SPECTRAL THEORY OF BIJECTIVE SUBSTITUTION SEQUENCES

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OBERWOLFACH, MARCH 5, 2009

1. Introduction

The dynamical systems of bijective substitution sequences in $\mathbb{Z}^d$ have a mixed dynamical spectrum, while many of their factors may only have a discrete part. Using as examples the well-known Thue-Morse and period-doubling substitutions, we will show what happens to the continuous part of the dynamical spectra through the factoring process.

2. Constant-length substitutions in $\mathbb{Z}^d$

When defining a constant-length substitution in $\mathbb{Z}^d$, the first order of business is to decide on the size of the rectangular blocks that the substitution will use. So we choose positive integers $l_1, l_2, ..., l_d$ and define

$$B = B(l_1, ..., l_d) = 0, 1, ..., l_1 - 1 \times ... \times 0, 1, ..., l_d - 1 \subset \mathbb{Z}^d$$

The block B defines an empty set of spaces for the substitution to fill in, sort of a "wire frame" structure waiting to be decorated (or colored in) by letters from some finite alphabet $\mathcal{A}$. To decide how to color in each space $\vec{\jmath} \in B$, we next choose a map $p_{\vec{\jmath}} : \mathcal{A} \to \mathcal{A}$. This gives a *substitution* $\mathcal{S} = (p_{\vec{\jmath}})_{\vec{\jmath} \in B}$, which assigns to each $a \in \mathcal{A}$ a block of letters of size B. The substitution may be iterated; we call a *level-n block* a letter which has been substituted n times.

Example 1. Let $B = B(2) = \{0, 1\}$ and let $\mathcal{A} = \{a, b\}$. The *period-doubling* substitution takes $a \to a\,b$ and $b \to a\,a$. In our notation, we see that the map p_0 takes both a and b to a, where p_1 is the map taking a to b and b to a.

Example 2. Again let $B = B(2) = \{0, 1\}$ and let $\mathcal{A} = \{a, b\}$. The *Thue-Morse* substitution takes $a \to a\,b$ and $b \to b\,a$. In our notation, we see that the map p_0 is the identity and p_1 is again the map taking a to b and b to a.

For details, examples, and a spectral analysis of multidimensional constant-length substitution sequences, see [2].

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[1] 2000 *Mathematics Subject Classification.* Primary 37B50; Secondary 37A30.

[2] *Key words and phrases.* Self-similar tilings, substitution sequences, dynamical spectrum

A substitution is said to be *bijective* if for each $\vec{\jmath} \in B$, p_j is a bijection on $\mathcal{A}$. Notice that if a substitution is bijective, then there can never be coincidences in the sense of Dekking [1], and therefore will have a mixed dynamical spectrum.

2.1. **Substitution dynamical systems.** Once a substitution is decided upon, we define the *hull* X of the substitution as the space of all sequences in $\mathcal{A}^{\mathbb{Z}^d}$, all of whose subblocks appear somewhere in a level-n block. Translation by elements of $\mathbb{Z}^d$ give a multidimensional action that is known, when the substitution is primitive, to be uniquely ergodic with probability measure we will call μ.

The Thue-Morse and period-doubling substitution dynamical systems, denoted $(X_{TM}, \mathbb{Z}, \mu_{TM})$ and $(X_{PD}, \mathbb{Z}, \mu_{PD})$ respectively, are our main examples. It is known that the Thue-Morse system factors onto the period-doubling system. It is also known that the Thue-Morse system has a mixed dynamical spectrum while the period-doubling system has pure point spectrum. We will show what becomes of the continuous part of the Thue-Morse spectrum during the factoring process.

3. Spectral theory of substitution sequences in $\mathbb{Z}^d$

Consider the unitary $\mathbb{Z}^d$-action on a Hilbert space given by $U^{\vec{\jmath}} : L^2(X, \mu) \to L^2(X, \mu)$ with $U^{\vec{\jmath}}(f(\mathcal{T})) = f(\mathcal{T} - \vec{\jmath})$ for all $\vec{\jmath} \in \mathbb{Z}^d$. We can analyze the action of $\mathbb{Z}^d$ on X by consideration of the action of $U^{\vec{\jmath}}$ on $L^2(X, \mu)$. The *spectral coefficients* of an $L^2(X, \mu)$ function are given, for each $\vec{\jmath} \in \mathbb{Z}^d$, by

$$\hat{f}(\vec{\jmath}) = \langle U^{\vec{\jmath}} f, f \rangle = \int_X U^{\vec{\jmath}} f(\mathcal{T}) \overline{f(\mathcal{T})} d\mu(\mathcal{T}). \tag{1}$$

It is known that these coefficients form a positive definite sequence and that therefore there is a unique measure σ_f on the d-torus [3] with:

$$\hat{f}(\vec{\jmath}) = \int_{\mathbb{T}^d} z^{\vec{\jmath}} d\sigma_f(z), \tag{2}$$

where $z^{\vec{\jmath}} = z_1^{j_1} \cdot \ldots \cdot z_d^{j_d}$.

It is hard to visualize these measures, but we know that they must decompose relative to Lebesgue measure into pieces that are atomic (discrete), singular continuous, and absolutely continuous. It is much easier to consider functions in L^2 and draw conclusions based on their spectral coefficients only, as we do in the case of eigenfunctions below.

An *eigenvalue* of U is an $\vec{\alpha} \in \mathbb{R}^d$ such that there is an $f \in L^2(X, \mu)$ for which $U^{\vec{\jmath}}(f) = \exp(2\pi i \vec{\alpha} \cdot \vec{\jmath}) f$ for all $\vec{\jmath} \in \mathbb{Z}^d$. (Equivalently, $f(\mathcal{T} - \vec{\jmath}) = \exp(2\pi i \vec{\alpha} \cdot \vec{\jmath}) f(\mathcal{T})$ for all $\mathcal{T} \in X$). It is not hard to check that the spectral measure of an eigenfunction is an atomic measure. Thus we call the closure of the linear span of eigenfunctions $H_D \subseteq L^2(X, \mu)$ the *discrete spectrum* of U. A substitution is said to have *pure point spectrum* if $H_D = L^2(X, \mu)$.

3.1. **Odometer structure and eigenfunctions.** The underlying box $B = B(l_1, ..., l_d)$ provides a wire-frame structure of the level-n blocks of any sequence in the hull X as follows (see [2] for details). For each $n = 1, 2, ...$ we define a map $\mathcal{O}_n : X \to \mathbb{Z}^d$ by $\mathcal{O}_n(\mathcal{T})$ = the position of the level-$(n-1)$-block of $\mathcal{T}$ containing the origin inside its level-n block. Each $\mathcal{T} \in X$ has a coding by level-n blocks given by the sequence $\{\mathcal{O}_n(\mathcal{T})\}$. The action of translation by elements of $\mathbb{Z}^d$ acts as an odometer on the space of level-n codings.

Odometer actions are know to have pure point spectrum, and the substitution dynamical system factors onto the odometer action, thus inheriting its eigenfunctions. Under the (relatively mild) condition of "trivial height" (see [2] for definition), the odometer system forms the maximal equicontinuous factor of the substitution system and so gives all the the eigenfunctions. In fact the eigenvalues must then take the form $\vec{\alpha} = \left(\frac{m_1}{l_1^{n_1}}, ..., \frac{m_d}{l_d^{n_d}}\right)$, where the l_i's remain the lengths of the substitution that define the block B.

Example 3. Consider either the PD or the TM substitution, so that $l_1 = 2$, and let $\vec{\alpha} = 1/2$. We have the eigenfunction given by

$$g(\mathcal{T}) = \begin{cases} 1 & \text{if } \mathcal{O}_1(\mathcal{T}) = 0 \\ -1 & \text{if } \mathcal{O}_1(\mathcal{T}) = 1 \end{cases}$$

(i.e. it is 1 if the origin is in the left-hand side of its level-1 block and it is -1 if it is in the right-hand side.) The reader should check that g is an eigenfunction with eigenvalue 1/2.

An important thing to notice is that the eigenfunctions only "see" the odometer structure given by B, not the labellings the substitution has decided to include. Thus if a substitution is pure point spectrum, the odometers must "see" everything there is to know about the hull X.

3.2. **Continuous spectrum in bijective substitutions.** Given a bijective substitution of trivial height, it is easy to write down functions in the orthocomplement of H_D. Let $F : \mathcal{A} \to \{1, 2, ..., |\mathcal{A}|\}$, and define

$$f(\mathcal{T}) = \exp\left(2\pi i \frac{F(\mathcal{T}(\vec{0}))}{|\mathcal{A}|}\right)$$

Obviously f only cares about the symbol at the origin in any sequence.

The fact that this is orthogonal to each eigenfunction is proved in [2], but it is instructive to consider the specific case of the Thue-Morse substitution and the eigenfunction defined in our previous example.

Example 4. For each $\mathcal{T} \in X_{TM}$, define

$$f(\mathcal{T}) = \begin{cases} 1 & \text{if } \mathcal{T}(0) = a \\ -1 & \text{if } \mathcal{T}(0) = b \end{cases}$$

We can show that this function is orthogonal to the eigenfunction g constructed in Example 3. To do this, we write X_{TM} as the union of four sets, $X_{0,a}, X_{1,a}, X_{0,b}$, and $X_{1,b}$, where $\mathcal{T} \in X_{i,e}$ if $\mathcal{O}_1(\mathcal{T}) = i$ and $\mathcal{T}(0) = e$. It is not difficult to show these sets have equal measure; moreover the product $g(\mathcal{T})f(\mathcal{T})$ is constant on each set. Thus we compute

$$\begin{aligned} < g, f > &= \int_{X_{TM}} g(\mathcal{T})f(\mathcal{T})d\mu \\ &= \int_{X_{0,a}} g(\mathcal{T})f(\mathcal{T})d\mu + \int_{X_{1,a}} g(\mathcal{T})f(\mathcal{T})d\mu + \int_{X_{0,b}} g(\mathcal{T})f(\mathcal{T})d\mu \\ &\qquad + \int_{X_{1,b}} g(\mathcal{T})f(\mathcal{T})d\mu \\ &= \int_{X_{0,a}} 1 \cdot 1 d\mu + \int_{X_{1,a}} -1 \cdot 1 d\mu + \int_{X_{0,b}} 1 \cdot -1 d\mu + \int_{X_{1,b}} -1 \cdot -1 d\mu \\ &= .25 - .25 - .25 + .25 = 0 \end{aligned}$$

Because f is in fact orthogonal to all of the eigenfunctions, this means the eigenfunctions in $L^2(X_{TM})$ cannot "see" the color at the origin. Moreover we get a spectral decomposition $L^2(X_{TM}) = H_D \oplus \overline{Span(f)}$.

4. Conclusion

We know that the Thue-Morse system factors onto that of the period-doubling, so why does f have a continuous spectral measure for the Thue-Morse system and an atomic one for the period-doubling substitution? The answer is simply that for the period-doubling substitution, the odometer coding can tell you whether a sequence has a a or a b at the origin. To see this, notice that if $\mathcal{O}_1(\mathcal{T}) = 0$, meaning that the level-0 block containing the origin is in the left of its level-1 block, then $\mathcal{T}(0)$ must equal a and thus $f(\mathcal{T}) = 1$. If the coding of $\mathcal{T}$ begins by 1 0 then $\mathcal{T}(0) = b$ and so $f(\mathcal{T}) = -1$. Indeed, the reader can check that if the coding begins with n 1's and then a 0, then $f(\mathcal{T}) = -1^n$.

In this way we see that the period-doubling substitution has not really altered the odometer at all, but the Thue-Morse substitution has.

References

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