SUBSTITUTION SEQUENCES IN $\mathbb{Z}^d$ WITH A NONSIMPLE LEBESGUE COMPONENT IN THE SPECTRUM

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Abstract. We construct d-dimensional substitution sequences for which the continuous part of the spectrum is generated by measures equal to Lebesgue measure. A special case is the Rudin-Shapiro substitution sequence. The construction uses Hadamard matrices in an essential way, so the dimension and size of a substitution is restricted by the size of the Hadamard matrix defining it. Each such substitution automatically has a *dual substitution*, which is defined by the same Hadamard matrix, and which retains a Lebesgue spectral component. We also see that random application of our substitutions produces sequences with a Lebesgue component in their spectrum. Finally, we see that any d-dimensional substitution with $d > 1$ can be "unraveled" into lower-dimensional substitutions which still have Lebesgue spectral components.

1. Introduction

Substitution sequences and self-similar tilings give mathematical examples of *quasicrystals:* objects exhibiting a high degree of order, but not enough to satisfy the crystallographic restriction. We can study these sequences and tilings from an ergodic theoretic perspective by looking at the large-scale structure of the objects. In this perspective, we associate a dynamical system and investigate its ergodicity and mixing properties. We can also study spectral properties by considering the function space associated to the sequence or tiling dynamical system. In this method, a function is used to create a spectral measure on the torus that can be compared to the spectral measures of other functions, and to Lebesgue measure. The examples given in this paper are the first known nontrivial substitution sequences in $\mathbb{Z}^d$ ($d > 1$) that have a nonsimple Lebesgue spectral component.

In order to arrive at the main result of the paper, we construct a (highly specialized) substitution $\mathcal{S}$ and consider (X, μ) to be the sequence space associated to $\mathcal{S}$. Writing H_D as the discrete spectrum and $Z(f)$ as the cyclic subspace associated to a function $f \in L^2(X, \mu)$, we will show that:

Theorem. Let $(X, \mathbb{Z}^d, \mu)$ be the dynamical system associated to a nonperiodic $\mathbb{Z}^d$ substitution satisfying (R1), (R2), and (H). Then there exist $f_1, f_2, ... f_K \in L^2(X, \mu)$, each with spectral measure equal to Lebesgue measure, so that

$$L^2(X, \mu) = H_D \oplus Z(f_1) \oplus Z(f_2) \oplus \cdots \oplus Z(f_K).$$

A fairly old ergodic theory problem is to find a dynamical system (usually a $\mathbb{Z}$-action) whose continuous spectral component is Lebesgue of a given multiplicity $K \geq 1$. The classical Rudin-Shapiro ± 1 sequence and its related substitution sequence give rise to dynamical systems that have Lebesgue components of multiplicity $K = 2$ [Q2]. In fact, the "Toeplitz $\mathbb{Z}_2$-extensions" in [Le] provide examples for any even number K. Our construction can be used to create simple d-dimensional substitutions ($d \geq 1$) giving rise to Lebesgue spectral components of even multiplicity (although not for every even number). It is still open whether any given odd multiplicity can be attained. In particular, the "Banach problem" is still unsolved: whether there exists a system with simple Lebesgue spectrum ($K = 1$) in the orthocomplement of the constants.

In the paradigm where sequences and/or tilings are considered models for the atomic structure of solids, one finds interest in the correlation measure of the sequences. These are closely related to the *diffraction image* of physical quasicrystals, explained in plain language in [Se]. We imagine shining x-rays or other rays with an appropriate wavelength through a configuration of points given by the sequence or tiling, and collecting the resulting diffraction image. The diffraction image is given mathematically by a measure which indicates the scattering per unit volume. The special case of the Rudin-Shapiro sequence is known to factor onto a sequence which has Lebesgue (totally diffuse) correlation measures. As noted in [HB]: "Amazingly, though the (Rudin-Shapiro) structure is completely deterministic, its two-point correlations are destroyed systematically so that only a constant diffuse background remains in (the measure)". Our construction shows that this can be extended into higher dimensions. Additional physical motivation comes from the study of Ising models with nonperiodic order. The use of one-dimensional block substitutions are used in this context in [He]; higher-dimensional models are a natural next step.

The paper is organized as follows. In Section 2 we set down the notation for $\mathbb{Z}^d$ sequences, put the Rudin-Shapiro sequence into the framework of our substitutions, give the construction of higher-dimensional sequences, and discuss the property which will cause the substitutions to have Lebesgue spectrum. In Section 3 we outline the ergodic theory of substitution sequences and explain how our dynamical systems can be seen as two-point extensions of products of adic transformations. This will be enough to show that there is a nontrivial discrete part of the spectrum in addition to the Lebesgue component mentioned above. In Section 4 we outline how spectral theory can be applied to substitution dynamical systems, and prove that the spectral type of our systems have in part nonsimple Lebesgue spectrum. We conclude in Section 5 with a discussion of some related constructions and questions.

The author would like to thank Boris Solomyak for pointing out the fact that symbol matrices for property-(H) substitutions are known as Hadamard matrices.

2. Definitions

2.1. **$\mathbb{Z}^d$ sequences.** Let $\mathcal{A}$ be a finite set, d a positive integer, and consider a *sequence on* $\mathbb{Z}^d$ to be a function $\mathcal{T} : \mathbb{Z}^d \to \mathcal{A}$. The set of all such sequences is denoted $\mathcal{A}^{\mathbb{Z}^d}$. From a tiling dynamical systems viewpoint, we may wish to consider such a sequence as a *tiling of* $\mathbb{R}^d$ by determining a coloring on the set $\mathcal{A}$ and considering $\mathcal{T}$ the tiling of unit square tiles with lowest corners at $\vec{\jmath} \in \mathbb{Z}^d$ which are colored by $\mathcal{T}(\vec{\jmath})$. Using the tiling model allows us access to some of the results found in [RW, So1, So2], and we will use these freely.

The *translation* or *shift* of $\mathcal{T}$ by some $\vec{k} \in \mathbb{Z}^d$ is the sequence $\mathcal{T} - \vec{k}$ given by $(\mathcal{T} - \vec{k})(\vec{\jmath}) = \mathcal{T}(\vec{\jmath} + \vec{k})$. We consider $\mathcal{T}$ to be *nonperiodic* if whenever $\mathcal{T} - \vec{k} = \mathcal{T}$, then $\vec{k} = \vec{0}$. We say that $\mathcal{T}$ is *almost periodic* if any block B appearing in $\mathcal{T}$ does so with bounded gap: there is a radius R_B such that for any $\vec{\jmath} \in \mathbb{Z}^d$, a copy of B appears in $\{\mathcal{T}(\vec{\jmath} + \vec{k}), |\vec{k}| \leq R_B\}$, where we denote $|\vec{k}| = \max_{i \in 1,\ldots,d}\{|k_i|\}$.

One general method for constructing a self-similar sequence is to specify a substitution rule $\mathcal{S}$ sending each letter of $\mathcal{A}$ to an $l_1 \times l_2 \times ... \times l_d$ block of letters from $\mathcal{A}$. The substitution is then iterated and a limiting sequence is obtained, which, if nonperiodic and almost periodic, will be considered a substitution sequence. It will be useful to frame a well-known example that is a special case of our construction in the notation we will be using.

2.2. **A famous example: the Rudin-Shapiro substitution sequence.** The Rudin-Shapiro sequence is discussed as a substitution in [Q1, Q2], in another format in [MN2], with generalizations in [AL, MT], and in numerous other contexts which differ substantially from our perspective. In the one-dimensional case, the connection to Hadamard matrices is investigated in [AL]. In [Q2], the Rudin-Shapiro substitution is on the alphabet $\mathcal{A} = \{0, 1, 2, 3\}$, and the substitution assigns:

$$\mathcal{S}(0) = 02, \quad \mathcal{S}(1) = 32, \quad \mathcal{S}(2) = 01, \quad \mathcal{S}(3) = 31$$

so that $\mathcal{S}^2(0) = 0201$, $\mathcal{S}^3(0) = 02010232$, and so on. A substitution sequence $\mathcal{T}$ is a limiting sequence for $\mathcal{S}$. To establish that this sequence has some Lebesgue spectrum, a factor of $\mathcal{T}$ on the alphabet $\{-1, 1\}$ is considered, the factor which is obtained by assigning $0, 1 \to 1$ and $2, 3 \to -1$. Details of this analysis are found in [Q2]; our proof of Theorem 4.1 applies to give the same result.

We could instead use the alphabet $\mathcal{A} = \{1, \bar{1}, 2, \bar{2}\}$, where $0 \to 1, 1 \to 2, 2 \to \bar{2}$ and $3 \to \bar{1}$. In this case the substitution becomes:

$$\mathcal{S}(1) = 1\bar{2}, \quad \mathcal{S}(2) = \bar{1}\bar{2}, \quad \mathcal{S}(\bar{2}) = 12, \quad \mathcal{S}(\bar{1}) = \bar{1}2.$$

There are three important things to note about this representation. First, a number now not only underlies two letters in the alphabet, but also indicates in which position those letters can occur in the substitution. Second, the *bar status* of a number (unbarred or barred) tells us whether the symbol will be identified with 1 or -1. Third, considering barred and unbarred to be *opposite* symbols when they have the same underlying number, we see that the substitution of a symbol and its opposite yield opposite blocks. The first and third properties will be utilized in the sequel; the second is what is used to produce a sequence on $\{-1, 1\}$ which has a Lebesgue correlation measure.

We should note that in [Q2], the bar notation is also used, but not the same way: with the alphabet $\{0, 1, 2, 3\}$, that paper puts $\bar{1} = 3$ and $\bar{0} = 2$. Thus the bar status does not determine whether a symbol will be associated to 1 or -1.

2.3. **Our construction of substitution sequences.** Fix a dimension $d \in \mathbb{Z}^+$, (where $\mathbb{Z}^+ = 1, 2, ...$), and *lengths* $l_1, l_2, ..., l_d > 1 \in \mathbb{Z}^+$ for the substitution. The *location set* for the substitution is the d-dimensional array

$$\mathcal{I}^d = \{\vec{j} = (j_1, ..., j_d), j_i \in \{0, 1, ..., l_i - 1\} \text{ for all } i = 1, ..., d\}.$$

The array $\mathcal{I}^d$ has a total of $l_1 \cdot l_2 \cdots l_d = K$ elements that we number $\{1, 2, ..., K\}$ in some order which we then consider fixed; we denote the number given to $\vec{j} \in \mathcal{I}^d$ as $n_{\vec{j}}$. Define the alphabet $\mathcal{A} = \{1, 2, ..., K, \bar{1}, \bar{2}, ..., \bar{K}\}$; every letter in $\mathcal{A}$ is considered to have an obvious *underlying number* which is in $1, 2, ..., K$. We abuse notation and refer to a number in $1, 2, ..., K$ as both a location in $\mathcal{I}^d$ and a letter in $\mathcal{A}$. The *bar status* of a letter is either unbarred or barred; for $e \in \mathcal{A}$ we define $sgn(e) = -1$ if e is barred and $sgn(e) = 1$ otherwise. Barring can be considered to be an order two action on letters, blocks, or entire sequences.

The substitution $\mathcal{S}$ is a map from $\mathcal{A}\times\mathcal{I}^d$ into $\mathcal{A}$. For each $\vec{\jmath} \in \mathcal{I}^d$, the substitution $\mathcal{S}$ assigns a map we will denote $\mathcal{S}_{\vec{\jmath}} : \mathcal{A} \to \mathcal{A}$ according to the following restrictions:

(R1) For each $e \in \mathcal{A}$, $\mathcal{S}_{\vec{\jmath}}(e) = n_{\vec{\jmath}}$ or $\overline{n_{\vec{\jmath}}}$.

(R2) For each $e \in \mathcal{A}$, $\mathcal{S}_{\vec{\jmath}}(e) = \overline{\mathcal{S}_{\vec{\jmath}}(\bar{e})}$.

Restriction (R1) implies that each letter in $\mathcal{A}$ is allowed to appear only in the slot given by its underlying number, while restriction (R2) implies that opposite letters are substituted by opposite blocks. Once $\mathcal{S}$ has been specified for the unbarred elements, the barred elements have also been specified. Figure 1 shows a substitution rule $\mathcal{S}$ in the two-dimensional case with $l_1 = l_2 = 2$, using the conventions that $\mathcal{S}(e)$ is the substitution of $e \in \mathcal{A}$ over all of $\mathcal{I}^2$, and that the origin is at the lower left corner of each substitution block.

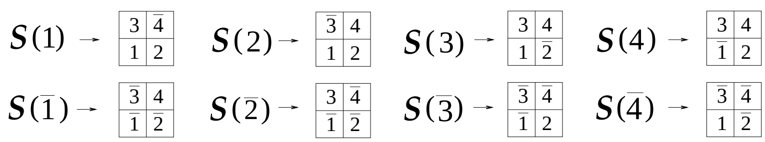

FIGURE 1. A substitution with $l_1 = l_2 = d = 2$ satisfying (R1) and (R2).

We can consider the substitution $\mathcal{S}$ as an action on $\mathcal{A}^{\mathbb{Z}^d}$ by assigning, for $\mathcal{T} \in \mathcal{A}^{\mathbb{Z}^d}, \vec{\jmath} \in \mathbb{Z}^d$, and $\vec{k} \in \mathcal{I}^d$ a sequence $\mathcal{S}\mathcal{T} \in \mathcal{A}^{\mathbb{Z}^d}$, which has at $\vec{w} = (l_1 j_1 + k_1, l_2 j_2 + k_2, ..., l_d j_d + k_d) \in \mathbb{Z}^d$

$$\mathcal{S}\mathcal{T}(\vec{w}) = \mathcal{S}_{\vec{k}}(\mathcal{T}(\vec{\jmath})). \tag{1}$$

Sometimes it will be convenient to consider $\mathcal{S}^m(e)$, the mth iteration of the substitution applied to a letter $e \in \mathcal{A}$. In this case we will consider $\mathcal{S}^m : \mathcal{A} \times (\mathcal{I}^d)^m \to \mathcal{A}$, where $(\mathcal{I}^d)^m = \{\vec{\jmath} = (j_1, ..., j_d), j_i \in \{0, 1, ..., l_i^m - 1\} \text{ for all } i = 1, ..., d\}$. When we write $\mathcal{S}^m_{\vec{\jmath}}(e)$ to denote the $\vec{\jmath}$th element of the mth substitution of the element e, it will be understood that $\vec{\jmath} \in (\mathcal{I}^d)^m$. We will call $\mathcal{S}^m(e)$ a *level-m block of type e* for the substitution.

Definition 2.1. If $\mathcal{S}$ is a substitution defined with restrictions (R1) and (R2) above, a *substitution sequence for $\mathcal{S}$* is a nonperiodic, almost periodic sequence invariant under the action of $\mathcal{S}^k$ for some positive integer k.

Figure 2 shows part of a substitution sequence for the $\mathcal{S}$ from in Figure 1.

2.4. **Condition (H).** A restriction on substitutions of unbarred elements of $\mathcal{A}$:

(H) For every $\vec{\jmath} \neq \vec{k} \in \mathcal{I}^d$, $\Sigma_{n=1}^{K} sgn(\mathcal{S}_{\vec{\jmath}}(n)) sgn(\mathcal{S}_{\vec{k}}(n)) = 0$.

Condition (H) requires that exactly $K/2$ unbarred elements have substitutions with a change in bar status between elements $\vec{\jmath}$ and $\vec{k}$. Note that this property forces at least one l_i to be an even number. Furthermore, it is clear that property (H) extends to the barred elements of $\mathcal{A}$ as well. Condition (H) is the condition from which a Hadamard matrix will be seen to arise in subsection 2.5.

The example pictured in Figure 1 has property (H). In the case $d = 3$ we give another example of a substitution with property (H). We take $l_1 = l_2 = l_3 = 2$, so that there will be 8 locations in the 2 by 2 by 2 grid for substitution. We number them as in Figure 3 and assign the substitution according to assignments (2).

3	4	3	4	3	4	3	4	$\bar{3}$	$\bar{4}$	$\bar{3}$	$\bar{4}$	$\bar{3}$	$\bar{4}$	$\bar{3}$	$\bar{4}$
1	$\bar{2}$	$\bar{1}$	2	1	$\bar{2}$	$\bar{1}$	2	$\bar{1}$	2	1	$\bar{2}$	$\bar{1}$	2	1	$\bar{2}$
3	$\bar{4}$	3	$\bar{4}$	$\bar{3}$	4	$\bar{3}$	4	$\bar{3}$	4	$\bar{3}$	4	3	$\bar{4}$	3	$\bar{4}$
1	2	$\bar{1}$	$\bar{2}$	$\bar{1}$	$\bar{2}$	1	2	$\bar{1}$	$\bar{2}$	1	2	1	2	$\bar{1}$	$\bar{2}$
3	4	$\bar{3}$	$\bar{4}$	3	4	$\bar{3}$	$\bar{4}$	3	4	$\bar{3}$	$\bar{4}$	3	4	$\bar{3}$	$\bar{4}$
1	$\bar{2}$	1	$\bar{2}$	1	$\bar{2}$	1	$\bar{2}$	1	$\bar{2}$	1	$\bar{2}$	1	$\bar{2}$	1	$\bar{2}$
3	$\bar{4}$	$\bar{3}$	4	$\bar{3}$	4	3	$\bar{4}$	3	$\bar{4}$	$\bar{3}$	4	$\bar{3}$	4	3	$\bar{4}$
1	2	1	2	$\bar{1}$	$\bar{2}$	$\bar{1}$	$\bar{2}$	1	2	1	2	$\bar{1}$	$\bar{2}$	$\bar{1}$	$\bar{2}$
3	4	3	4	$\bar{3}$	$\bar{4}$	$\bar{3}$	$\bar{4}$	$\bar{3}$	$\bar{4}$	$\bar{3}$	$\bar{4}$	3	4	3	4
1	$\bar{2}$	$\bar{1}$	2	$\bar{1}$	2	1	$\bar{2}$	$\bar{1}$	2	1	$\bar{2}$	1	$\bar{2}$	$\bar{1}$	2
3	$\bar{4}$	3	$\bar{4}$	3	$\bar{4}$	3	$\bar{4}$	$\bar{3}$	4	$\bar{3}$	4	$\bar{3}$	4	$\bar{3}$	4
1	2	$\bar{1}$	$\bar{2}$	1	2	$\bar{1}$	$\bar{2}$	$\bar{1}$	$\bar{2}$	1	2	$\bar{1}$	$\bar{2}$	1	2
3	4	$\bar{3}$	$\bar{4}$	$\bar{3}$	$\bar{4}$	3	4	3	4	$\bar{3}$	$\bar{4}$	$\bar{3}$	$\bar{4}$	3	4
1	$\bar{2}$	1	$\bar{2}$	$\bar{1}$	2	$\bar{1}$	2	1	$\bar{2}$	1	$\bar{2}$	$\bar{1}$	2	$\bar{1}$	2
3	$\bar{4}$	$\bar{3}$	4	3	$\bar{4}$	$\bar{3}$	4	3	$\bar{4}$	$\bar{3}$	4	3	$\bar{4}$	$\bar{3}$	4
1	2	1	2	1	2	1	2	1	2	1	2	1	2	1	2

(0,0)

FIGURE 2. Four iterations of the letter 1

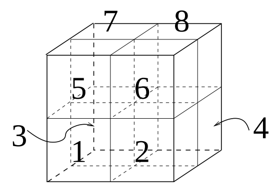

FIGURE 3. The numbering chosen for $\mathcal{I}^3$ with $L = 2$.

$$
\begin{array}{cccccccccc}
1 & \to & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\
2 & \to & 1 & 2 & 3 & 4 & \bar{5} & \bar{6} & \bar{7} & \bar{8} \\
3 & \to & 1 & \bar{2} & 3 & \bar{4} & 5 & \bar{6} & 7 & \bar{8} \\
4 & \to & \bar{1} & \bar{2} & 3 & 4 & \bar{5} & \bar{6} & 7 & 8 \\
5 & \to & \bar{1} & 2 & \bar{3} & 4 & 5 & \bar{6} & 7 & \bar{8} \\
6 & \to & 1 & \bar{2} & \bar{3} & 4 & 5 & \bar{6} & \bar{7} & 8 \\
7 & \to & 1 & 2 & \bar{3} & \bar{4} & \bar{5} & \bar{6} & 7 & 8 \\
8 & \to & \bar{1} & 2 & 3 & \bar{4} & 5 & \bar{6} & \bar{7} & 8
\end{array} \tag{2}
$$

Note the substitution can be "seen" by coloring the subcubes black to denote a barred element and white to denote an unbarred element. In this scenario the substitution on symbols 2, 3, and 4 yields the nonequivalent half-cubes, the substitution on symbols 5, 6, and 7 yields the three nonequivalent quarter-cube pairs, and the last one is a checkerboard.

2.5. **Symbol matrices, Hadamard matrices, dual substitutions.** We find it convenient to write the substitution as one-dimensional strings as we see in Equation 2. Since we have already decided what the underlying numbers will be, we really only need to keep track of the bar status. Thus the substitution given by assignments (2) could be rewritten as the *symbol matrix*:

$$M = \begin{pmatrix} + & + & + & + & + & + & + & + \\ + & + & + & + & - & - & - & - \\ + & - & + & - & + & - & + & - \\ - & - & + & + & - & - & + & + \\ - & + & - & + & + & - & + & - \\ + & - & - & + & + & - & - & + \\ + & + & - & - & - & - & + & + \\ - & + & + & - & + & - & - & + \end{pmatrix}. \tag{3}$$

We may choose to consider the symbols + and − to indicate unbarred and barred elements respectively, or we may choose to consider them as representing +1 or −1 respectively. The matrix M^T defines a d-dimensional substitution of length L which we denote by $\mathcal{S}^T$ and which we call the *dual substitution* to $\mathcal{S}$.

A Hadamard matrix is a square matrix with entries +1 or −1 for which all columns are orthogonal. If a substitution satisfies condition (H), then its symbol matrix M is a Hadamard matrix. Elementary linear algebra shows that the columns of M^T are also orthogonal, and so the dual substitution is also a condition (H) substitution.

Note that the symbol matrices for the Rudin-Shapiro substitution as given in Subsection 2.2 and for the substitution pictured in Figure 1 are

$$\begin{pmatrix} + & - \\ - & - \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} + & + & + & - \\ + & + & - & + \\ + & - & + & + \\ - & + & + & + \end{pmatrix}.$$

3. Ergodic theory and the adic representation

3.1. **The dynamical system associated to $\mathcal{S}$.** Writing $\mathcal{A}^{\mathbb{Z}^d}$ to be the set of all $\mathbb{Z}^d$-sequences on $\mathcal{A}$, the action of translation becomes a $\mathbb{Z}^d$ action on $\mathcal{A}^{\mathbb{Z}^d}$. For any sequences $\mathcal{T}$ and $\mathcal{T}' \in \mathcal{A}^{\mathbb{Z}^d}$ with $\mathcal{T} \neq \mathcal{T}'$ we write $N(\mathcal{T},\mathcal{T}') = \inf\{n \geq 0$ such that $\mathcal{T}(\vec{\jmath}) \neq \mathcal{T}'(\vec{\jmath})$ for some $|\vec{\jmath}| = n\}$, and we define the metric $d(\mathcal{T},\mathcal{T}') = \exp(-N(\mathcal{T},\mathcal{T}'))$. For $\mathcal{T} = \mathcal{T}'$ the distance is defined to be zero. This metric is similar to those seen in [Q1, RW], and it can be seen that $\mathcal{A}^{\mathbb{Z}^d}$ is compact in this metric. Considering a *block* B to be a function from some rectangular subset of $\mathbb{Z}^d$ into the alphabet $\mathcal{A}$, we can define a *cylinder set* $X(B)$ as the set $\{\mathcal{T} \in \mathcal{A}^{\mathbb{Z}^d} : B \text{ is a subblock of } \mathcal{T}\}$. In our metric, these sets form a basis for the topology. The action of translation is continuous with respect to the metric topology. Thus we have a *topological dynamical system* given by $(\mathcal{A}^{\mathbb{Z}^d}, \mathbb{Z}^d)$.

Let $\mathcal{T}_0$ be a fixed sequence in $\mathcal{A}^{\mathbb{Z}^d}$. We create the closed set

$$X = \overline{\{\mathcal{T}_0 - \vec{\jmath} \text{ for } \vec{\jmath} \in \mathbb{Z}^d\}}. \tag{4}$$

The set is compact and translation-invariant, and we have a d-dimensional *subshift* of the space $\mathcal{A}^{\mathbb{Z}^d}$ given by $(X, \mathbb{Z}^d)$. When $\mathcal{T}_0$ is a substitution sequence, the dynamical system is *uniquely ergodic* (see [So1] and references therein): there is a unique probability measure μ which is invariant with respect to the action of $\mathbb{Z}^d$. This measure is the frequency measure in that for any block B and its cylinder set $X(B)$, we find $\mu(X(B))$ is the frequency of occurrence of B in any sequence in X. Note in particular that the results in [So1] imply that $\mu(X(\bar{B})) = \mu(X(B))$ since for any sequence $\mathcal{T}$ with B at a particular location, the sequence $\bar{\mathcal{T}}$ has $\bar{B}$ at that same location. A detailed explanation of how the results concerning the frequency measure in [So1] apply to the situation of $\mathbb{Z}^d$ substitution sequences is found in [Fr].

A consequence of the construction technique used here is that if $\mathcal{S}$ admits a substitution sequence (which is by definition a nonperiodic sequence), then it cannot admit any periodic sequences and $(X, \mathbb{Z}^d, \mu)$ is automatically a minimal system (and $\mathcal{S}$ satisfies the *primitivity* condition given in [Q1, So1]). We call $\mathcal{S}$ a *nonperiodic substitution* and $(X, \mathbb{Z}^d, \mu)$ the *dynamical system associated to* $\mathcal{S}$.

3.2. **Recognizability or the unique composition property.** (Primitive) substitution sequences in one dimension and (primitive) self-similar tilings in any dimension have the property that the substitution or inflation rule that defined them can be "undone" [Mo, So2]. This is the notion of *unique composition* for tilings [So2], which is called *recognizability* [Mo] in the one-dimensional sequence case.

Definition 3.1. A $\mathbb{Z}^d$ substitution $\mathcal{S}$ has the *unique composition* property if for every $\mathcal{T} \in X$, there exists a unique $\mathcal{T}^1 \in X$ and a unique $\vec{\jmath} \in \mathcal{I}^d$ such that $\mathcal{T} = \mathcal{S}(\mathcal{T}^1) - \vec{\jmath}$.

It is possible to use the tiling model to see that the substitutions in this paper are primitive and so X has the unique composition property [So2]. Unique composition implies that for any $M \in \mathbb{Z}^+$ there exists a unique $\mathcal{T}^M \in X$ and $\vec{\jmath} \in (\mathcal{I}^d)^M$ such that $\mathcal{T} = \mathcal{S}^M(\mathcal{T}^M) - \vec{\jmath}$. That is, $\mathcal{T}$ decomposes into level-M blocks in a unique way. Each level-M block B of $\mathcal{T}$ is composed of K level-$(M-1)$ blocks of $\mathcal{T}$ whose positions inside of B are indexed by $\mathcal{I}^d$ in the obvious manner. We define functions $\mathcal{O}_M : X \to \mathcal{I}^d$ as follows: for $\mathcal{T} \in X$, $\mathcal{O}_M(\mathcal{T})$ = the location of the level-$(M-1)$ block of $\mathcal{T}$ containing $\vec{0}$ in its level-M block. Trivially we write $\mathcal{T}^0 = \mathcal{T}$, so that $\mathcal{O}_1(\mathcal{T})$ is the position of $\mathcal{T}(\vec{0})$ in its level-1 block.

3.3. **Factoring onto a product of d odometers.** Dynamical systems corresponding to substitution sequences can be seen as extensions over products of odometers (or adic transformations); we now explain how to see this for our examples. It is possible to show that the systems in this paper and in [Fr] are in fact isomorphic to skew products over odometers by pairing with an appropriate cocycle, but we omit the details and refer the reader to [Fr]. The procedure is fairly standard and is described in the one-dimensional case in [K, Q1].

Consider the sequence space $\Delta_0(i) = \{0, 1, ..., l_i - 1\}^{\mathbb{Z}^+}$. The adic transformation is an invertible transformation on $\Delta(i)$, the subspace of all sequences $x = \{x_n\}$ for which there is no N such that $x_n = 0$ for all $n \geq N$ or $x_n = l_i - 1$ for all $n \geq N$. For $x \in \Delta(i)$, we define $\eta_i(x) = \min\{n : x_n < l_i - 1\}$. The *odometer* or *adic transformation* $V_i : \Delta(i) \to \Delta(i)$ is the invertible map:

$$V_i(x)_n = \begin{cases} 0 & n < \eta_i(x) \\ x_n + 1 & n = \eta_i(x) \\ x_n & n > \eta_i(x). \end{cases} \tag{5}$$

The odometer is aptly named because it augments the first augmentable element, resets the previous ones to 0, and leaves the rest alone. Putting the usual product topology on $\Delta_0(i)$, we can take as a V_i-invariant measure ν_i the measure assigning $1/l_i$ to any cylinder set corresponding to a fixed letter.

Let $\Sigma_0 = \Delta_0(1) \times \Delta_0(2) \times ... \times \Delta_0(d)$, $\Sigma = \Delta(1) \times \Delta(2) \times ... \times \Delta(d)$, and $\nu_\Sigma = \nu_1 \times \nu_2 .... \times \nu_d$. The map $\Psi : (X, \mu) \to (\Sigma_0, \nu_\Sigma)$ defined by $\Psi(\mathcal{T}) = \{\mathcal{O}_M(\mathcal{T})\}_{M=1}^\infty$ can be seen to be a measure-preserving map by considering the measures of cylinder sets. Fixing $\vec{k} \in \mathcal{I}^d$ and $N \in \mathbb{Z}^+$, the cylinder set $[\vec{k}]_N \subset \Sigma$ is the set of all sequences with $\vec{k}$ in the Nth position, and it has measure $(1/l_1) \cdot (1/l_2) \cdots (1/l_d) = 1/K$. We see that $\Psi^{-1}([\vec{k}]_N)$ is all of the sequences in X which have the origin somewhere inside of the level-N block which occupies the $\vec{k}$th spot in its level-$(N+1)$ block, and the measure of this set is also $1/K$.

The mapping Ψ forms a factor map of the dynamical systems $(X, \mathbb{Z}^d, \mu)$ and $(\Sigma, \mathbb{Z}^d, \nu_\Sigma)$ by intertwining the action of translation by the ith basis vector $\vec{e_i}$ on X and the adic V_i on Σ, so that $\Psi(\mathcal{T} - \vec{e_i}) = V_i(\Psi(\mathcal{T}))$ for all $\mathcal{T} \in X$. It is possible to show that Ψ is an almost two-to-one mapping and that there is a skew product representation of $(X, \mathbb{Z}^d, \mu)$ over the odometers. The factor map can be used to show that the product of adic transformations is the maximal equicontinuous factor of the system.

4. The spectral theory of the construction

For $\vec{\jmath} \in \mathbb{Z}^d$, the unitary operator $U^{\vec{\jmath}} : L^2(X, \mu) \to L^2(X, \mu)$ is given by $U^{\vec{\jmath}}(f(\mathcal{T})) = f(\mathcal{T} - \vec{\jmath})$ for all $\vec{\jmath} \in \mathbb{Z}^d$. We can analyze the action of $\mathbb{Z}^d$ on X by consideration of the action of $U^{\vec{\jmath}}$ on $L^2(X, \mu)$, (a unitary operator on a Hilbert space). Recall that the spectral coefficients of an $L^2(X, \mu)$ function are given, for each $\vec{\jmath} \in \mathbb{Z}^d$, by

$$\hat{f}(\vec{\jmath}) = \langle U^{\vec{\jmath}} f, f \rangle = \int_X U^{\vec{\jmath}} f(\mathcal{T}) \overline{f(\mathcal{T})} d\mu(\mathcal{T}). \tag{6}$$

It is known that these coefficients form a positive definite sequence and that therefore there is a unique measure σ_f on the d-torus (see [Ru]) with:

$$\hat{f}(\vec{\jmath}) = \int_{\mathbb{T}^d} z^{\vec{\jmath}} d\sigma_f(z), \tag{7}$$

where $z^{\vec{\jmath}} = z_1^{j_1} \cdot ... \cdot z_d^{j_d}$. For a fixed $f \in L^2(X, \mu)$, we consider the *cyclic subspace* generated by the closed linear span of f as $Z(f) = \overline{span\{U^{\vec{\jmath}}(f) : \vec{\jmath} \in \mathbb{Z}^d\}}$. The action of U restricted to $Z(f)$ is unitarily equivalent to the action $W^{\vec{\jmath}} : L^2(\mathbb{T}^d, \sigma_f) \to L^2(\mathbb{T}^d, \sigma_f)$ given by $W^{\vec{\jmath}}(g(\vec{z})) = z^{\vec{\jmath}} g(\vec{z})$. In the context of one-dimensional dynamical systems, much work has been done; a survey of recent spectral results in this area with an extensive bibliography appears in [Go].

Any Borel measure σ on the torus can be decomposed into at most three mutually singular parts, a discrete part corresponding to purely atomic measure, a singular continuous part which is nonatomic but singular with respect to Lebesgue measure, and a part which is absolutely continuous with respect to Lebesgue measure. Every substitution sequence in $\mathbb{Z}^d$ has functions whose spectral measures are purely discrete, and this is due to the underlying product of adic transformations described above. Some substitution sequences have a mixed spectrum, including the ones discussed in this paper, those discussed in [Fr] and the "table" in [Ro]. What makes the construction given here particularly interesting is that the continuous part of the spectrum is entirely composed of measures equivalent to Lebesgue. We describe the two parts of the spectrum next.

4.1. **The discrete spectrum.** We refer to a constant $\vec{\alpha} \in \mathbb{R}^d$ as an *eigenvalue* of the action $U^{\vec{\jmath}}$ if there is a function $f \in L^2(X, \mu)$ for which

$$U^{\vec{\jmath}} f = \exp(2\pi i(\vec{\alpha} \cdot \vec{\jmath}))f \tag{8}$$

for all $\vec{\jmath} \in \mathbb{Z}^d$. (Here $\vec{\alpha} \cdot \vec{\jmath}$ denotes the usual dot product in $\mathbb{R}^d$.) It is not too difficult to check that the spectral measure of such an eigenfunction f is the atomic measure supported on $\exp(2\pi i\vec{\alpha}) = (\exp(2\pi i\alpha_1), ..., \exp(2\pi i\alpha_d)) \in \mathbb{T}^d$. Every function with an atomic spectral measure is in the linear span of the eigenfunctions, which we denote H_D and call the *discrete component* of the spectrum. Since the spectral measures of distinct eigenfunctions are mutually singular, a nontrivial result from spectral theory is that H_D can be written as a single cyclic subspace.

The discrete component of $L^2(X, \mu)$ for the substitution sequences constructed in this paper are easily defined in two ways. First, it is well-known that the eigenfunctions for the adic system $(\Sigma, \mathbb{Z}^d, \mu_\Sigma)$ are easily specified and have eigenvalues given by $\vec{\alpha} = (\frac{m_1}{l_1^{n_1}}, ..., \frac{m_d}{l_d^{n_d}})$, where m_i and n_i are in $0, 1, 2, ...$ Alternatively, one can refer to [So1] for an explicit construction of the eigenfunctions using the tiling model, which shows the same eigenvalue group. Since $(X, \mathbb{Z}^d, \mu)$ is ergodic, each eigenvalue can have only one normalized eigenfunction, so if $g \in H_D$, then $g(\mathcal{T})$ depends only on the coding of $\mathcal{T}$ into Σ_0. We will make use of that fact below.

4.2. **The continuous spectrum.**

Theorem 4.1. *Let $(X, \mathbb{Z}^d, \mu)$ be the dynamical system associated to a nonperiodic $\mathbb{Z}^d$ substitution satisfying (R1), (R2), and (H). Then there exist $f_1, f_2, ...f_K \in L^2(X, \mu)$, each with spectral measure equal to Lebesgue measure, so that*

$$L^2(X, \mu) = H_D \oplus Z(f_1) \oplus Z(f_2) \oplus \cdots \oplus Z(f_K).$$

Proof. Suppose $e \in \mathcal{A}$ and define the indicator function $\chi_e \in L^2(X, \mu)$ by $\chi_e(\mathcal{T}) = 1$ if $\mathcal{T}(\vec{0}) = e$, and 0 otherwise. For any $n \in \{1, 2, ..., K\}$, we define the indicator function $\chi_{n\cup\bar{n}} = \chi_n + \chi_{\bar{n}}$, and let $f_n \in L^2(X, \mu)$ be given by $f_n(\mathcal{T}) = \chi_n(\mathcal{T}) - \chi_{\bar{n}}(\mathcal{T}) = sgn(\mathcal{T}(\vec{0}))\chi_{n\cup\bar{n}}(\mathcal{T})$.

First we will show that the spectral measure for f_n is equivalent to Lebesgue measure. It is clear that $\hat{f}_n(\vec{0}) = 1/K$. We must show that for all $\vec{\jmath} \neq \vec{0}$,

$$\hat{f}_n(\vec{\jmath}) = \int_X sgn((\mathcal{T} - \vec{\jmath})(\vec{0}))\chi_{n\cup\bar{n}}(\mathcal{T} - \vec{\jmath})sgn(\mathcal{T}(\vec{0}))\chi_{n\cup\bar{n}}(\mathcal{T})d\mu(\mathcal{T}) = 0.$$

This will show that the spectral measure of f_n is equal to a constant times Lebesgue measure; multiplication of f_n by $\sqrt{K}$ will give exactly Lebesgue measure.

If there is an i for which $\vec{\jmath}$ has $j_i \neq 0 \mod l_i$, then $\chi_{n\cup\bar{n}}(\mathcal{T})$ and $\chi_{n\cup\bar{n}}(\mathcal{T} - \vec{\jmath})$ cannot both be nonzero, and automatically we obtain $\hat{f}_n(\vec{\jmath}) = 0$. If $\vec{\jmath}$ has $j_i \equiv 0 \mod l_i$ for all $i = 1, 2, ..., d$, then we must appeal to property (H) to obtain $\hat{f}_n(\vec{\jmath}) = 0$.

We restrict our attention to those $\mathcal{T} \in X$ with $\mathcal{T}(\vec{0})$ and $\mathcal{T}(\vec{\jmath})$ equal to n or $\bar{n}$. For each $M \in 1, 2, ...$, let $X(M)$ be the set of all $\mathcal{T}$ such that M is the smallest integer with $\mathcal{T}(\vec{0})$ and $\mathcal{T}(\vec{\jmath})$ in the same level-M block. The set of sequences that have no such M has measure zero, and so $\int_X \chi_{n\cup\bar{n}}(\mathcal{T}-\vec{\jmath})\chi_{n\cup\bar{n}}(\mathcal{T})d\mu(\mathcal{T}) = \sum_{M=1}^{\infty} \mu(X(M))$.

We need to decompose each $X(M)$ further, specifying exactly where in the level-M block $\mathcal{T}(\vec{0})$ is located for each $\mathcal{T} \in X(M)$. Let $P(M)$ be the set of all locations $\vec{p}$ in a level-M block for which: (1) $\vec{p} + \vec{\jmath}$ is also in the same level-M block, but in a different level-$(M-1)$ block, and (2) $\vec{p}$ and $\vec{p} + \vec{\jmath}$ are at n or $\bar{n}$. Note that $P(M)$ is independent of the type of level-M block and depends only on the coding into the odometer space Σ. For $\vec{p} \in P(M)$ and $l \in \mathcal{A}$, write $X(l, M, \vec{p})$ to be the set of all $\mathcal{T} \in X(M)$ for which $\mathcal{T}(\vec{0})$ is at location $\vec{p}$ in a level-M block of type l. We have $X(M) = \bigcup_{\vec{p}\in P(M)} \bigcup_{l\in\mathcal{A}} X(l, M, \vec{p})$, and moreover we have that f_n is constant over $X(l, M, \vec{p})$. It is clear that $\mu(X(l, M, \vec{p})) = C(M, \vec{p})$ is independent of l, and so

$$\begin{aligned}
\hat{f}_n(\vec{\jmath}) &= \sum_{M=0}^{\infty} \sum_{\vec{p}\in P(M)} \sum_{l\in\mathcal{A}} \int_{X(l,M,\vec{p})} sgn((\mathcal{T} - \vec{\jmath})(\vec{0}))sgn(\mathcal{T}(\vec{0}))d\mu(\mathcal{T}) \\
&= \sum_{M=0}^{\infty} \sum_{\vec{p}\in P(M)} \sum_{l\in\mathcal{A}} sgn(\mathcal{S}^M_{\vec{p}+\vec{\jmath}}(l))sgn(\mathcal{S}^M_{\vec{p}}(l))\mu(X(l, M, \vec{p})) \\
&= \sum_{M=0}^{\infty} \sum_{\vec{p}\in P(M)} C(M, \vec{p}) \sum_{l\in\mathcal{A}} sgn(\mathcal{S}^M_{\vec{p}+\vec{\jmath}}(l))sgn(\mathcal{S}^M_{\vec{p}}(l)).
\end{aligned}$$

By property (H), we know that half of the level-M blocks will have a change in bar status between the level-$(M - 1)$ blocks containing $\mathcal{T}(\vec{0})$ and $\mathcal{T}(\vec{\jmath})$, and half will not. This extends by construction to show that $\sum_{l\in\mathcal{A}} sgn(\mathcal{S}^M_{\vec{p}+\vec{\jmath}}(l))sgn(\mathcal{S}^M_{\vec{p}}(l)) = 0$.

To show that $Z(f_n) \perp Z(f_m)$ whenever $m \neq n$, it suffices to show that for all $\vec{\jmath} \in \mathbb{Z}^d$, we have $\int_X f_n(\mathcal{T} - \vec{\jmath})f_m(\mathcal{T})d\mu(\mathcal{T}) = 0$. This argument mimics the previous one, except that now we define $X(M)$ to be the set of all $\mathcal{T}$ with $\mathcal{T}(\vec{0}) = m$ or $\bar{m}$ and $\mathcal{T}(\vec{\jmath}) = n$ or $\bar{n}$, and setting $P(M)$ and $X(l, M, \vec{p})$ accordingly.

Finally we must show that the cyclic subspaces $Z(f_n)$ along with H_D generate $L^2(X, \mu)$. We have that $H_D \perp Z(f_n)$ for all $n \in 1, 2, ..., K$ since they have mutually singular spectral types, so we need only show that every function in $L^2(X, \mu)$ is in the above decomposition. It will be sufficient to show that the indicator functions for a basis for the topology can be written that way. The cylinder sets formed by the level-M blocks for all $M = 0, 1, 2, ...$ generates the topology.

Note that for any $n \in \{1, ..., K\}$, letting $g_n = \chi_n + \chi_{\bar{n}}$, we have that $\chi_n = 1/2f_n + 1/2g_n$ and $\chi_{\bar{n}} = -1/2f_n + 1/2g_n$. We know g_n is in the discrete spectrum part because it depends only on the coding of $\mathcal{T}$ into Σ_0. So the indicator functions for all individual letters in any location are in $H_D \oplus Z(f_1) \oplus ... \oplus Z(f_K)$. For $e \in \mathcal{A}$, let $\chi_{\mathcal{S}(e)}$ represent the presence of the level-1 block $\mathcal{S}(e)$. For any $n \in \{1, 2, ..., K\}$, create the function

$$f_{\mathcal{S}(n)} = \frac{1}{K} \sum_{\vec{\jmath}\in\mathcal{I}^d} sgn(\mathcal{S}_{\vec{\jmath}}(n))U^{\vec{\jmath}} f_{n_{\vec{\jmath}}}.$$

It is not difficult to check that if $\chi_{\mathcal{S}(n)}(\mathcal{T}) = 1$, then $f_{\mathcal{S}(n)}(\mathcal{T}) = 1$, and that if $\chi_{\mathcal{S}(\bar{n})}(\mathcal{T}) = 1$, then $f_{\mathcal{S}(n)}(\mathcal{T}) = -1$. Recalling the definition of $\mathcal{O}_1(\mathcal{T})$ given in subsection 3.2, if $\mathcal{O}_1(\mathcal{T}) \neq \vec{0}$, we see that $f_{\mathcal{S}(n)}(\mathcal{T}) = 0$, and if $\mathcal{O}_1(\mathcal{T}) = \vec{0}$ but $\mathcal{T}$ is in the cylinder set for $\mathcal{S}(m)$, with $m \neq n$, then property (H) applied to the dual substitution indicates that $f_{\mathcal{S}(n)}(\mathcal{T}) = 0$. Let $g_{\mathcal{S}(n)}$ be the indicator function for the cylinder set given by $\mathcal{S}(n) \cup \mathcal{S}(\bar{n})$; this is again in H_D because it depends only on the coding into Σ_0. Then we get that $\chi_{\mathcal{S}(n)} = 1/2f_{\mathcal{S}(n)} + 1/2g_{\mathcal{S}(n)}$ and $\chi_{\mathcal{S}(\bar{n})} = -1/2f_{\mathcal{S}(n)} + 1/2g_{\mathcal{S}(n)}$, which are clearly in $H_D \oplus Z(f_1) \oplus ... \oplus Z(f_n)$.

Now let $\mathcal{S}^M(n)$ represent the level-M block of type n located at the origin. To get the indicator function for $\mathcal{S}^M(n)$, build the function

$$f_{\mathcal{S}^M(n)} = \frac{1}{K^M} \sum_{\vec{\jmath}\in(\mathcal{I}^d)^M} sgn(\mathcal{S}^M_{\vec{\jmath}}(n))U^{\vec{\jmath}} f_{n_{\vec{\jmath}}}.$$

Defining the discrete spectrum function $g_{\mathcal{S}^M(n)}$ as the indicator function for the cylinder set for $\mathcal{S}^M(n) \cup \mathcal{S}^M(\bar{n})$, we see that $\chi_{\mathcal{S}^M(n)} = 1/2 f_{\mathcal{S}^M(n)} + 1/2 g_{\mathcal{S}^M(n)}$, and $\chi_{\mathcal{S}^M(\bar{n})} = -1/2 f_{\mathcal{S}^M(n)} + 1/2 g_{\mathcal{S}^M(n)}$. □

As a corollary, we can show that the correlation measure of the $\mathbb{Z}^d$ sequence $sgn(\mathcal{T})$ given by $sgn(\mathcal{T})(\vec{\jmath}) = sgn(\mathcal{T}(\vec{\jmath}))$ is Lebesgue measure for every $\mathcal{T} \in X$. The general definition of correlation measure can be found in [Q1] and a discussion of its relationship to diffraction theory can be found in [BH, Hof] and references therein. In our situation we see that the correlation measure of $sgn(\mathcal{T})$ is the measure on the d-torus with coefficients:

$$\gamma(\vec{\jmath}) = \lim_{N\to\infty} \frac{1}{(2N+1)^d} \sum_{|\vec{k}|\le N} sgn(\mathcal{T}(\vec{k}))sgn(\mathcal{T}(\vec{k}-\vec{\jmath})), \tag{9}$$

where we are assured of the existence of the limit using arguments such as are found in [LP, Fr].

Corollary 4.2. *The correlation measure of $sgn(\mathcal{T})$ is Lebesgue measure.*

Proof. Consider the function $F : X \to X$ given by $F = \sum_{n=1}^{K} f_n$, where the f_n's are as in the proof of Theorem 4.1. Then $sgn(\mathcal{T}(\vec{\jmath})) = F(\mathcal{T} - \vec{\jmath})$, and we have that

$$\gamma(\vec{\jmath}) = \lim_{N\to\infty} \frac{1}{(2N+1)^d} \sum_{|\vec{k}|\le N} F(\mathcal{T}-\vec{k})F(\mathcal{T}-(\vec{k}-\vec{\jmath})) = \int_X F(\mathcal{T})F(\mathcal{T}-\vec{\jmath})d\mu(\mathcal{T})$$

by the pointwise ergodic theorem, since μ is the unique ergodic measure. But the integral is the $\vec{\jmath}$th spectral coefficient of F, and so it must be zero. □

5. Related constructions and questions

5.1. **Unraveling substitutions into lower dimensions.** In the assignment (2) of the three-dimensional substitution given in Section 2, we found it convenient to write the assignment for each letter as a string that was 2^3 letters long, rather than attempting to draw the cube $\mathcal{I}^3$ with the letters of the substitution inside. It is immediately clear that writing the substitution in this manner also defines a one-dimensional, property (H) substitution to which Theorem 4.1 applies. An interesting thing to note is that although the one-dimensional action is related to the three-dimensional action, the relationship is complicated. The action of the shift on the one-dimensional sequence winds its way through the three-dimensional space in a space-filling fashion. Conversely, the action of the three-dimensional shift acts on the one-dimensional space in a manner depending on the coding in Σ.

One could also unravel the substitution from three dimensions down to just two, perhaps by slicing the cube $\mathcal{I}^d$ in half and laying the two pieces next to one another in $\mathbb{Z}^2$ to create a 2 by 4 substitution. In general, one should note that any d-dimensional substitution of length L could be unraveled into an m-dimensional substitution, for $1 \le m < d$. Since there are many ways to arrange the elements of $\mathcal{I}^d$ into m-dimensional arrays, there arise many m-dimensional substitutions satisfying property (H). In general, it does not seem that their dynamical systems are isomorphic.

5.2. **Isomorphic Hadamard matrices.** It is conjectured that there are Hadamard matrices of order 0, 2, and $4n$, $n = 1, 2, 3, \dots$, and the conjecture has been verified for all $n < 107$ [VW]. Two Hadamard matrices of the same order are considered isomorphic if there is a signed permutation matrix taking one to the other. It is known that for sizes less than 16, there is only one Hadamard matrix up to isomorphism, but for sizes 16 and above there seem to be several, and this is an area of active research. The result in this paper shows that all Hadamard matrices of a given size produce substitutions with identical spectral decompositions, although they may not even produce conjugate dynamical systems. What differences can there be between the dynamical systems arising from isomorphic but nonequal Hadamard matrices, and can the differences be more substantial between those arising from nonisomorphic Hadamard matrices?

5.3. **Random substitutions.** Fix a dimension d and lengths $l_1, l_2, \dots, l_d$, and consider the set $\Gamma(d, K)$ of all substitutions of this size defined as in Section 2 and satisfying property (H). Clearly this is a finite set of substitutions; under mild conditions it is possible to construct a substitution sequence $\mathcal{T}_0$ which is given by random applications substitutions from this set. Consideration of such random substitutions in one dimension can be seen in [K]. Following the work in that paper, one sees that the eigenvalues of the dynamical system would remain as before d-tuples of l_i-adic numbers in $\mathcal{T}^d$, and it would seem the proof of Theorem 4.1 could be adapted to this situation, since the crux of the argument lies in the fact that property (H) is satisfied on all level-M blocks, which would still be true.

5.4. **The direct product question.** We saw in Section 3 that $(X, \mathbb{Z}^d, \mu)$ is a two-point extension of a direct product of adic transformations. However, it is unknown whether the system is dominated by a direct product of one-dimensional systems. That is, it is unknown whether there exist d one-dimensional systems whose direct product factors onto $(X, \mathbb{Z}^d, \mu)$. Although it is possible for the construction given in this paper to produce such a system, we conjecture that many are not factors of direct products. The system pictured in Figures 1 and 2 would seem to be a candidate for such a system, as it has in the vertical and horizontal directions a periodic sequence. While one spectral property of our systems satisfies a condition on the spectrum of direct products given in [Fi], this does not imply that our system is a direct product. Further work should be done on this problem, because it is an open problem in the study of higher-dimensional substitution sequences in general.

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