathcal{L}_{F_{i_m}} + g_{i_m}))$. To obtain the final result, we will show that for all $j = \{1, ..., m\}, \phi^k(\mathcal{L}_{F_{i_j}} + g_{i_j}) \subset \mathcal{L}_r$.

Let q' be any element of $\phi^k(\mathcal{L}_{F_{i_j}} + g_{i_j})$, and let $q \in \mathcal{L}_r \cap \phi^k(\mathcal{L}_{F_{i_j}} + g_{i_j})$. Since F_{i_j} was the outer patch of a ball of radius $\rho\lambda^{l+1}$, we have that any copy of $\phi^k(F_{i_j})$ contains the outer $\mathcal{T}$-patch of a ball of radius $\rho\lambda^{k+l+1}$. There is a copy of P_r at q, and because of the size of F_{i_j}, this copy of P_r is inside the copy of $\phi^k(F_{i_j})$. There is a copy of $\phi^k F_{i_j}$ at q', and by the ϕ-subdividing property of $\mathcal{T}$ this means that the $\mathcal{T}$-patch appearing there is the same as the $\mathcal{T}$- patch appearing at q, implying that there is a copy of P_r at q'. This implies that there is a translate of P_r centered at q' contained in $[\phi^k(F')]^{\mathcal{T}} - q'$, making $q' \in \mathcal{L}_r$. $\square$

It should be noted that the proof of the previous proposition holds for any self-affine tiling, given an appropriate version of Lemma 5.3. However, the next corollary is not true unless ϕ is a similarity—Voronoï constructions are sensitive to this. A Delaunay set and its image under a similarity produce Voronoï tilings which are the same modulo this similarity.

Corollary 5.5. *Let $\mathcal{T}$ be a self-similar tiling of $\mathbb{R}^d$ with expansion map ϕ and expansion factor $\lambda = |\det \phi|^{1/d}$. Let $F_1, ..., F_N$ and $g_1, ..., g_N$ be as in Proposition 5.4. For any $r \geq 0$ there exist $i_1, ..., i_m \in \{1, ..., N\}$ so that the Voronoï tiling given by the locator set $\mathcal{L}_r$ is similar to the Voronoï tiling of the Delaunay set $(\mathcal{L}_{F_{i_1}} + g_{i_1}) \cup (\mathcal{L}_{F_{i_2}} + g_{i_2}) \cup ... \cup (\mathcal{L}_{F_{i_m}} + g_{i_m})$.*

Proof. Fix an r and set $\mathcal{L}_r = \phi^k((\mathcal{L}_{F_{i_1}} + g_{i_1}) \cup (\mathcal{L}_{F_{i_2}} + g_{i_2}) \cup ... \cup (\mathcal{L}_{F_{i_m}} + g_{i_m}))$, the result of Proposition 5.4. Since ϕ^k is a similarity, the relative distances between elements of $(\mathcal{L}_{F_{i_1}} + g_{i_1}) \cup (\mathcal{L}_{F_{i_2}} + g_{i_2}) \cup ... \cup (\mathcal{L}_{F_{i_m}} + g_{i_m})$ are simply multiplied by λ^k, so the Voronoï tiling of $\mathcal{L}_r$ is its image under ϕ^k. That is, the Voronoï cell $\mathrm{supp}(t_q)$ of a locator point $q \in \mathcal{L}_r$ is given by $\phi^k(\mathrm{supp}(t_{\phi^{-k}(q)}))$, where $\mathrm{supp}(t_{\phi^{-k}(q)})$ is the Voronoï cell of $\phi^{-k}(q)$ in the Voronoï tiling of $\mathcal{L}_E$. $\square$

We have shown that there are only a finite number of similarity equivalence classes for the DV tilings of a self-similar tiling. It is not difficult to establish that there are only a finite number of ways that these tilings could have been labelled by $\mathcal{H}_r$-patches; we do this next.

Claim 5.1. For any positive numbers $r \leq s$ such that for the set of integers $i_1, ..., i_m$,

$$\mathcal{L}_r = \phi^k((\mathcal{L}_{F_{i_1}} + g_{i_1}) \cup (\mathcal{L}_{F_{i_2}} + g_{i_2}) \cup ... \cup (\mathcal{L}_{F_{i_m}} + g_{i_m}))$$

and

$$\mathcal{L}_s = \phi^j((\mathcal{L}_{F_{i_1}} + g_{i_1}) \cup (\mathcal{L}_{F_{i_2}} + g_{i_2}) \cup ... \cup (\mathcal{L}_{F_{i_m}} + g_{i_m})),$$

the labelling of $\mathcal{T}_r$ factors onto that of $\mathcal{T}_s$.

Proof. The previous Corollary established that the similarity ϕ^{j-k} takes the supports of tiles of $\mathcal{T}_r$ onto those of $\mathcal{T}_s$; we must establish that the labels of $\mathcal{T}_r$-tiles factor onto those of $\mathcal{T}_s$. That is, we will show that for q, q' in $\mathcal{L}_r$ with $l(q) = l(q')$ in $\mathcal{T}_r$, $l(\phi^{j-k}(q)) = l(\phi^{j-k}(q'))$ in $\mathcal{T}_s$. (Note that since $r \leq s$, we have $k \leq j$.)

Let R_r be the almost periodicity radius of P_r. Since $\mathcal{L}_s = \phi^{j-k}(\mathcal{L}_r)$, we have that the almost periodicity radius of P_s must be given by $\lambda^{j-k}R_r$. Recall that the almost periodicity radius is used to determine the size of the $\mathcal{H}_r$-patches and hence the labelling of $\mathcal{T}_r$. That is, for any $q \in \mathcal{L}_r$, $l(q) = i$ if and only if $[B_{2R_r}(q)]^{\mathcal{T}} \sim H_i$, where H_i is a representative of a translation equivalence class in $\mathcal{H}_r$. Recall the notation $H_r(q) = [B_{2R_r}(q)]^{\mathcal{T}}$ for the $\mathcal{H}_r$-patch of q.

Similarly, the labelling for $\mathcal{T}_s$ is given by (translation) equivalence classes of patches in $\mathcal{H}_s = \{[B_{2\lambda^{j-k}R_r}(q)]^{\mathcal{T}} : q \in \mathcal{L}_s\}$. We have that for $q, q' \in \mathcal{L}_s, l(q) = l(q')$ if and only if $[B_{2\lambda^{j-k}R_r}(q)]^{\mathcal{T}} \sim [B_{2\lambda^{j-k}R_r}(q)]^{\mathcal{T}}$ (that is, $H_s(q) \sim H_s(q')$). We are ready to show that the labels of points in $\mathcal{L}_r$ factor onto the labels of points in $\mathcal{L}_s$.

Let $q, q' \in \mathcal{L}_r$ such that $l(q) = l(q')$. Then $H_r(q) \sim H_r(q')$, and by the ϕ-subdividing property of $\mathcal{T}$, $]\phi^{j-k}(\mathrm{supp}(H_r(q)))[^{\mathcal{T}} \sim]\phi^{j-k}(\mathrm{supp}(H_r(q')))[^{\mathcal{T}}$. These are $\mathcal{T}$-patches which contain balls of radius $2\lambda^{j-k}R_r$, so they contain $H_s(\phi^{j-k}(q))$

and $H_s(\phi^{j-k}(q'))$, respectively. It follows that $H_s(\phi^{j-k}(q)) \sim H_s(\phi^{j-k}(q'))$, and so $l(\phi^{j-k}(q)) = l(\phi^{j-k}(q'))$, as desired. □

For any set $\{i_1, ..., i_m\} \subset \{1, ..., N\}$ for which there is an r with $\mathcal{L}_r = \phi^k((\mathcal{L}_{F_{i_1}} + g_{i_1}) \cup (\mathcal{L}_{F_{i_2}} + g_{i_2}) \cup ... \cup (\mathcal{L}_{F_{i_m}} + g_{i_m}))$, we can fix a minimal r. The labelling of the tiling $\mathcal{T}_r$, which is on a finite alphabet depending on both r and $\mathcal{T}$, factors onto the labelling of any other $\mathcal{T}_s$ for which $\mathcal{L}_s$ is made from the same combination of F_{i_j}'s. Thus, there is a finite list of tilings which $\mathcal{T}_r$ can factor onto in this way. Since there are only a finite number of combinations $\{i_1, ..., i_m\}$ of distinct integers in $\{1, ..., N\}$, there are only a finite number of DV tilings $\mathcal{T}_r$ with minimal r. This establishes that $\mathcal{F}(\mathcal{T})$ is ϕ-finite, as desired.

5.2. **The proof of Theorem 5.2.** Next we will prove that if a nonperiodic, almost periodic tiling of $\mathbb{R}^d$ has a finite number of similarity classes of DV tilings, then it is pseudo-self-similar. The proof is in two main steps. First, we will show that there exists an $r \geq 0$ and an integer I for which $\phi^I(\mathcal{T}_r)$ is mutually locally derived from $\mathcal{T}_r$, showing that $\mathcal{T}_r$ is pseudo-self-similar with expansion map ϕ^I. Then we will show that since $\mathcal{T}_r$ is pseudo-self-similar with expansion map ϕ^I, $\mathcal{T}$ is also pseudo-self-similar with the same expansion map.

Suppose that the family $\mathcal{F}$ is ϕ-finite. Since there are an infinite number of DV tilings in $\mathcal{F}$, it must be that there are infinitely many r for which $\mathcal{T}_r = \phi^k(\mathcal{S}_i)$ for some fixed i. Choose real numbers r and u so that $\mathcal{T}_r = \phi^k(\mathcal{S}_i)$ and $\mathcal{T}_u = \phi^j(\mathcal{S}_i)$ and u much larger than r. Then $\mathcal{T}_u = \phi^{j-k}(\mathcal{T}_r)$. Lemma 4.4 implies that u can be chosen so that $j-k$ is arbitrarily large, since $\mathcal{T}$ is not a periodic tiling.

Set $I = j - k$ and let $\lambda = |\det \phi|^{1/d}$ (so λ is the expansion factor of ϕ). We can assume $\lambda^I \geq 2$, since we can take a larger u to make it so if it is not already true. We will use $\mathcal{T}_u$ to establish a local code between $\mathcal{T}_r$ and $\phi^I(\mathcal{T}_r)$.

For any $q \in \mathcal{L}_r$ and corresponding tile $t_q \in \mathcal{T}_r$, we have that $\phi^I(t_q) = t_g$, a tile in $\mathcal{T}_u$. (In many cases it must be that $g = \phi^I(q)$, but it is not possible to assume that.) Let R_r be the almost periodicity radius of P_r. Then we see that $R_u = \lambda^I R_r$, since we can determine the almost-periodicity radius of a patch by looking at the maximum diameter of a return tile (cf. Lemma 4.3). So the $\mathcal{H}_u$-patch $H_u(g) = [B_{2R_u}(g)]^{\mathcal{T}}$ is given by $[B_{2\lambda^I R_r}(g)]^{\mathcal{T}}$. Define a local code from $\mathcal{T}_u$ to $\mathcal{T}_r$ by:

$$C(t_g) = C(\phi^I(t_q)) = \bigcup_{q' \in \mathcal{L}_r \cap \phi^I(B_{\lambda^I R_r}(g))} t_{q'}. \tag{21}$$

Since the tile $\phi^I(t_q)$ is very large relative to $\mathcal{T}_r$-tiles, the $b^I R_r$-patch code is given by a single tile in $\mathcal{T}_u$ factoring onto a patch of $\mathcal{T}_r$-tiles.

To show that C forms a local code from $\mathcal{T}_u = \phi^I(\mathcal{T}_r)$ to $\mathcal{T}_r$, we must show that for any $g, g' \in \mathcal{L}_u$ with $l(g) = l(g')$, the patches $C(t_g)$ and $C(t_{g'})$ are equivalent. This is from the choice of I: since $\lambda^I \geq 2$ we have that $\lambda^I R_r + 2R_r \leq 2\lambda^I R_r$, so the $\mathcal{H}_r$-patch of any $q' \in \mathcal{L}_r \cap \phi^I(B_{\lambda^I R_r}(g))$ will be completely contained in $H_u(g)$. If $l(g) = l(g')$, then the $\mathcal{T}_r$-patches given by $C(t_g)$ and $C(t_{g'})$ are equivalent since the $\mathcal{H}_u$-patches they are defined by are equivalent. It is clear that C can be extended to form a local code on $\mathcal{T}_u = \phi^I(\mathcal{T}_r)$.

Conversely, there is a local code from $\mathcal{T}_r$ onto T_u. Let $\{P_1, ...P_m\}$ be a representative set of $\mathcal{T}_r$-patches so that any $[B_{2\lambda^I R_r}(g)]^{\mathcal{T}_r}$ for $g \in \mathcal{L}_u$ is equivalent to a patch in $\{P_1, ...P_m\}$. Patches of this form cover $\mathcal{T}_r$; for each i we have a $g_i \in \mathcal{L}_u$ such that $P_i = [B_{2\lambda^I R_r}(g_i)]^{\mathcal{T}_r}$. Define a $2\lambda^I R_r$-patch code $\mathcal{D}$ from $\mathcal{T}_r$ to $\mathcal{T}_u$ by letting

$$\mathcal{D}(P_i) = t_{g_i}. \tag{22}$$

The simplest way to extend $\mathcal{D}$ to a map on all $\mathcal{T}_r$-patches of size $2\lambda^I R_r$ is to map any patch of size $2\lambda^I R_r$ which is not equivalent to any P_i onto the empty patch. We do not need contributions from these patches since $\mathbb{R}^d$ is already covered by the tiles in $\mathcal{T}_u$, and $\mathcal{D}$ maps onto all of those. Once again we see that if two $\mathcal{T}_r$-patches are equivalent they map onto equivalent $\mathcal{T}_u$-tiles by passing via $\mathcal{H}_r$-patches to $\mathcal{H}_u$-patches. Any P_i is sufficiently large so that the $\mathcal{T}$-patch generated by the union of the $\mathcal{H}_r$-patches contained in it contains the $\mathcal{H}_u$-patch of g_i, determining the tile t_{g_i} uniquely. Any translate of P_i does the same for its corresponding locator point.

This establishes a local code from $\mathcal{T}_r$ onto $\mathcal{T}_u$, and so we have shown that $\mathcal{T}_r$ and $\mathcal{T}_u$ are mutually locally derivable. Since $\mathcal{T}_u = \phi^I(T_r)$, we have proved that $\mathcal{T}_r$ is pseudo-self-similar with expansion map ϕ^I.

It remains to prove that $\mathcal{T}$ is pseudo-self-similar with expansion map ϕ^I. Since $\mathcal{T}$ and $\mathcal{T}_r$ are mutually locally derivable, it is clear that $\phi^I(\mathcal{T})$ and $\phi^I(\mathcal{T}_r)$ also are mutually locally derivable. But $\phi^I(\mathcal{T}_r)$ is mutually derived from $\mathcal{T}_r$, which makes $\mathcal{T}_r$ and $\phi^I(\mathcal{T})$ mutually locally derivable. Since $\mathcal{T}_r$ and $\mathcal{T}$ are mutually locally derivable, it follows that $\mathcal{T}$ and $\phi^I(\mathcal{T})$ are mutually locally derivable. This finishes the proof of Theorem 5.2.

5.3. **For further study.** The following paragraphs outline some questions and conjectures brought about by the method of analysis used in this work. Much information is yet to be acquired about the interaction between the various forms of hierarchy for tilings and the various possibilities for generalizations of derived Voronoï tilings.

We could use DV tilings to try to solve the question from Section 3.2: is every pseudo-self-similar tiling $\mathcal{T}$ mutually locally derivable from a self-similar tiling? Several authors (see e.g. [1, 5]) have proved results involving the refinement of pseudo-self-similar tilings into tilings made from "fractiles", using self-similar sets and fractal theory to prove the conjecture for classes of examples. Although they do not use DV tilings in their proofs, the use of DV tilings might simplify the problem in general because of the geometric properties of such tilings.

We conjecture that an extension of our results to tilings such as the pinwheel tiling—tilings which have an infinite number of tile orientations but otherwise satisfy the self-similarity condition—can be seen to have a finite number of DV tilings up to similarity. (The DV tiles would also come in an infinite number of orientations.) Instead of locating only translates of P_r to form $\mathcal{L}_r$, we could locate all images of P_r under allowable isometry. Since every large ball in such a tiling is contained in a rotation of one of finitely many large substituted tiles, there would be a finite number of DV tiles up to allowable isometry. The missing elements needed to mimic the proof of Theorem 5.1 are the technical estimates on the sizes of DV tiles. Converse theorems could be developed using DV tilings made from a finite number of tile shapes in an infinite number of orientations.

How can the results be extended to include self-affine tilings—tilings which are expanded like self-similar tilings but with an arbitrary expanding linear map ϕ instead of a similarity? Could alterations in the shapes of initial patches, such as considering $[\phi^I(B_1(0))]^{\mathcal{T}}$, letting I vary, instead of $[B_r(0)]^{\mathcal{T}}$, letting r vary, be useful? It is possible to prove reasonably accurate estimates on the size and shape of return tiles for such an initial patch in a self-affine tiling. Might it also be useful to assign weights to the locator points and then construct the Laguerre tiling—a variant of the Voronoï tiling which takes the weights into account? These tilings are known to be edge-to-edge and made of convex polygonal tiles, so they would lend themselves to computation. Perhaps these techniques would help circumvent the important question which prevented the generalization of our results to the self-affine case: in general, what sort of interaction is there between Voronoï tilings and arbitrary expanding linear maps?

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