DETECTING COMBINATORIAL HIERARCHY IN TILINGS USING DERIVED VORONOÏ TESSELLATIONS

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ABSTRACT. Tilings of $\mathbb{R}^2$ can display hierarchy similar to that seen in the limit sequences of substitutions. Self-similarity for tilings has been used as the standard generalization, but this viewpoint is limited because such tilings are analogous to limit points of constant-length substitutions. To generalize limit points of non-constant-length substitutions, we define hierarchy for infinite, labelled graphs, then extend this definition to tilings via their dual graphs. Examples of combinatorially substitutive tilings that are not self-similar are given. We then find a sufficient condition for detecting combinatorial hierarchy that is motivated by the characterization by F. Durand of substitutive sequences. That characterization relies upon the construction of the "derived sequence"—a recoding in terms of reappearances of an initial block. Following this, we define the "derived Voronoï tiling"—a retiling in terms of reappearances of an initial patch of tiles. Using derived Voronoï tilings, we obtain a sufficient condition for a tiling to be combinatorially substitutive.

1. INTRODUCTION

In one dimension, it can be a simple matter to make a substitution on some finite alphabet and obtain an infinite sequence by repeated iterations of the substitution. A famous example is the *Fibonacci substitution* $0 \to 01, 1 \to 0$, which after a few iterations yields

$$0 \to 01 \to 010 \to 01001 \to 01001010 \to 0100101001001 \cdots .$$

Once a substitutive sequence is created, it can be analyzed using ergodic-theoretic and spectral methods [10].

In higher dimensions, it is unclear how to proceed in the creation of such a hierarchical structure. We are motivated by the premise that tilings of $\mathbb{R}^2$ using a finite "prototile set"are a viable generalization of sequences on a finite alphabet. Unlike the situation for sequences, arbitrary concatenation of prototiles may not result in a tiling at all, so whether or not a given prototile set can actually form a tiling is an important question [16, 23]. Self-similarity for tilings, discussed from differing viewpoints in e.g. [13, 14, 20], has been seen as a generalization of substitutive sequences. But we know that this notion really only corresponds to constant-length substitution, where the replacements of the letters all have the same length.

In the first part of this paper, we create a less geometrically rigid generalization of substitution for sequences. To begin this task, we define what it means to be a substitution on a labelled, plane graph, and what it means for that graph to be a fixed point of the substitution. (The reader is referred to [6] for a different viewpoint on graph substitution). Since every normal tiling of $\mathbb{R}^2$ has an associated labelled,

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plane "dual graph", we can define a combinatorially substitutive tiling (CST) as being one with a substitutive dual graph. Every self-similar tiling is a CST, but not every CST is self-similar; we present three examples of the latter. Unfortunately, few such examples are known.

The work of F. Durand [1] characterizing primitive substitutive sequences motivates the second part of this paper. Given a minimal sequence X and a finite block u from X, the "derived sequence" $D_u(X)$ is defined to be a recoding of X in terms of the occurrences of u in X. The alphabet of $D_u(X)$ corresponds to the "return words" for u in X: the words beginning and ending in u and containing no other copy of u. Thus the derived sequence contains all information about the layout of copies of u inside X, and what goes between these copies. He proves the following characterization.

Theorem (Durand). *X is a primitive substitutive sequence if and only if the number of derived sequences $D_u(X)$ is finite, as u ranges throughout all possible finite initial words of X.*

In order to generalize this result to the tilings case we define "derived Voronoï (DV) tilings". Given a tiling $\mathcal{T}$ and a patch $P \subset \mathcal{T}$, we form a new tiling $\mathcal{T}_P$ using a Voronoï construction on the set of occurrences of P in $\mathcal{T}$. The resulting DV tiling contains information about the layout of translates of P in $\mathcal{T}$ and provides a notion of which translates of P are "neighbors" of one another. (Derived Voronoï tilings are developed in [7] and used to characterize pseudo-self-similar tilings in [8, 9]; a variant of the construction was independently discovered and used to study aperiodic $\mathbb{Z}^d$-actions on Cantor systems in [3].) We define $\mathcal{F}(\mathcal{T}) = \{\mathcal{T}_r : r \geq 0\}$, where P_r is the patch of tiles in $\mathcal{T}$ determined by the ball of radius r around the origin. We prove the following sufficient condition for a tiling to be a CST.

Theorem. *Let $\mathcal{T}$ be a nonperiodic, almost periodic tiling of $\mathbb{R}^2$ for which $\mathcal{F}(\mathcal{T})$ is finite up to combinatorial isomorphism. Then $\mathcal{T}$ is combinatorially substitutive.*

A close inspection of the proof indicates that the DV tiling is a useful tool for detecting hierarchy on a more practical level. Given a tiling $\mathcal{T}$, we need not necessarily construct the entire DV family $\mathcal{F}(\mathcal{T})$. Instead, we need only establish that there exist real numbers r and u, with r sufficiently smaller than u, such that the graphs of $\mathcal{T}_r$ and $\mathcal{T}_u$ are isomorphic. Following the second part of the proof, we can establish that $\mathcal{T}$ is combinatorially substitutive.

The results presented here are based on the author's Ph.D. dissertation [7] written at the University of North Carolina at Chapel Hill under the direction of Karl E. Petersen. Thanks go to him and also Michael U. Kart for many helpful discussions.

2. Definitions

2.1. **Tilings.** We begin with the definitions of prototiles, tiles, and tilings that will be in use throughout this work. Given a set $A \subset \mathbb{R}^2$ homeomorphic to the closed unit disk $\{x \in \mathbb{R}^2 : \|x\| \leq 1\}$ and an integer $l \in \{1, 2, ..., L\}$, define a *prototile* t as the pair (A, l). The *support* of t is $\text{supp}(t) = A$ and the *label* (or *tile type*) of t is $l(t) = l$. We label the prototiles so that we may distinguish prototiles having congruent supports, if that is desirable. Given a prototile t and an $x \in \mathbb{R}^2$, a *tile* T is a pair $(\text{supp}(t) - x, l(t))$, having the obvious support and label. A *prototile set* is a finite set τ of distinctly labelled prototiles. Given a prototile set τ, a collection of tiles

(1) $\mathcal{T} = \{T_j = (\mathrm{supp}(t_{i_j} - x_j), l(t_{i_j})) \text{ for } j \in \mathbb{N}, t_{i_j} \in \tau, \text{ and } x_j \in \mathbb{R}^2\}$,

is a *τ-tiling* if $\mathbb{R}^2 = \bigcup_j \mathrm{supp}(T_j)$ and $\mathrm{int}(T_i) \cap \mathrm{int}(T_j) = \emptyset$ for $i \neq j$. Note that this definition of tiling could be generalized in a variety of ways, including allowing rotations of prototiles or using a higher-dimensional setting. Tilings found in [4] encompass a wide variety of possibilities.

A *$\mathcal{T}$-patch* P is a finite configuration of tiles in $\mathcal{T}$. The *outer patch* of a subset U of $\mathbb{R}^2$ is given by

(2) $$[U]^{\mathcal{T}} = \{T \in \mathcal{T} \text{ such that } \mathrm{supp}(T) \cap U \neq \emptyset\}.$$

A notable type of outer patch is the one associated to a point $y \in \mathbb{R}^2$: the *elementary patch* $[y]^{\mathcal{T}}$. An elementary patch can be a single tile, two tiles meeting along an edge, or several tiles that share a common vertex. Another notable type of outer patch is $[B_R(x)]^{\mathcal{T}}$, the set of all tiles intersecting the closed ball of radius R centered at x.

Tiles (and therefore patches and tilings) can be acted upon by translation: given a tile T and an $x \in \mathbb{R}^2$, define the tile $T - x = (\mathrm{supp}(T) - x, l(T))$. This induces an equivalence relation $T \sim T - x$, which extends naturally to patches and tilings.

Remark 2.1. A tiling space X can be defined as a translation-invariant set of tilings made from a given prototile set τ. Tiling spaces are metrizable and are studied as dynamical systems under the action of translation. For a description of this viewpoint and many dynamical and ergodic-theoretic results, see [8], [11, 12, 15], [17],[20, 21].

In this work, we require three conditions of our tilings: normality, local finiteness, and almost periodicity. *Normality* is defined in [4] as the requirement that all tiles in a tiling be uniformly bounded topological disks that intersect in connected sets. A tiling $\mathcal{T}$ is *locally finite* if for any $R > 0$ there is an integer n and $\mathcal{T}$-patches $P_1, ..., P_n$ such that for any $x \in \mathbb{R}^2$, $[B_R(x)]^{\mathcal{T}}$ is translation equivalent to P_i for some $i \in \{1, ..., n\}$. A tiling $\mathcal{T}$ of $\mathbb{R}^2$ is *periodic* if there exists a basis x_1, x_2 of $\mathbb{R}^2$ so that $\mathcal{T} - x_i = \mathcal{T}$ for $i = 1, 2$. It will be considered *nonperiodic* if there is no nonzero $x \in \mathbb{R}^2$ with $\mathcal{T} - x = \mathcal{T}$. A tiling $\mathcal{T}$ is called *almost periodic* if for any patch $P \in \mathcal{T}$ there is a real number R such that for any $x \in \mathbb{R}^2$ there is a $\mathcal{T}$-patch P' such that $\mathrm{supp}(P') \subset B_R(x)$ and $P' \sim P$. The minimum such R, denoted $R(P)$, is called the *almost-periodicity radius* of P.

2.2. **Graphs.** A *graph* Γ is given by a pair $(V(\Gamma), E(\Gamma))$, where the *vertex set* $V(\Gamma)$ is any nonempty, at most countable set, and $E(\Gamma) \subset \{\{v_1, v_2\} : v_1, v_2 \in V(\Gamma)\}$ is the *edge set* of Γ. Note that at most a single undirected edge connecting any two vertices is allowed by this definition. We will refer to a vertex $v \in V(\Gamma)$ or an edge $e \in E(\Gamma)$ as an *element* of Γ; the set of elements is $\mathcal{E}(\Gamma) = E(\Gamma) \cup V(\Gamma)$. The *order* $||\Gamma||$ of a graph is the cardinality of the vertex set. A graph Γ will be assumed to be *labelled* by a map $l : \mathcal{E}(\Gamma) \to \{1, 2, ...L\}$, where $l(V(\Gamma)) \cap l(E(\Gamma)) = \emptyset$.

A *path* P of *length* n is an ordered sequence of vertices v_0, v_1, ...,v_n in $V(\Gamma)$ such that $\{v_i, v_{i+1}\} \in E$ for all $i \in \{0, 1, ..., n-1\}$; we denote the length of the path by $||P|| = n$. We will always assume that Γ is *connected*: for any $v, w \in V$ with $v \neq w$, there is a path $P = v_0, v_1, ..., v_n$ in Γ with $v_0 = v$ and $v_n = w$. The distance between any distinct vertices v and w in Γ is given by $d(v, w) = \min\{||P|| : P =$

$v, ..., w$ is a path in $\Gamma\}$. If $v = w$ we define $d(v, w) = 0$, so that d forms a metric on Γ.

A *subgraph* $G \subset \Gamma$ is a graph $(V(G), E(G))$ such that $V(G) \subset V(\Gamma)$, $E(G) \subset E(\Gamma)$, and whenever $e = \{v_1, v_2\} \in E(G)$, then $v_1, v_2 \in V(G)$. Let $S \subset V(\Gamma)$ be any subset of the vertex set of Γ, and let E_S be the subset of $E(\Gamma)$ given by $E_S = \{\{v, w\} : v, w \in S \text{ and } \{v, w\} \in E(\Gamma)\}$. The *induced subgraph of* S is $\Gamma_S = (S, E_S)$. The *ball of radius* N *centered at the vertex* v is the subgraph of Γ induced by the set of vertices w for which $d(v, w) \leq N$.

Definition 2.1. Let Γ be a graph and let $\mathcal{P}(\mathbb{R}^2)$ the set of all homeomorphic images of $[0, 1]$ in $\mathbb{R}^2$. A *drawing* of Γ is a function $\mathcal{D} : \mathcal{E}(\Gamma) \to \mathbb{R}^2 \cup \mathcal{P}(\mathbb{R}^2)$ such that

- i.) $\mathcal{D}(v) \in \mathbb{R}^2$ for any vertex $v \in V(\Gamma)$ and $\mathcal{D}(v) \neq \mathcal{D}(w)$ for $v \neq w \in V(\Gamma)$,
- ii.) for $\{v_1, v_2\} \in E(\Gamma)$, $\mathcal{D}(\{v_1, v_2\}) \in \mathcal{P}(\mathbb{R}^2)$ is a simple curve in $\mathbb{R}^2$ that has as endpoints $\mathcal{D}(v_1)$ and $\mathcal{D}(v_2)$.

The graph Γ is *planar* if and only if there is a drawing $\mathcal{D} : \mathcal{E}(\Gamma) \to \mathbb{R}^2$ such that for all edges e and f in $E(\Gamma)$, $\mathcal{D}(e) \cap \mathcal{D}(f) = \mathcal{D}(e \cap f)$. Once a planar graph has been drawn in this fashion, it is called a *plane graph* and posseses additional elements called *facets*. A facet f is a finite subset of $V(\Gamma)$ such that the drawing of the subgraph induced by f encloses a simply connected region, the interior of which contains the drawing of no other elements of Γ. The facet set of Γ is denoted $F(\Gamma)$, and the element set of Γ is expanded to contain it so that $\mathcal{E}(\Gamma) = V(\Gamma) \cup E(\Gamma) \cup F(\Gamma)$. All plane graphs are assumed to posess labellings $l : \mathcal{E}(\Gamma) \to \{1, 2, ...L\}$ with $l(V(\Gamma)), l(E(\Gamma))$, and $l(F(\Gamma))$ pairwise disjoint. Note the inclusion relationship among the elements of a graph: a vertex can be contained in an edge, and both can certainly be contained in a facet.

Let Γ and Γ' be plane graphs and suppose $\Phi : \mathcal{E}(\Gamma) \to \mathcal{E}(\Gamma')$ is a map such that $\Phi(V(\Gamma)) = V(\Gamma')$ and $\Phi(E(\Gamma)) = E(\Gamma')$. We call Φ *inclusion-preserving* if whenever $a, b \in \mathcal{E}(\Gamma)$ and $a \subset b$, then $\Phi(a) \subset \Phi(b)$. We call Φ *label-preserving* if it is inclusion-preserving and if, for all elements $g, h \in \mathcal{E}(\Gamma)$ with $l(g) = l(h)$, it is true that $l'(\Phi(g)) = l'(\Phi(h))$, and in this case we call Γ' a *label factor* of Γ. If Φ is a label-preserving bijection, then it is a *graph isomorphism* and we may write $\Gamma \cong \Gamma'$.

Conversely, given two plane graphs Γ and Γ', a map $\Psi : \mathcal{E}(\Gamma) \to \mathcal{E}(\Gamma')$ is called *inclusion-reversing* if for all $a, b \in \mathcal{E}(\Gamma)$, $\Psi(a) \subset \Psi(b)$ if and only if $a \supset b$. The plane graph Γ is considered *dual* to the plane graph Γ' if and only if there exists an inclusion-reversing bijection $\Psi : \mathcal{E}(\Gamma) \to \mathcal{E}(\Gamma')$ such that for $a, b \in \mathcal{E}(\Gamma)$, $l(a) = l(b)$ if and only if $l'(\Psi(a)) = l'(\Psi(b))$. A plane graph Γ and its dual graph G' are *dually situated* if for every vertex $v \in V(\Gamma)$, the drawing of v in $\mathbb{R}^2$ is inside the region defined by the drawing of the facet that is its image in G'. It is a theorem [4, 22] that connected plane graphs always have duals. Moreover, a plane graph and its dual graph can always be drawn dually situated [4] p. 171-172.

2.3. **Tilings as plane graphs.** We base our definitions on those in [4]. A *vertex* in a tiling $\mathcal{T}$ is a point in $\mathbb{R}^2$ contained in three or more tiles in $\mathcal{T}$, and we denote the vertex set $V(\mathcal{T})$. The *edge set* $E(\mathcal{T})$ is given by the set of all subsets $\{x, y\} \subset V(\mathcal{T})$ for which there exist $T, S \in \mathcal{T}$ with $\{x, y\} = \text{supp}(T) \cap \text{supp}(S) \cap V(\mathcal{T})$. The *facet set* of $\mathcal{T}$ is the set given by $F(\mathcal{T}) = \{\text{supp}(T) \cap V(\mathcal{T}) : T \in \mathcal{T}\}$. Let $\mathcal{E}(\mathcal{T}) = V(\mathcal{T}) \cup E(\mathcal{T}) \cup F(\mathcal{T})$ denote the set of *elements* of $\mathcal{T}$. We will consider $G(\mathcal{T})$ to be the plane graph with this element set, and the drawing provided by the tiling $\mathcal{T}$. Facets in $F(\mathcal{T})$ contain at least three vertices, since we have assumed that $\mathcal{T}$ is a normal tiling.

We choose a labelling for $G(\mathcal{T})$ using elementary patches $[x]^{\mathcal{T}}$. Let $E_1, ..., E_{N(\mathcal{T})}$ be a set of representatives of all of the translation equivalence classes of elementary patches in $\mathcal{T}$. Let e be any element in $\mathcal{E}(\mathcal{T})$. We define the label of e to be

$$l(e) = i \quad \text{if and only if } \bigcap_{x \in e} [x]^{\mathcal{T}} \sim E_i. \tag{3}$$

For a facet $f \in F(\mathcal{T})$ corresponding to the tile $T \in \mathcal{T}$, we see that $\bigcap_{x\in f}[x]^{\mathcal{T}}$ equals T, and so we arrange to have $l(f) = l(T)$, reordering the set $\{E_i\}$ as necessary. For an edge $e = \{v, w\} \in E(\mathcal{T})$, $\bigcap_{x\in e}[x]^{\mathcal{T}} = [v]^{\mathcal{T}} \cap [w]^{\mathcal{T}}$, a patch consisting of two adjacent tiles. Thus the label of e reveals exactly how the tiles connected by e are sitting next to each other.

Two tilings $\mathcal{T}_1, \mathcal{T}_2$ are said to be *combinatorially equivalent* if the graphs $G(\mathcal{T}_1)$ and $G(\mathcal{T}_2)$ are isomorphic as unlabelled graphs, and they are said to be *combinatorially isomorphic* if $G(\mathcal{T}_1)$ and $G(\mathcal{T}_2)$ are isomorphic as labelled graphs. Combinatorially equivalent tilings have the same adjacency structure but are not necessarily made out of prototile sets of the same size. The following fundamental theorem is adapted from [4], p. 169.

Theorem 2.1. *If $\mathcal{T}_1$ and $\mathcal{T}_2$ are normal tilings, then $\mathcal{T}_1$ is combinatorially equivalent to $\mathcal{T}_2$ if and only if there exists a homeomorphism of $\mathbb{R}^2$ taking $\mathcal{T}_1$ onto $\mathcal{T}_2$ ($\mathcal{T}_1$ and $\mathcal{T}_2$ are topologically equivalent).*

We see from the next theorem that it is possible to use dual graphs to define combinatorial equivalence [4] p. 171.

Theorem 2.2. *If each of two tilings $\mathcal{T}_1$ and $\mathcal{T}_2$ is dual to the same tiling $\mathcal{T}^*$, then $\mathcal{T}_1$ and $\mathcal{T}_2$ are combinatorially equivalent.*

Combinatorially isomorphic tilings are combinatorially equivalent and hence topologically equivalent, but the converse does not always hold. Wildly different tilings with the same combinatorial structure but different element labellings are given as examples in Chapter 4 of [4].

It is clear from our original assumptions on tilings that connected, isomorphic, labelled subgraphs of $G(\mathcal{T})$ have translationally congruent drawings. It is not difficult to construct a drawing for the dual graph $\mathcal{G}(\mathcal{T})$ with this property, and that is dually situated with $G(\mathcal{T})$. We will always assume this drawing for $\mathcal{G}(\mathcal{T})$.

3. Graph substitution

We are ready to define graph substitutions, beginning by outlining the sort of graph that admits a substitution. A labelled, plane graph Γ is said to be *locally finite* if for any positive integer N there are only a finite number of balls of radius N appearing in Γ (up to isomorphism). Given an element $a \in \mathcal{E}(\Gamma)$, denote by Γ_a the subgraph of Γ induced by the vertices contained in a. We say that Γ is *consistently labelled* if there exists a family of isomorphisms $I = \{I_{a,b} : \Gamma_a \to \Gamma_b$ with $a, b \in \mathcal{E}(\Gamma)$ and $l(a) = l(b)\}$ such that:

i.) for all $e \in \mathcal{E}(\Gamma_a)$, we have $l(e) = l(I_{a,b}(e))$,
ii.) $I_{a,a}$ is the identity map on Γ_a, and
iii.) If $a, b, c \in \mathcal{E}(\Gamma)$ with $l(a) = l(b) = l(c)$, then $I_{b,c} \circ I_{a,b} = I_{a,c}$.

We say Γ is *substitutable* if it is consistently labelled and locally finite. Note that the dual graph of a locally finite tiling, when drawn and labelled as described in Section 2.3, is automatically substitutable.

Let $\mathcal{E}(\Gamma)$ be the element set of a substitutable graph Γ with label set $\{1, ..., L\}$ and define $\mathcal{G}^*$ to be the set of all finite, connected, plane graphs with label set $\{1, ..., L\}$. Let $\varphi : \mathcal{E}(\Gamma) \to \mathcal{G}^*$ represent an element substitution map such that whenever elements a and b of Γ have the same label, then there exists an isomorphism $\Psi_{a,b} : \mathcal{E}(\varphi(a)) \to \mathcal{E}(\varphi(b))$, satisfying

i.) $\Psi_{a,a}$ is the identity on $\varphi(a)$, and
ii.) if $a, b, c \in \mathcal{E}(\Gamma)$ with $l(a) = l(b) = l(c)$, then $\Psi_{b,c} \circ \Psi_{a,b} = \Psi_{a,c}$.

That is, elements with the same label are substituted by isomorphic graphs, with the precise isomorphism recorded in $\Psi = \{\Psi_{a,b} : a, b \in \mathcal{E}(\Gamma) \text{ with } l(a) = l(b)\}$.

With φ and Ψ as defined above, we require that whenever $a \subset b \in \mathcal{E}(\Gamma)$, there is a one-to-one inclusion-preserving map $\Phi_{a,b}$ in a family of maps Φ taking $\varphi(a)$ onto a subgraph of $\varphi(b)$ and satisfying

i.) If $a \subset b \subset c$ in $\mathcal{E}(\Gamma)$, then $\Phi_{a,c} = \Phi_{b,c} \circ \Phi_{a,b}$,
ii.) For any facet or edge a, $\varphi(a)$ may not have any vertices that are not already in $\Phi_{v,a}(\varphi(v))$ for some vertex $v \subset a$, and
iii.) if $a_i \subset b_i \in \mathcal{E}(\Gamma), i = 1, 2$ with $l(b_1) = l(b_2)$ and $I_{b_1,b_2}(a_1) = a_2$, then

$$\Phi_{a_2,b_2} \circ \Psi_{a_1,a_2}(\varphi(a_1)) = \Psi_{b_1,b_2} \circ \Phi_{a_1,b_1}(\varphi(a_1)). \tag{4}$$

We now use the families Φ and Ψ to define an equivalence relation on the discrete union of vertices $\hat{V} = \bigsqcup_{v \in V(\Gamma)} V(\varphi(v))$. This will establish how the individual substitutions $\varphi(e)$ are put together to form a single graph $\mathcal{S}(\Gamma)$. Let v and w be in $\hat{V}$ such that $v \in \varphi(a)$ and $w \in \varphi(b)$ for a and b in $V(\Gamma)$. Then $v \sim' w$ if and only if there exists $c \in \mathcal{E}(\Gamma)$ with $a, b \subset c$ and $\Phi_{a,c}(v) = \Phi_{b,c}(w)$. Symmetry and reflexivity of the relation are clear; extend $\sim'$ by transitivity to form an equivalence relation. Write the equivalence class of $v \in \hat{V}$ as $[v]$. Because all elements of $\varphi(e)$ for any $e \in \mathcal{E}(\Gamma)$ have vertex set contained in $\hat{V}$, the relation on the vertex set $\hat{V}$ can be extended to an equivalence relation on $\hat{E} = \bigsqcup_{a \in \mathcal{E}(\Gamma)} E(\varphi(a))$. For $e_1 = \{v_1, w_1\}$ and $e_2 = \{v_2, w_2\} \in \hat{E}$, we define $e_1 \sim e_2$ if $\{[v_1], [w_1]\} = \{[v_2], [w_2]\}$. This forms an equivalence relation on $\hat{E}$; we write the equivalence class of an edge as $[e] = \{[v], [w]\}$. Note that it is not possible to extend to facets—the resulting graph may not be planar.

Definition 3.1. The pair $(\varphi, \sim)$ forms a *graph substitution* $\mathcal{S}$ *on* Γ. The graph $\mathcal{S}(\Gamma)$ has vertex set $V(\mathcal{S}(\Gamma)) = \{[w] : w \in \hat{V}\}$ and edge set $E(\mathcal{S}(\Gamma)) = \{[e] : e \in \hat{E}\}$. For any element $a \in \mathcal{E}(\Gamma)$, we can define the subgraph $\mathcal{S}(a)$ of $\mathcal{S}(\Gamma)$ to be the subgraph induced by the vertex set of $\varphi(a)$ under the equivalence relation $\sim$. We say that Γ is a *fixed point* of the substitution $\mathcal{S}$ if $\Gamma \cong \mathcal{S}(\Gamma)$.

If Γ is a fixed point of the graph substitution $\mathcal{S}$, then it is possible to consider powers $\mathcal{S}^m$ of the substitution. For a subgraph G of Γ, we define $\mathcal{S}(G)$ to be the subgraph of Γ that is given by the substitution on each of the elements of Γ. The graph $\mathcal{S}^m(G)$ is defined as being the application of $\mathcal{S}$ to $\mathcal{S}^{m-1}(G)$.

Definition 3.2. Let Γ be a fixed point of the substitution $\mathcal{S}$. We call $v \in V(\Gamma)$ an *expanding vertex* if for every $N \in \mathbb{Z}^+$, there exists an $m \in \mathbb{Z}$ and a vertex

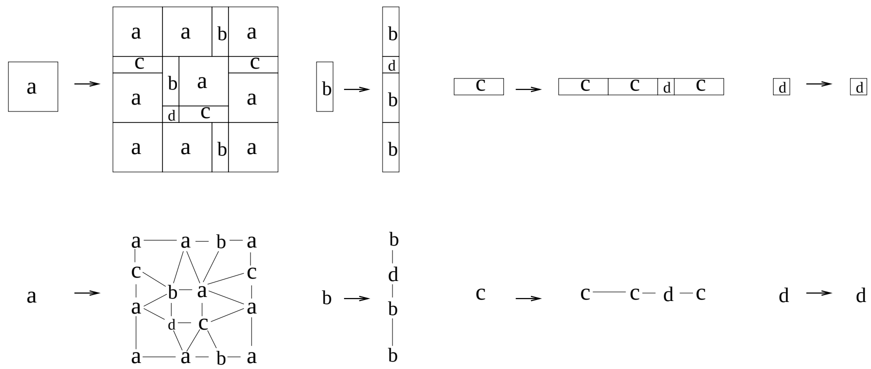

FIGURE 1. A two-dimensional Chacon-type substitution.

w in $\mathcal{S}^m(v)$ so that $B_N(w) \subset \mathcal{S}^m(v)$. We call Γ the *fixed point of an expanding substitution* if it has an infinite number of expanding vertices.

We define combinatorial substitution for tilings in terms of their dual graphs.

Definition 3.3. The tiling $\mathcal{T}$ is called a *combinatorially substitutive tiling* if and only if there exists a fixed point Γ of an expanding substitution for which $\mathcal{G}(\mathcal{T})$ is a label factor of Γ.

Proposition 3.1. *If Γ is connected, then the graph $\mathcal{S}(\Gamma)$ is connected.*

Proof. Let $[v],[w]$ be vertices in $V(\mathcal{S}(\Gamma))$. Choose $a,b \in V(\Gamma)$ so that $v \in \varphi(a)$ and $w \in \varphi(b)$. Since Γ is connected, there is a path $a, c_1, ..., c_n, b$ taking a to b in Γ. Since $\mathcal{S}(a)$ is connected, there is a path taking $[v]$ to an element $[z_1]$ of $\mathcal{S}(a) \cap \mathcal{S}(c_1)$. Since $\mathcal{S}(c_1)$ is connected, there is a path from $[z_1]$ to some $[z_2] \in \mathcal{S}(c_1) \cap \mathcal{S}(c_2)$. This argument can be repeated to connect each $\mathcal{S}(c_i)$ to $\mathcal{S}(c_{i+1})$, producing a path from $[v]$ to $[w]$. □

4. EXAMPLES

We present three examples of combinatorially substitutive tilings that are not self-similar. In Example 1, we present the construction and show that the tiling is combinatorially substitutive. In Examples 2 and 3, we simply present the tilings and leave the details to the reader. Each example illustrates the necessity of certain parts of the definition of graph substitution.

Example 1. We begin with an example that is similar in spirit to the "Chacon $\mathbb{Z}^2$-actions" examined in [5] from an ergodic-theoretic perspective. In Figure 1 we show a substitution rule defined on a set of four tile types, along with the corresponding definition of φ on the vertex set of $\mathcal{G}(\mathcal{T})$. We use the convention that the drawing of a vertex is indicated by its label, and we suppress edge and facet labels.

In Figure 2 we show the result when the substitution is applied to the tile type a two times. One can obtain a tiling of $\mathbb{R}^2$ that is invariant under the substitution by centering a two-by-two array of a tiles at the origin. Holding this patch fixed and applying the substitution ad infinitum will result in an invariant tiling $\mathcal{T}$. In order to show that $\mathcal{G}(\mathcal{T})$ is a fixed point of some graph substitution $\mathcal{S}$, we must

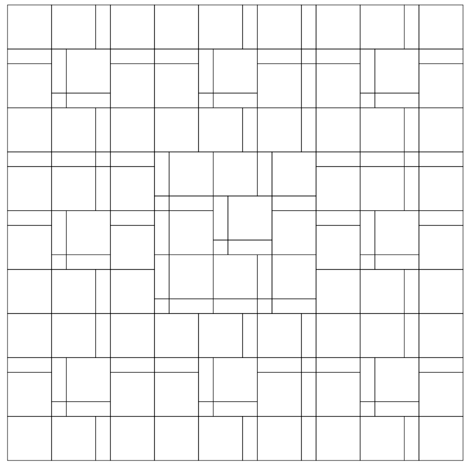

FIGURE 2. Two iterations of the tile a.

establish the map φ on the rest of the element set of $\mathcal{G}(\mathcal{T})$. Once this is done, the families Ψ and Φ are defined by our specified drawing of $\mathcal{G}(\mathcal{T})$.

There are 26 edge types in $\mathcal{T}$, and it is possible to list the substitution of each edge type to obtain the definition of φ on $E(\mathcal{G}(\mathcal{T}))$. We show in Figure 3 a small sampling of the edges along with their substitutions, drawing the elements of $\mathcal{G}(\mathcal{T})$ embedded in their associated tiles. One can check that no new edges are introduced in $\varphi(f)$ for any facet type f that were not already in $\varphi(e)$ for some edge $e \subset f$. Therefore it is somewhat unnecessary to write out the list of facet substitutions in this example.

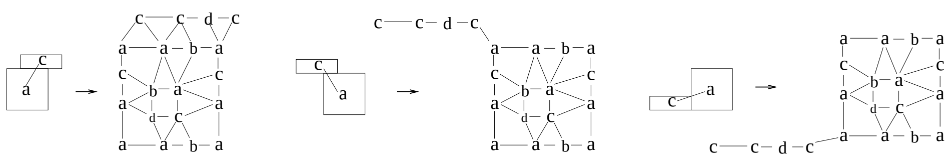

FIGURE 3. The definition of φ for a few edge types.

The isomorphisms $\Psi_{a,b}$ for any a and b with $l(a) = l(b)$ are given by translation of the drawing of $\varphi(a)$ onto that of $\varphi(b)$. The inclusions $\Phi_{a,b}$ for any $a \subset b$ are given by the inclusion of the drawing of $\varphi(a)$ into that of $\varphi(b)$. By construction, these maps satisfy their required properties and so $(\varphi, \sim)$ forms a substitution $\mathcal{S}$ on $\mathcal{G}(\mathcal{T})$ under which it is invariant.

Note that in this example, only vertices labelled a are expanding vertices.

Example 2. Consider a prototile set composed of four unit square tiles, labelled a, b, c, and d, and define a replacement rule $\mathcal{R}$ as in Figure 4. Let F_n denote the nth Fibonacci number (so that $F_1 = 1, F_2 = 1, F_3 = 2$, and so on). Using induction, one can show that $\mathcal{R}^n(a)$ is a F_{n+2} by F_{n+2} array of tiles, $\mathcal{R}^n(b)$ is a F_{n+2} by F_{n+1} array of tiles, $\mathcal{R}^n(c)$ is a F_{n+1} by F_{n+2} array of tiles, and $\mathcal{R}^n(d)$ is a F_{n+1} by F_{n+1} array of tiles. In Figure 5 we show $\mathcal{R}^5(a)$.

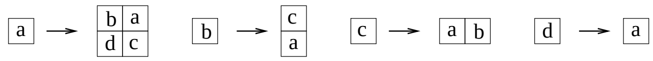

FIGURE 4. A replacement rule for the prototile set.

c	b	a	a	b	b	a	c	a	b	c	b	a
a	d	c	b	a	d	c	a	b	a	a	d	c
a	a	b	d	c	c	b	a	d	c	a	a	b
a	b	c	b	a	a	d	c	b	a	b	a	c
b	a	a	d	c	a	a	b	d	c	d	c	a
d	c	a	a	b	c	b	a	c	b	a	a	b
b	a	b	a	c	a	d	c	a	d	c	b	a
d	c	d	c	a	a	a	b	a	a	b	d	c
a	b	c	b	a	a	b	c	b	a	b	a	c
b	a	a	d	c	b	a	a	d	c	d	c	a
d	c	a	a	b	d	c	a	a	b	c	b	a
b	a	b	a	c	b	a	b	a	c	a	d	c
d	c	d	c	a	d	c	d	c	a	a	a	b

FIGURE 5. Iterating the replacement rule five times on the tile a.

It is tedious but not difficult to verify that there is a tiling $\mathcal{T}$ of $\mathbb{R}^2$ that is invariant under six iterations of $\mathcal{R}$ (it has the array $\begin{matrix} b & d \\ a & c \end{matrix}$ centered at the origin). We may therefore use $\mathcal{R}^6$ to define φ, Ψ, and Φ on the elements of $\mathcal{G}(\mathcal{T})$.

Note that knowing the adjacency type of two tiles is not enough to specify precisely how their substitutions fit together (consider two horizontally adjacent a tiles). Thus it is necessary to relabel the edges of $\mathcal{G}(\mathcal{T})$ to form a graph $\mathcal{G}'$ on which a substitution $\mathcal{S}$ can be defined.

Example 3. This example is, to the best of our knowledge, an original tiling. The prototile set has twelve tiles, six labelled $a - f$ and their reflections about the line $y = -x$. We show the first six prototiles and their substitutions in Figure 6; the substitutions for the reflected prototiles are just the corresponding reflections of these. There is a tiling $\mathcal{T}$ that is invariant under two iterations of this substitution,

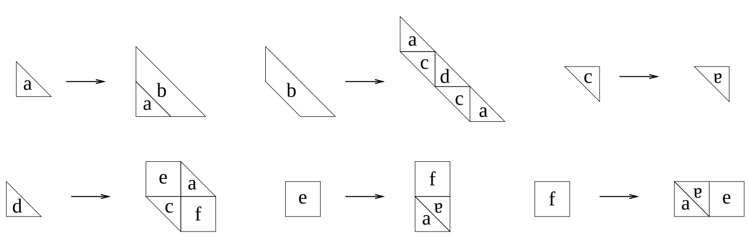

FIGURE 6. Six of the prototiles with their replacements.

with a fixed patch at the origin shown in Figure 7.

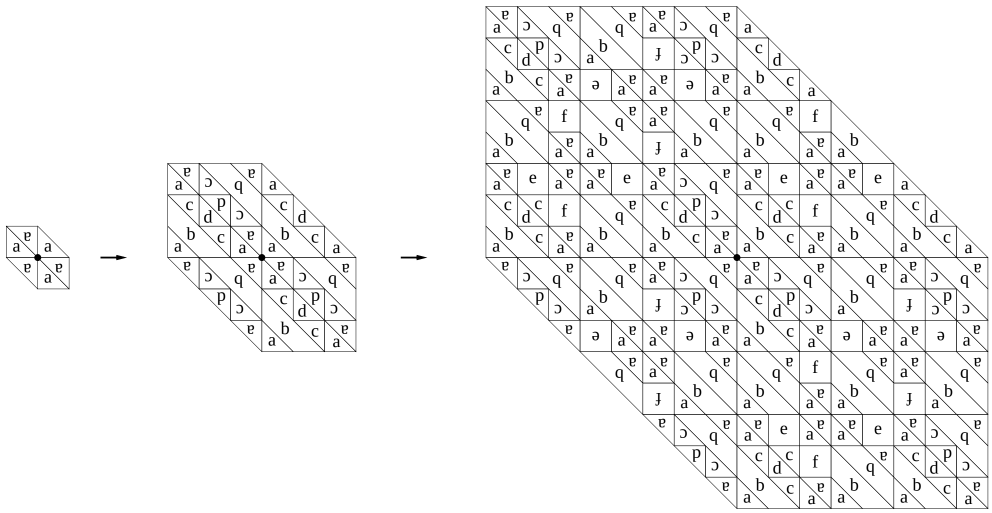

FIGURE 7. Part of $\mathcal{T}$ near the origin.

In this example it is not necessary to consider a relabelling of the dual graph $\mathcal{G}(\mathcal{T})$; the standard labelling is enough to specify the graph substitution. The importance of defining φ on the facet set of $\mathcal{G}(\mathcal{T})$ as well as the edge and vertex set is illustrated by looking carefully at one of the facets that occurs in $\varphi^2(e)$. The facet, shown with dotted lines in Figure 8, has an extra edge in its substitution (also shown with a dotted line) that is not present in any of the substituted edges or vertices that comprise the facet.

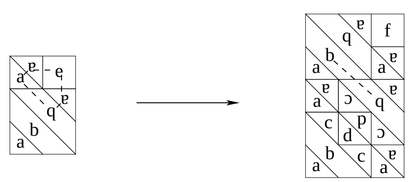

FIGURE 8. A facet of $\mathcal{G}(\mathcal{T})$ that generates an extra edge when substituted.

5. DERIVED VORONOÏ TILINGS

In the introduction, a result of F. Durand [1] characterizing limit sequences of symbolic substitutions was discussed. In the characterization, sequences are recoded in terms of *return words* of a fixed prefix u—words beginning and ending in u and containing no other copy of u. We define a similar process for tilings.

Fixing a nonempty patch P in $\mathcal{T}$, we define the *locator set* $\mathcal{L}_P$ to be

$$\mathcal{L}_P = \{q \in \mathbb{R}^2 : \text{ there exists } P' \subset \mathcal{T} \text{ with } P = P' - q\}. \tag{5}$$

The elements of this set pinpoint the locations of all equivalent copies of P in the tiling $\mathcal{T}$. Since the tilings under consideration here are assumed to be locally finite and almost periodic, $\mathcal{L}_P$ forms a *Delaunay set* [18]: a relatively dense set whose elements are uniformly bounded away from each other. This is exactly the type of set for which it is possible to form a normal Voronoï tessellation [18]. The *Voronoï cell* for $q \in \mathcal{L}_P$ is given by $V_q = \{x \in \mathbb{R}^2 \mid d(x,q) \leq d(x,q') \text{ for all } q' \in \mathcal{L}_P\}$. We call two locator points *neighbors* if their Voronoï cells share edges.

Lemma 5.1. *Let $\mathcal{T}$ be a tiling of $\mathbb{R}^2$, let P be a $\mathcal{T}$-patch, and let R be the almost-periodicity radius of P. Then for any $q \in \mathcal{L}_P$ and Voronoï cell $V_q \in \mathcal{T}_P$, we have that $V_q \subset B_R(q)$. All points that are neighbors of q are contained in $B_{2R}(q)$.*

Proof. Let $w \in V_q$ so that $d(w,q) \leq d(w,q')$ for all $q' \in \mathcal{L}_P$. If $d(w,q) > R$, then $d(w,q') > R$, so there are no copies of P in $B_R(w)$. This contradiction shows that for all $w \in \text{supp}(V_q), d(w,q) \leq R$. □

For any $q \in \mathcal{L}_P$ there will be a *return tile* t_q with support V_q and label to be described below. Fix an $R \geq 2R(P)$. The set of $\mathcal{H}_P(\mathcal{T}, R)$*-patches* is given by $\{[B_R(q)]^{\mathcal{T}} : q \in \mathcal{L}_P\}$. Considering both $\mathcal{T}$ and R fixed, we refer only to the set $\mathcal{H}_P$. This set has a finite number of patches up to translation. Let $H_1, H_2, \ldots H_{N(P)}$ denote representatives of the equivalence classes of patches in $\mathcal{H}_P$.

Definition 5.1. For $q \in \mathcal{L}_P$, the *return tile* t_q is defined to be the tile with support $\text{supp}(t_q) = V_q$ and label $l(t_q) = i$, where $[B_R(q)]^{\mathcal{T}} \sim H_i$. A *derived Voronoï tiling for the patch* P is given by

$$\mathcal{T}_P(R) = \bigcup_{q \in \mathcal{L}_P} t_q\,. \tag{6}$$

If $R(P)$ is the almost periodicity radius, denote the DV tiling $\mathcal{T}_P(2R(P))$ as $\mathcal{T}_P$.

The DV tiling $\mathcal{T}_P$ has the smallest possible label set that allows it to be mutually locally derivable from $\mathcal{T}$, and that implies that the dynamical systems associated to $\mathcal{T}$ and $\mathcal{T}_P$ are topologically conjugate [7, 8]. Of importance to this work is the fact that the dual graph of $\mathcal{G}(\mathcal{T}_P)$ is a label factor of $\mathcal{G}(\mathcal{T}_P(R))$ whenever $R \geq 2R(P)$. Note that $\mathcal{T}_P(R)$ inherits almost periodicity and local finiteness from $\mathcal{T}$. Additionally, the tile geometry has these simple known properties (see [18]):

Proposition 5.2. *Let $\mathcal{L}$ be a Delaunay set in $\mathbb{R}^d$, and let $\mathcal{T}(\mathcal{L})$ be the Voronoï tiling of $\mathcal{L}$. Then*

- i.) *The tiles of a $\mathcal{T}(\mathcal{L})$ are convex polytopes that intersect along whole faces; no two tiles have a common interior point;*
- ii.) *the points of $\mathcal{L}$ whose Voronoï tiles share a vertex v lie on a sphere, centered at v, that has no points of $\mathcal{L}$ in its interior.*

The field of computational geometry has provided a variety of algorithms for constructing the Voronoï tessellations of point sets in several dimensions. A convenient algorithm for local construction is to construct the perpendicular bisectors of the line segments qq', for $q, q' \in \mathcal{L}_P$. The smallest open convex region containing q and bounded by the bisectors is the interior of the tile t_q.

For a tiling $\mathcal{T}$ of the plane and a fixed $\mathcal{T}$-patch P, the graph $G(\mathcal{T}_P) = G_P$ has vertices and edges that correspond to the vertices and edges of the tiles. Since the tiles intersect along whole edges, we call $\mathcal{T}_P$ an *edge-to-edge* tiling. Translationally equivalent tiles have the same number of adjacent tiles in the tiling, so vertices in the dual graph of $\mathcal{T}_P$ with the same label have the same degree.

We assume the following drawing of the dual graph $\mathcal{G}(\mathcal{T}_P) = \mathcal{G}_P$. The vertex set of $\mathcal{G}_P$ is drawn as $\mathcal{L}_P$, so that $q \in V(\mathcal{G}_P)$ corresponds to the facet given by t_q in G_P. If $\text{supp}(t_q) \cap \text{supp}(t_{q'}) = e$ is an edge in G_P, the edge e is perpendicular to the line segment qq', which we take as the drawing of the dual edge in $\mathcal{G}_P$. With this drawing the two graphs are dually orthogonally situated. As noted earlier, the dual graph $\mathcal{G}_P$ is also substitutable.

Remark 5.1. A Delaunay (or Dirichlet) triangulation of a point set $\mathcal{L}$ is a triangulation of the plane using the point set $\mathcal{L}$ as vertices; it has the property that the triangles are as close to being equiangular as possible. It follows from the discussion in [2], p.301-302, that the dual graph $\mathcal{G}_P$ is a Delaunay triangulation of $\mathcal{L}_P$ if and only if the degree of every vertex in G_P is 3. Otherwise, $\mathcal{G}_P$ is a *Delaunay pretriangulation* [19]: it can be "completed" to form a Delaunay triangulation by adding some extra edges.

6. Hierarchy and Derived Voronoï Tilings

Given a fixed tiling $\mathcal{T}$, we consider the family of DV tilings of central patches of the form $P_r = [B_r(0)]^{\mathcal{T}}, r \geq 0$. We truncate the notation so that the derived Voronoï tiling $\mathcal{T}_{P_r}$ is simply $\mathcal{T}_r$, $\mathcal{H}_{P_r}$ is simply $\mathcal{H}_r$, and so on. Let

$$\mathcal{F}(\mathcal{T}) = \{\mathcal{T}_r \text{ such that } r \in [0, \infty)\}. \tag{7}$$

Define $R(\mathcal{T}_r) = \max\{m \in \mathbb{R} \text{ such that } B_m(q) \subset V_q \text{ for all } q \in \mathcal{L}_r\}$. This measure of the minimum tile size in $\mathcal{F}(\mathcal{T})$ has the following property.

Lemma 6.1. *As $r \to \infty$, $R(\mathcal{T}_r) \to \infty$.*

Proof. In search of a contradiction, suppose that there is an $R \in \mathbb{R}$ such that for all $r \in \mathbb{R}$ there is a $q \in \mathcal{L}_r$ with $B_R(q) \not\subset \text{supp}(t_q)$. For any such r and q, there must exist a $q' \in \mathcal{L}_r$ with $\|q - q'\| \leq 2R$. By local finiteness of the tiling $\mathcal{T}_0$, there are only a finite number of vectors $q - q'$ with $\|q - q'\| \leq 2R$ and $q, q' \in \mathcal{L}_0$. Since $\mathcal{L}_s \subset \mathcal{L}_0$ for all $s \geq 0$, there are only a finite number of vectors $q - q'$ of modulus not exceeding $2R$ with q, q' in $\mathcal{L}_s$. This implies that there is a $z \in \mathbb{R}^2$ with $\|z\| \leq 2R$ such that there are $q, q' \in \mathcal{L}_r$ with $z = q - q'$ for infinitely many r.

We will show that for all $T \in \mathcal{T}$, $T - z \in \mathcal{T}$, establishing that $\mathcal{T} - z = \mathcal{T}$ and contradicting the nonperiodicity of $\mathcal{T}$. Choose $r \in \mathbb{R}$ such that $\text{supp}(T)$ and $\text{supp}(T - z)$ are contained in $B_r(0)$. We have that $T \in P_r$ and must show that $T - z \in P_r$. Choose q and $q' \in \mathcal{L}_r$ such that $q - q' = z$. Then $P_r + q \subset \mathcal{T}$ and $P_r + q' \subset \mathcal{T}$; in particular $T + q \in \mathcal{T}$ and $T + q' \in \mathcal{T}$. But $T + q' = T + (q - z)$, so $(T - z) + q \in P_r + q$ by choice of r. Therefore $T - z \in P_r$, and hence in $\mathcal{T}$, as desired. □

So we see that $\mathcal{F}(\mathcal{T})$ is an infinite family and therefore is likely to have an infinite number of combinatorial isomorphism classes. If it does not, this implies combinatorial hierarchy in the original tiling $\mathcal{T}$.

Theorem 6.2. *Let $\mathcal{T}$ be a nonperiodic, almost periodic tiling of $\mathbb{R}^2$ for which $\mathcal{F}(\mathcal{T})$ is finite up to combinatorial isomorphism. Then $\mathcal{T}$ is combinatorially substitutive.*

Proof. The proof is in three steps. The first step is to show that there exist certain numbers r and u associated to a piecewise linear homeomorphism h that takes $\mathcal{T}_r$ onto $\mathcal{T}_u$. The second step is to relabel the tiles of $\mathcal{T}$ in terms of $\mathcal{T}_r$, producing a tiling $\mathcal{T}'$ that factors onto $\mathcal{T}$. The third step is to use the map h to help establish an expanding substitution $\mathcal{S}$ on the graph $\mathcal{G}(\mathcal{T}')$. Since $\mathcal{G}(\mathcal{T})$ is a label factor of $\mathcal{G}(\mathcal{T}')$, showing $\mathcal{G}(\mathcal{T}')$ is a fixed point of $\mathcal{S}$ will finish the proof.

Since there are a finite number of DV tilings up to combinatorial isomorphism, there is an $r > 0$ for which infinitely many P_u have DV tilings that are combinatorially isomorphic to $\mathcal{T}_r$. Fix such an r. By Lemma 6.1 we know that the size of the tiles of $\mathcal{T}_u$ goes to infinity as u does. Letting $R(s)$ denote the almost periodicity radius of P_s for any $s \in [0, \infty)$, choose u such that $R(u) \geq 3R(r)$ and $R(\mathcal{T}_u) \geq 2R(r)$. Since $\mathcal{T}_u$ and $\mathcal{T}_r$ are combinatorially isomorphic, by Theorem 2.1 there is a homeomorphism $h : \mathbb{R}^2 \to \mathbb{R}^2$ taking $\mathcal{T}_r$ onto $\mathcal{T}_u$.

The high degree of geometric structure in a Voronoï tessellation allows us to choose the homeomorphism h that takes $\mathcal{T}_r$ to $\mathcal{T}_u$ to be a piecewise linear map. The nature of the map is described below and is pictured in Figure 9. To each

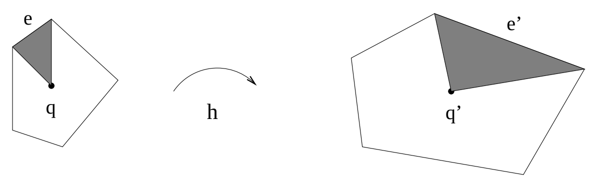

FIGURE 9. The map h is linear on the triangular pieces.

labelled edge e contained in the tile $t_q, q \in \mathcal{L}_r$, associate the triangle with corners at q and at the ends of e. When t_q is mapped onto its image $t_{q'}$ in $\mathcal{T}_u$, this triangle can be mapped via a linear map onto the one defined similarly for q' and the associated edge e'. Together these linear maps on the triangles define piecewise a homeomorphism h of $\mathbb{R}^2$ taking $\mathcal{T}_r$ to $\mathcal{T}_u$ such that $\mathcal{L}_r$ is mapped onto $\mathcal{L}_u$.

To form $\mathcal{T}'$, we relabel the tiles of $\mathcal{T}$ by their relationship to the tiling $\mathcal{T}_r$. For a prototile $t \in \tau$, there are a finite number of ways (up to translation equivalence) that a copy of t in $\mathcal{T}$ can be in the $\mathcal{H}_r$-patch of any $\mathcal{T}_r$-tile it intersects. Make a new label list for t given by $\{(l(t), i), i = 1..n(t)\}$, so that each copy of t appearing in $\mathcal{T}$ is given an integer i which uniquely identifies its position with respect to $\mathcal{T}_r$. Let $\tau' = \{t_i = (\text{supp}(t), (l(t), i)) : t \in \tau \text{ and } i \in 1, ..., n(t)\}$. The relabelling of $\mathcal{T}$ by this label set defines the tiling $\mathcal{T}'$, with prototile set τ', and it is clear that $\mathcal{G}(\mathcal{T})$ is a label factor of $\mathcal{G}(\mathcal{T}')$.

We use h to define a substitution on the graph $\mathcal{G}(\mathcal{T}')$. For any vertex $v_T \in \mathcal{E}(\mathcal{G}(\mathcal{T}'))$ corresponding to the tile T in $\mathcal{T}'$, let the vertex set of $\varphi(v_T)$ be given by

$$V(\varphi(v_T)) = \{v_{T_i} : T_i \in [h(\text{supp}(T))]^{\mathcal{T}'}\}.$$

Define the graph $\varphi(v_T)$ to be the induced subgraph of $\mathcal{G}(\mathcal{T}')$ given by this vertex set. Figure 10 depicts a $\mathcal{T}'$-patch creating $V(\varphi(v_T))$.

To show that this vertex substitution is well-defined, we must show that if $l(T_0) = l(T_1)$ for any T_0, T_1 in $\mathcal{T}'$, then $\varphi(v_{T_0}) \cong \varphi(v_{T_1})$. We will do this by choosing a $\mathcal{T}_r$-tile which intersects the support of T_0 and establishing that the substitution on v_{T_0} is entirely determined by the $\mathcal{H}_u$-patch of the corresponding tile in $\mathcal{T}_u$. Due to the labeling scheme for tiles in $\mathcal{T}'$, this will prove that the substitution on v_{T_1} is defined by a translationally equivalent patch of tiles, proving the desired result.

Choose $q \in \mathcal{L}_r$ so that $\text{supp}(T_0) \cap \text{supp}(t_q) \neq \emptyset$. Let ρ be the maximum diameter of any prototile in τ', and let λ be the maximum modulus of any eigenvalue of any of

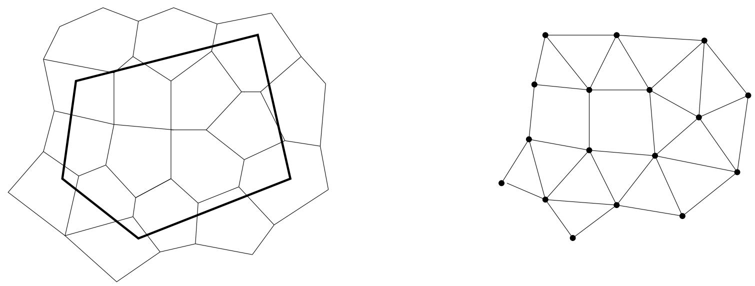

FIGURE 10. How $h(\mathrm{supp}(T))$ determines $\varphi(v_T)$.

the pieces of the map h. It is clear that since the map h, in the most extreme case, would take the interior radius $R(\mathcal{T}_r)$ onto the exterior radius $R(u)$, we will have $\lambda \leq R(u)/R(\mathcal{T}_r)$. As shown in Figure 11, we have that $h(\mathrm{supp}(T)) \subset B_{R(u)+\lambda\rho}(h(q))$, so $[h(\mathrm{supp}(T))]^{\mathcal{T}'} \subset B_{R(u)+\lambda\rho+\rho}(h(q))$. We may assume, by Lemma 6.1, that r is

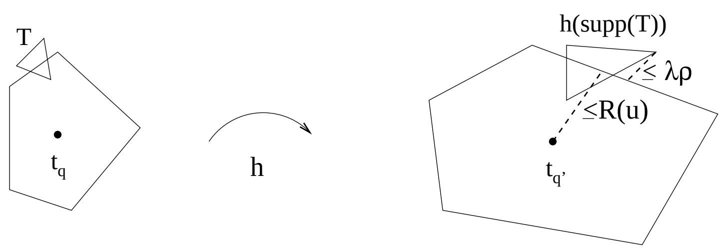

FIGURE 11. The intersection of $h(\mathrm{supp}(T))$ with the $\mathcal{T}_u$-tile t'_q.

sufficiently large to ensure that $\rho(1/R(\mathcal{T}_r) + 1/R(u)) \leq 1$, so that $\rho(R(u)/R(\mathcal{T}_r) + 1) \leq R(u)$, making $\lambda\rho + \rho \leq R(u)$. Thus $\varphi(v_{T_0})$ is determined by a $\mathcal{T}'$-patch which is entirely contained within the $\mathcal{H}_u$-patch of $h(q)$. Since $l(T_0) = l(T_1)$, we now have that $\varphi(v_{T_1})$ is created by a translationally equivalent patch of tiles, proving that the substitution is well-defined.

Similarly, the substitution on any element $e \in \mathcal{E}(\mathcal{G}(\mathcal{T}'))$ is given by the induced subgraph of the following vertex set in $\mathcal{G}(\mathcal{T}')$:

$$V(\varphi(e)) = \{v_{T_i} : T_i \in \bigcup_{v_t \subset e} [h(\mathrm{supp}(t))]^{\mathcal{T}'}\} \tag{8}$$

This substitution can be shown to be well-defined using analogous estimates as were used for the vertex substitution.

Next we exhibit the families of maps Ψ and Φ that coordinate the substitutions on elements. Let $a, b \in \mathcal{E}(\mathcal{G}(\mathcal{T}'))$ with $l(a) = l(b)$; by definition the subgraphs $\varphi(a)$ and $\varphi(b)$ are labelled-graph isomorphic (and each graph is connected). Since we have assumed the drawing for $\mathcal{G}(\mathcal{T}')$ as specified in subsection 2.3, we have that $\varphi(a)$ is translationally congruent to $\varphi(b)$. Let the map $\Psi_{a,b}$ be given by the translation mapping between the two subgraphs. For any $a \subset b \in \mathcal{E}(\mathcal{G}(\mathcal{T}'))$, we can define the map $\Phi_{a,b} : \mathcal{E}(\varphi(a)) \to \mathcal{E}(\varphi(b))$ by inclusion, since the vertex set of

$\varphi(a)$ is contained in that of $\varphi(b)$ by definition. It is clear that the families Φ and Ψ commute as required.

This establishes that there is a substitution $\mathcal{S}$ on $\mathcal{G}(\mathcal{T}')$ given by the map φ along with the family of isomorphisms Ψ and the family of inclusion maps Φ. All subgraphs were induced from $\mathcal{G}(\mathcal{T}')$ on its vertex set, so $\mathcal{S}(\mathcal{G}(\mathcal{T}')) \subset \mathcal{G}(\mathcal{T}')$. Since the mapping h covers $\mathbb{R}^2$, every element of $\mathcal{G}(\mathcal{T}')$ is contained in $\mathcal{S}(\mathcal{G}(\mathcal{T}'))$, proving that $\mathcal{G}(\mathcal{T}') = \mathcal{S}(\mathcal{G}(\mathcal{T}'))$. So $\mathcal{G}(\mathcal{T}')$ is a fixed point of the substitution $\mathcal{S}$.

Let T be any tile in $\mathcal{T}'$ which intersects $\mathcal{L}_r$ at a point q. Then the support of $[h(\mathrm{supp}(T)]^{\mathcal{T}'}$ contains an open set U around the point $h(q) \in \mathcal{L}_u$; i.e. the patch creating $\mathcal{S}(v_T)$ contains U. So the $\mathcal{T}'$-patch creating $\mathcal{S}^2(v_T)$ contains $h(U)$, and in general the $\mathcal{T}'$-patch creating $\mathcal{S}^n(v_T)$ contains $h^{n-1}(U)$. The map h is a piecewise linear map which expands outwards from elements of $\mathcal{L}_r$, by the assumption that $R(\mathcal{T}_u) \geq 2R(r)$. Thus given any $N \in \mathbb{Z}^+$, there is an $m \in \mathbb{Z}^+$ with $h^m(U)$ sufficiently large so that $[h^m(U)]^{\mathcal{T}'}$ induces a subgraph of $\mathcal{G}(\mathcal{T}')$ that contains $B_N(h^m(q))$. Hence $\mathcal{S}^m(v_T)$ contains $B_N(h^m(q))$, and since this can be done for any $N \in \mathbb{Z}^+$, we have shown that v_T is an expanding vertex. This argument can be applied to any tile in $\mathcal{T}'$ that intersects $\mathcal{L}_r$, so there are infinitely many expanding vertices in $\mathcal{G}(\mathcal{T}')$. This proves that $\mathcal{G}(\mathcal{T}')$ is the fixed point of an expanding substitution, and hence $\mathcal{T}$ is a combinatorially substitutive tiling. ☐

Suppose $\mathcal{T}$ is a nonperiodic, almost periodic tiling of $\mathbb{R}^2$ for which $\mathcal{F}(\mathcal{T})$ is finite up to combinatorial isomorphism. In the beginning of the proof of Theorem 6.2 we showed that there exist numbers r, u such that $\mathcal{T}_r$ is combinatorially isomorphic to $\mathcal{T}_u$ with the isomorphism specified by a piecewise linear map h. Moreover, r and u were chosen so that $R(u) \geq 3R(r)$ and $R(\mathcal{T}_u) \geq 2R(r)$. We consider $\mathcal{T}, r, u, \mathcal{T}_r, \mathcal{T}_u$ and h fixed in the following proposition.

Proposition 6.3. *The tiling $\mathcal{T}_r$ is a combinatorially substitutive tiling. Moreover, $\mathcal{G}_r$ is a fixed point of an expanding substitution.*

Proof. We define the substitution φ for the element $q \in V(\mathcal{G}_r)$ to be the subgraph of $\mathcal{G}_r$ induced by

$$V(\varphi(q)) = \mathcal{L}_r \cap [h(\mathrm{supp}(t_q))]^{\mathcal{T}_r} = \mathcal{L}_r \cap [\mathrm{supp}(t_{h(q)})]^{\mathcal{T}_r}. \tag{9}$$

Similarly, the substitution on any element $e \in \mathcal{E}(\mathcal{G}_r)$ can be given by the induced subgraph of

$$V(\varphi(e)) = \mathcal{L}_r \cap \bigcup_{q \subset e \cap \mathcal{L}_r} [h(\mathrm{supp}(t_q))]^{\mathcal{T}_r}. \tag{10}$$

Suppose $l(a) = l(b)$, for $a, b \in V(\mathcal{G}_r)$. Then there is a translation $x \in \mathbb{R}^2$ with $h(\mathrm{supp}(t_a)) = h(\mathrm{supp}(t_b)) - x$, since $h(\mathrm{supp}(t_a))$ and $h(\mathrm{supp}(t_b))$ are the supports of the $\mathcal{T}_u$-tiles $t_{h(a)}$ and $t_{h(b)}$. It follows that $H_u(h(a)) = H_u(h(b)) - x$. For any $q \in V(\varphi(a))$, since $B_{2R(r)}(q) \subset B_{2R(u)}(a)$, the $\mathcal{T}_r$-tile t_q is uniquely determined inside $H_u(h(a))$. Thus $q \in V(\varphi(a))$ if and only if $q+x \in V(\varphi(b))$ and $l(q) = l(q+x)$. That is, $[\mathrm{supp}(t_{h(a)})]^{\mathcal{T}_r} = [\mathrm{supp}(t_{h(b)})]^{\mathcal{T}_r} - x$, and we again let $\Psi_{a,b}$ be given by the translation by x taking the drawing of $\varphi(a)$ onto the drawing of $\varphi(b)$. A similar process can be used to establish the the family Ψ for the other elements of $\mathcal{G}_r$.

For any $a \subset b \in \mathcal{E}(\mathcal{G}_r)$, we can define the map $\Phi_{a,b} : \mathcal{E}(\varphi(a)) \to \mathcal{E}(\varphi(b))$ by inclusion, since the vertex set of $\varphi(a)$ is contained in that of $\varphi(b)$ by definition. It is clear that $\Phi_{a_1,b_2} \circ \Psi_{a_1,a_2} = \Psi_{b_1,b_2} \circ \Phi_{a_1,b_1}$. This establishes that there is a substitution $\mathcal{S}$ on $\mathcal{G}_r$ given by the map φ along with the family of isomorphisms Ψ and the family of inclusion maps Φ. All subgraphs were induced from $\mathcal{G}_r$ on this vertex set, and since $\mathcal{T}_u$ covers $\mathbb{R}^2$, every element of $\mathcal{G}_r$ is contained in $\mathcal{S}(\mathcal{G}_r)$, proving that $\mathcal{G}_r = \mathcal{S}(\mathcal{G}_r)$. Every vertex $q \in \mathcal{L}_r$ is expanding, since $h^n(\text{supp}(t_q))$ goes to infinity as n does. Hence $\mathcal{G}_r$ is the fixed point of an expanding graph substitution, so $\mathcal{T}_r$ is combinatorially substitutive. □

6.1. **Related questions.** There are many questions to investigate on the subject of tilings and their graphs. What property of an infinite graph allows it to be drawn in the plane as a tiling with a finite number of tile types? How much of the geometric structure of a tiling is carried in its labelled graph? How much is preserved when the drawing of the labelled graph is ignored, eliminating the notion of facets? Can there be an embedding of $G(\mathcal{T})$ producing a plane graph with facets that do not correspond to tiles in $\mathcal{T}$? What properties, dynamical or other, can be deduced from knowing the combinatorial structure of $G(\mathcal{T})$ without facets? Selected folklore theorems are discussed in [7].

There are also questions to be answered on the subject of graph substitution. The graphs generated by substitutions considered in Section 6 are all plane graphs, but what conditions on $\mathcal{S}$ and Γ will ensure that $\mathcal{S}(\Gamma)$ is planar? For substitutions on sequences and also for self-affine tilings, matrices can be defined which tell how many, and which kinds, of labels are present in the substitution of each element type. Results have been obtained which tie algebraic properties of these matrices to dynamical properties of the associated symbolic or tiling dynamical systems. Can a similar type of analysis be carried out for graph substitutions?

It may be the case that we can use the combinatorial substitutive property of DV tilings for more than establishing the piecewise linearity of the homeomorphism h. Perhaps simple conditions can be discovered which force h to be a single linear map. In those cases, a proof like that found in [8] could be used to show that one of the DV tilings of $\mathcal{T}$ is a pseudo-self-similar tiling, implying that $\mathcal{T}$ is pseudo-self-similar.

We conjecture that an extension of our results is possible to tilings such as the pinwheel tiling [12]—tilings that have an infinite number of orientations of the same tile type. In this case the DV tilings would still be made from a finite number of congruent tiles, but they would come in an infinite number of orientations.

It is well-known that there are tilings of higher-dimensional space, and there is a way to see tilings of $\mathbb{R}^d$ as embedded graphs with higher-dimensional "elements" using a definition like that given in [18], p. 139. There is a notion of duality, and orthogonally situated dual graphs can be defined. It seems clear that our construction would extend naturally to this case. In an arbitrary topological or metric space, perhaps a notion of "embeddable graphs" can also be defined. That could allow us to consider combinatorial substitution for graphs and tilings in a very general setting.

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