A History of Manifolds and Fibre Spaces[1]: Tortoises and Hares

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> The hare is an idealist: his preferred position is one of elegant and all embracing generality. He wants to build a new heaven and a new earth and no half-measures. The tortoise, on the other hand, takes a much more restrictive view. He says that his modest aim is to make a cleaner statement of known theorems . . .
>
> J.F. Adams

1. Introduction

During the early 1930's topology developed some of its most important notions. The first international conference on the young subject took place in Moscow 1935. Fibre spaces were introduced by H. Seifert (1907–1996). By 1950 the notions of fibre space and fibre bundle had become central in the study of algebraic topology. In that year in Bruxelles, and in 1953 at Cornell University, international conferences on topology focused on the study of these spaces. The 1949/50 *Séminaire* of H. Cartan (1904–) in Paris, an influential seminar in the spread of new ideas in topology, was dedicated to fibre spaces. In 1951, N.E. Steenrod (1910–1971) published the first textbook on the subject—this was also the first textbook in algebraic topology to give complete accounts of homotopy groups and cohomology groups. In this paper we will discuss how fibre spaces came to become basic objects in algebraic topology.

In his report and problem set from the Cornell University conference, W. S. Massey (1920–) listed five definitions of fibre space [30]: (a) fibre bundles in the American sense; (b) fibre spaces in the sense of Ehresmann and Feldbau; (c) fibre spaces as defined by the French school; (d) fibre spaces in the sense of Hurewicz and Steenrod, and (e) fibre spaces in the sense of Serre. Each of these competing definitions developed out of a mix of examples and problems of interest to the research community in topology, often marked by a national character. We will consider the origins of each of these strands and the relations among them (see also [68]).

This paper is about definitions, and about ordinary developments in twentieth century mathematics. The principal hare in the development of fibre spaces is H. Whitney

[1] This paper represents an expanded version of a talk given at Oberwohlfach during the meeting "The History of the Mathematics of the 20th Century", January 30–February 5, 2000. My thanks to the organizers for the invitation and directions to speak on this topic.

(1907–1989) whose introduction of sphere spaces was a sweeping advance for the study of manifolds. The principal tortoise is E. Stiefel (1909–1978) whose investigations of families of vector fields on manifolds firmly grounded new topological invariants for the study of geometric problems. The usual forces that play out in mathematical research—problems and examples—are the keys to our story. In the hands of able mathematicians in a supportive community, this basic stuff leads to refinements and new approaches. Adams's epigraph was meant to describe the development of stable homotopy theory, but it makes for a useful distinction among the makers of definitions.

2. Manifolds

Henri Poincaré (1854–1912) made his seminal contributions to algebraic topology in the celebrated paper *Analysis situs* (1895), and its subsequent five *Compléments* (1899, 1900, 1902, 1904) [38–44]. His interest in topological methods can be traced to his initial work on the three-body problem and his qualitative approach to dynamical systems. The main objects of study in *Analysis situs* are manifolds and Poincaré offered several concrete ways to define them (see also [47]). These include:

1) The zero set of a differentiable function $f\colon (A \subset \mathbb{R}^{n+k}) \to \mathbb{R}^k$. Here the manifold is given by $M = f^{-1}(0, \ldots, 0)$.
2) The manifold M is a union of parameterized subsets. This is a generalization of analytic continuation on the overlaps.
3) The manifold is given by the quotient of a geometric cell by boundary identifications.
4) The manifold is the union of two homeomorphic handle bodies identified along their boundaries, that is, a manifold may be defined by its Heegaard diagram (see the fifth complément [44]).

For Poincaré, the examples of manifolds were rich enough to explore the basic questions of their topological properties and classification. The basic example for classification (*an elementary paradigm* [47]) was the case of surfaces for which A. F. Möbius (1790–1868) and C. Jordan (1838–1921) had demonstrated that the topological invariants of genus and orientation were complete invariants. Poincaré introduced the invariants of homology and the fundamental group in an effort to generalize the successful classification of closed surfaces to higher dimensional manifolds.

In contrast with the concrete handling of manifolds by Poincaré, the notion of a manifold required an axiomatic definition by other mathematicians (of the Göttinger persuasion). D. Hilbert (1862–1943) sought an axiomatic characterization of the plane sufficient for the foundations of geometry in Appendix IV (1902) to the *Grundlagen der Geometrie* [21]. The definition of the plane, according to Hilbert, begins with a system of neighborhoods that satisfy certain topological conditions. The use of locally Euclidean neighborhoods became the basis of many refinements that eventually led to the definition of manifold used today. Among his topological axioms, Hilbert assumes the existence of large neighborhoods (for any pair of points in the plane, there is a neighborhood containing them). This assumption was dropped by H. Weyl (1885–1955) in his 1913 *Der Idee der Riemannschen Fläche*. There Weyl based his definition of surface on a neighborhood system that gave a basis for the topology on the surface and satisfied an open map condition. Absent from the definition schemes of Hilbert and Weyl is the Hausdorff condition on the underlying topological space—a fault pointed out by Hausdorff in his axiomatic treatment of topological spaces [20].

Scholz, in his discussion of the development of axiomatic definitions of a manifold [47], mentions a foray of Weyl into the development of a concept of surfaces from a constructivist perspective. I bring this point up to illustrate a key feature of definitions in modern mathematics. It is not only a striving for precision that motivated axiomatics, but the potential to describe new objects that leads to new mathematics. A case in point is the introduction of non-Archimedean structures in Hilbert's *Grundlagen der Geometrie*. This point holds for the development of fibre spaces as well.

In 1924 H. Kneser (1898–1973) addressed the Innsbruck meeting of the *Gesellschaft deutscher Naturforscher und Ärtze* on the topology of manifolds [29]. He called for a balance between the familiar examples (Riemann surfaces and the associated images of analytic functions of one or more variables, regular surfaces in spaces, and phase spaces of dynamical systems), whose particular properties might not be shared by all manifolds, and an axiomatic definition sharp enough to work with. Kneser isolated the questions of the existence of a triangulation on a given manifold and the existence of a common refinement of two triangulations as the *Hauptvermutung* for the topology of manifolds. From the *Hauptvermutung* follows the fact that combinatorial invariants are topological invariants. Kneser put forward his own balanced approach in the notion of combinatorial complexes (*Zellgebäude*)—defined inductively in terms of ordinary n-cells and their boundaries S^{n-1}, giving a cell complex. Combinatorial manifolds satisfy an additional condition, that the cellular neighborhood complex of a 0-cell is a copy of S^{n-1}. This structure admitted combinatorial and inductive proofs with which Kneser proved that two complexes with isomorphic decompositions are homeomorphic.

In 1929 B. L. van der Waerden (1903–1996) reported to a meeting in Prague on progress in Kneser's combinatorial ideas since the Innsbruch address. He described combinatorial topology as *"ein Kampfplatz verschiedener Methoden, die ich in Anschluß an Kneser als ,,rein-kombinatorische" Methode und ,,méthode mixte" bezeichnen werde."* In this report[2] [59] van der Waerden mentioned five definitions of a combinatorial manifold from the most purely set-theoretic, to the most useful (due to Van Kampen [28]). For van der Waerden, the measure of a definition is in line with the wishes of a tortoise—that it recover known theorems. In particular, a benchmark was Poincaré duality for which the use of stars of simplices offered a means to a proof. Van Kampen's definition was based directly on homology and the representation of stars as cycles and so emphasized invariants over point-set constructions. This direction of development, away from objects toward their classifying invariants, is common in algebraic topology and a feature of the history of fibre spaces.

3. Seifert's fibre spaces

Fibre spaces were introduced by H. Seifert in a lengthy article [49] that constituted his second doctoral thesis, officially written under van der Waerden at Leipzig, but written independently. The main problem addressed by the work is one taken up by Seifert in his first thesis (written under Threlfall in Dresden), the construction of closed three-dimensional

[2] The bibliography of this paper is exceptional in breadth and depth.

manifolds. Motivated by the Poincaré conjecture, Seifert described, in exacting detail, a class of three-dimensional closed manifolds that admit a *fibring*, that is, a decomposition of the manifold into closed curves, one through any given point, and topologized by a neighborhood system made up of identifiable spaces.

The most important new idea in the paper is the notion of the decomposition surface (*die Zerlegungsfläche* [49, p. 155]). For Seifert, the availability of different definitions of a manifold was instrumental. For each fibre (a closed curve) in a fibre space F, there is a fibre neighborhood, and the system of fibre neighborhoods satisfies the axioms for a surface chez Weyl. With this in place, Seifert derived many properties of the manifold F from its decomposition surface. In particular, since surfaces were known to be triangulable ([45]), he could prove the triangulability of F by lifting the triangulation from f ([49, p. 163]).

Seifert examined two important examples. The first (§3) is the Hopf map, recently published in [23], where $F = S^3$. The decomposition of the three-sphere into circles is given explicitly and the decomposition surface shown to be S^2. Seifert observes that the mapping $F \to f$ sending a fibre to its corresponding point in the decomposition surface is the same mapping as studied by Hopf in [23]. The second example (§12) is the dodecahedral space of Poincaré. In this example, he applied the methods developed in the paper to compute the fundamental group of the fibred space by using covering spaces and the decomposition space.

To close the paper, Seifert identified a condition that prevents a space from being fibred; he showed that the topological sum of two three-manifolds is not fibred in general.

Around the same time that Seifert introduced fibre spaces, Threlfall had constructed three-manifolds by taking the projective lines associated to the tangent space to a surface [57]. Threlfall remarked at the beginning of the paper that *"die Topologie is noch keine klassiche Dizciplin wie die Funktionentheorie; deshalb sei der Ort der folgenden Untersuchungen innerhalb der Topologie bezeichnet."* By this remark, he meant to emphasize the need for rigorous definitions and proofs in his account. He also made clear that he was placing his work in a context, even forging a type of discourse. In this period, Seifert and Threlfall wrote their influential textbook [51].

Threlfall remarked in a footnote that these examples of three-manifolds could be understood in the cadre of Seifert's work. He also mentioned that Hopf had informed him of similar work by Hotelling eight years earlier [24]. (We will discuss Hotelling's work below.)

In 1936 Seifert published another contribution of importance to our discussion. In this paper [50] he considered the problem of when an n-dimensional manifold M may be approximated by an algebraic manifold—that is, one that is the zero set of a family of polynomials. He required an embedding $M^n \subset \mathbb{R}^{n+k}$ for which there exists k independent normal vector fields on M. This led to the *neighborhood manifold*, a tubular neighborhood of M. Seifert showed that you can get an algebraic manifold by deforming M slightly in the tubular neighborhood.

The examples and techniques of analysis of the topological properties of fibre spaces in the work of Seifert and Threlfall solidly laid the groundwork in the context of investigations of three-manifolds and the Poincaré conjecture.

4. Fibre spaces in the United States

Poincaré was led to topology by the theory of dynamical systems, particularly the three-body problem. It is through G. D. Birkhoff (1884–1944) that some of the topological ideas of Poincaré were developed in the United States. Birkhoff was interested in dynamical systems from the start of his career and gained prominence with his proof of a key theorem (Poincaré's geometric theorem) for the three-body problem (published in 1913 [1], a year after the death of Poincaré). In a major paper [2], awarded the first Bôcher award from the American Mathematical Society, Birkhoff extended Poincaré's qualitative analysis of equations of motion in the case of a dynamical system with two degrees of freedom. The manifold of states of motion of such a system is a three-manifold on which a closed stream line represents a periodic orbit of the system. Birkhoff associated to this three-manifold a *surface of section*, bounded by a collection of closed stream lines. Seifert [49, p. 155] noted that these surfaces had nothing to do with his *Zerlungsfläche*.

Birkhoff's focus on dynamical systems and the translation of dynamical properties into geometric ones forged a link between the topology of a manifold and the interpretation of structures like the collection of all unit tangents to the manifold. H. Hotelling (1895–1973) wrote Ph.D. thesis at Princeton [24], under O. Veblen (1880–1960) with some help from J. W. Alexander (1888–1971), in which he studied such spaces for surfaces. He observed the construction gave new examples of three-dimensional manifolds, not given by products, that were accessible to analysis via Heegaard diagrams, a tool being developed by Alexander at the time. His examples of three manifolds that are locally products, but not globally so, were known to Hopf and to Threlfall. Hotelling did little more in topology, but is well known for his contributions to mathematical economics and statistics.

Another strand of axiomatic development was the effort by E. H. Moore (1862–1932) at the University of Chicago to involve his Ph.D. students in the kind of mathematics done by Hilbert in the *Grundlagen der Geometrie*. This group, later known as the *Postulationalists*, included Veblen who continued his interest in geometry and topology in Princeton. In 1932, he published a treatise [60] on the foundations of differential geometry with his then graduate student, J. H. C. Whitehead (1904–1960). In this influential text the strands of analysis situs, the Erlangen Program, and axiomatics come together to give a definition of an n-dimensional manifold that makes it the basic object of geometry. The local coordinate systems of Hilbert and Weyl were extended to include the data of a structural group. A thorough discussion of the tangent space to a manifold at a point makes up Chapter V in which topics like the Ricci-calculus and general relativity appear.

The principal hare in our story opened up the potential examples of fibre spaces by focusing on the geometric examples of the tangent bundle and the normal bundle to an embedding. H. Whitney took a Ph.D. at Harvard in 1929 under G. D. Birkhoff with a thesis on graph coloring. Awarded a National Science Foundation Fellowship in 1931 he came to Princeton during a period of considerable activity in topology. Alexander, Veblen, and Lefschetz were at the University, and Alexandroff and Hopf visited during 1931. After this time Whitney devoted his considerable energies to topology, with a particular interest in manifolds.

By 1935 Whitney had introduced two important ideas for the study of manifolds. The first is his embedding theorem. In his words:

"A differentiable manifold is generally defined in two ways; as a point set with neighbor-

hoods homeomorphic with Euclidean space E_n, ..., or as a subset of E_n, defined near each point by expressing some of the coördinates in terms of the others by differentiable functions.

The first fundamental theorem is that the first definition is no more general than the second; any differentiable manifold may be imbedded in Euclidean space."

The proof returns the study of manifolds from the abstract manipulation of combinatorial invariants to the study of concrete objects—subsets of Euclidean spaces, together with all of the associated structure. An important corollary is the existence of a Riemannian metric on arbitrary smooth manifolds, which follows from the embedding.

The second idea is a generalization of fibre spaces of Seifert [49]. In a short note in the Proceedings of the National Academy of Science [61], Whitney introduced **sphere spaces**, which were spaces "in which the points themselves are spaces of some simple sort, for instance spheres. . . ." The definition given in [61] was based on neighborhoods covering a *base space* K (a manifold or complex) over each of these neighborhoods one has a homeomorphic copy of the product of that neighborhood and a sphere of fixed dimension. The glueing data on the overlapping neighborhood are given by diffeomorphisms, and even can be given by orthogonal transformations, for which the sphere space is called *regular*. The union of the all the spheres is a space $\mathfrak{S}(K)$ dubbed the *total space* by Whitney.

Whitney cited the three-manifolds of Hotelling, the fibred spaces of Seifert and the three-manifolds of Threlfall as examples of sphere spaces. His insight was to emphasize starting with a base space and associating a sphere space to it. For Seifert, the fibred space as a total space comes first and the base space (*Zerlegungsfläche*) is derived from it. By fixing the base space, the classification problem can be more clearly stated—to find invariants that distinguish between sphere spaces over the base.

Assuming a fine enough triangulation of the base space of a regular sphere space, Whitney associated invariants to the sphere space by extending choices of orthogonal sections over skeleta of the base. The invariants take their values in the homology, of spaces Q_s^l of ordered sets of s orthogonal points on S^l. Whitney states without proof that the homology of the Q_s^l is given by the integers or the integers mod 2 in the crucial dimensions. These classes, indexed over the cells of the base space, were soon seen to be more easily expressed in terms of cohomology [63] and they are now called the *Stiefel-Whitney classes* of the sphere space.

Though the paper [61] lacked explicit proofs, it gave examples showing the breadth of the generalization. In particular, if a manifold or complex is embedded in a Euclidean space, then it has a normal sphere space associated to it. Topological invariants of that sphere space are invariants of the embedding. A companion note appeared in 1940 [64], once again telegraphic, in which relations among the invariants were given. In particular, given two sphere spaces over a fixed base space, one can form a product sphere space whose invariants are related in an explicit manner. For an embedded manifold, the product of the tangent and normal sphere spaces is trivial and so Whitney obtained a formula, the *Whitney sum formula* in [33], that eases the computation of these invariants. This extended the work of Stiefel (see below) considerably. In a set of lecture notes [65], Whitney gave a computational account of these relations for bundles of low rank on surfaces where geometric arguments could be made.

Whitney planned a book-length treatment of characteristic classes with complete proofs of the theorems in his notes. The editors of his collected works [67] described the work in progress as difficult, and Whitney claimed that the proof of the sum formula to have been the most difficult argument he ever produced. Later developments in algebraic topology cast Whitney's work into a more manageable context in which complete proofs could be given easily.

5. Fibre spaces in Switzerland

Poincaré left his mark on the beginning of algebraic topology in another paper that appeared before his series on *Analysis situs*. In 1885, in the second memoir on the qualitative theory of differential equations, he proved the Poincaré index theorem for surfaces. This theorem shows how the topology of a surface controls qualitative aspects of the calculus on the surface. In particular, if there is a vector field on the surface with a finite number of isolated singular points, then each singular point may be assigned an index according to the local behavior of the vector field at that point, and the sum of these indices is the Euler characteristic of the surface. It follows that there are no nonsingular (everywhere nonzero) vector fields on the sphere S^2 (the so-called *Hairy Ball Theorem*), and that the torus is the only surface with a continuous everywhere nonzero vector field.

Among his applications of the notion of the degree of a mapping, L. E. J. Brouwer (1881–1966) [3] extended the work of Poincaré to the n-dimensional sphere. Around the same time, J. Hadamard (1865–1963) [19] stated without proof that, for a closed compact n-dimensional manifold embedded in some Euclidean space $\mathbb{R}^{n+k}$, the sum of the indices of a continuous vector field on the manifold is a topological invariant of the manifold.

In his Habilitationschrift, H. Hopf (1894–1971) [22] gave complete proofs of the general case of the Poincaré index theorem for n-dimensional manifolds. His proof is a model of clarity. One first shows, by a homotopy argument, that any pair of vector fields on a manifold with finite numbers of isolated singularities have the same index. Then one constructs a particular vector field associated to a given triangulation for which it is evident that the sum of the indices is the Poincaré-Euler characteristic.

Hopf's beautiful argument is the point of departure for the Zürich Ph.D. thesis of his student E. Stiefel. The problem set by Hopf for Stiefel was a generalization of the Poincaré index theorem—given an n-dimensional manifold M, are there m ($m \leq n$) continuous everywhere nonzero vector fields on M that are linearly independent at each point of M? The case of $m = 1$ is the index theorem of Poincaré and Hopf, for which the Poincaré-Euler characteristic is a complete invariant.

Stiefel considered as a special case the geometric problem of *Fernparallelismus* (teleparallelism, absolute parallelism, parallelizability [48]). This problem concerns the properties of parallel transport on a manifold: If a manifold is parallelizable, then the tangent space at each point is isomorphic to every other tangent space by an isomorphism induced by a parallel transport along a curve, but that isomorphism is independent of the curve. In particular, the behavior of tangents locally can be moved out across the entire manifold by transport. Examples of parallelizable manifolds are given by Lie groups for which parallel transport can be effected by the group operation.

Fernparallelismus was a notion of recent interest in the thirties[3] because of a paper published in 1928 by Einstein [15]. In this attempt at a unification of gravitation with electromagnetism, Einstein worked out the consequences of parallelizability for the Riemann curvature tensor and identified an associated tensor that he hoped would give the electromagnetic potential. The reputation of Einstein at this time was such that the *New York Times* carried a story on this work, that he was 'on the verge of a great discovery' [34] with Fernparallelismus. The idea of parallelizability did not originate with Einstein—it was studied by É. Cartan and J. A. Schouten in their studies of Lie groups ([4], [5]) and others as well ([48]). In letters between Cartan and Einstein, Cartan pointed out to Einstein that they had discussed such a geometry together in 1922.

Stiefel's approach to the problem of linearly independent vector fields followed Hopf's proof of the index theorem. Given m vector fields on a manifold M, the locus of points where they fail to be linearly independent determines an $(m-1)$-dimensional subcomplex of M. By defining a notion of index along this chain, Stiefel identified a homology class in $H_{m-1}(M;R)$ where the coefficients R are determined by the homology groups of the universal examples of spaces of m-frames, the manifolds $V_{n,m}$ consisting of m-frames in $\mathbb{R}^n$. These manifolds are analyzed by Stiefel in detail, and bear his name in the present literature. The coefficients are given by the integers in certain dimensions and by the integers mod 2 in others.

The main theorem of Stiefel's thesis is the generalization of Hopf's theorem: There are classes F^0, F^1, $\ldots$, F^{m-1}, in the homology of M with appropriate coefficients, which must vanish if a collection of m everywhere linearly independent vector fields on M exists. The corollaries of this theorem include the fact that all closed orientable three-dimensional manifolds are parallelizable, and the vanishing of these topological invariants for Lie groups. Stiefel analyzed the *characteristic classes* F^i for the real projective space $\mathbb{R}P^{4k+1}$ to prove that there does not exist a pair of everywhere linearly independent vector fields on this space ($F^1 \neq 0$).

Stiefel's thesis offers a stark contrast with the papers of Whitney. Stiefel was a tortoise by Adams's definition, careful to focus on the well known and to push a good argument to new applications. The fibre spaces for Stiefel were tangent spaces for which he did not define a new object, but worked with the accepted objects associated to manifolds. The most interesting new feature in the thesis is the introduction and analysis of the Stiefel manifolds $V_{n,m}$. Whitney, by contrast, defined first a new category of objects, sphere spaces, for which he could analyze the classification problem in terms of Grassmann manifolds, his tool corresponding to the Stiefel manifolds. His intuition ran ahead of his techniques and he published complete proofs only in the geometrically accessible cases. Others picked up the threads of these researches.

6. Moscow 1935

Topology as a field of mathematics was well established at Princeton in the early

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[3] My thanks to Jim Ritter for pointing this out to me. It would be interesting to know when Hopf became aware of the problem of *Fernparallelismus*—he spent summers in Göttingen where relativity was a focus of research, and earlier, in 1928, he had been in Princeton on a Rockefeller fellowship.

1930's[4]. Activity around Brouwer in Amsterdam, Hopf in Zürich, and Alexandroff in Moscow made for an international community of researchers in this specialty. In order to foster communication in this community, P. S. ALEXANDROFF (1896–1982) organized the First International Topological Congress, held 4–10 September, 1935. Alan Tucker reported on the conference in the Bulletin of the American Mathematical Society [58]:

"There were thirty-seven delegates, twelve from the Soviet Union and twenty-five from nine foreign countries, including ten from the America; there were also about twenty visitors in attendance, mostly from parts of the USSR. The group was noteworthy for its youthfulness and vigor; in its small number is included a large proportion of the active topologists in the world."

The list of participants included Lefschetz, Borsuk, Čech, Freudenthal, P. A. Smith, Tychonoff, Alexander, Kolmogoroff, Hopf, Whitney, Kuratowski, Sierpinski, van Kampen, von Neumann, Pontryagin, Weil, Heegaard, de Rham, Zariski, Hurewicz and others.

Much has been written about this conference[5] as a crucial moment in the development of topology. Three aspects of the conference are important to our story:

I. Kolmogoroff and Alexander each presented talks introducing cohomology together with its cup product structure. Whitney and Čech, both at the conference, immediately took up this new structure, corrected errors in its original formulation, and found new applications. Whitney, in fact, gave a talk in Moscow reformulating results of Hopf in terms of cohomology [67].

II. Hopf presented Stiefel's work on vector fields and characteristic classes, and Whitney presented his more general notion of sphere spaces, together with his work on the classification problem.

III. Hurewicz spoke on the definition of the higher homotopy groups of a space [25]. Among the participants at the conference, Čech had already presented the definition of these invariants at the 1932 International Congress in Zürich, and Alexander and van Dantzig claimed earlier unpublished investigations of these groups.

What made Hurewicz's presentation the seminal one for higher homotopy groups? In his four notes of 1935-36, Hurewicz outlined his investigation of classical invariants of mapping spaces—in particular, the space of based mappings $(X, x_0)^{(S^{n-1}, e_0)}$ has as its fundamental group $\pi_n(X, x_0)$. Hurewicz concerned himself with the point-set topology of general mapping spaces and applied these results to the case where the domain is a sphere. His celebrated theorem relating the higher homotopy groups to the homology groups is another bridge between the most classical invariants of a space and these new invariants.

For the theory of fibre spaces, the introduction of higher homotopy groups played a key role—Hurewicz studied the case of a closed subgroup H of a Lie group G to compare the mapping spaces H^X, G^X, and G^X/H^X. Since G and H are topological groups, so are the mapping spaces G^X and H^X. In the case of based mappings and $X = S^{n-1}$, he showed in *Satz XII* some of the relations between $\pi_n(G)$, $\pi_n(H)$, and $\pi_n(G/H)$. Together with the

[4] An oral history of the research activity in Princeton during the early 1930's was prepared by Alan Tucker and collaborators. Visit the website http://www.princeton.edu/mudd/math.

[5] See [31], [67], and [27], particularly [27, p. 840] for a picture of some of the participants.

then recent result of Hopf [23], that $\pi_3(S^2) \cong \mathbb{Z}$, computations of the new invariants seemed possible. Furthermore, the description of the properties of higher homotopy groups posed an exciting new vista for topologists to explore.

7. Fibre spaces in France

In the month after the Moscow conference, a second international conference met in Geneva, "Colloque sur quelques questions de Géométrie et de Topologie" ([27]). Though there were some shared speakers at Moscow and Geneva, there was a marked difference in emphasis. Roughly, in Moscow, general invariants of general spaces were discussed, while in Geneva, particular aspects of particular spaces (the spaces of importance in geometry) were presented. The lecturers from France included É. Cartan (1869–1951), A. Weil (1906–1998), and Ch. Ehresmann (1905–1979), all of whom spoke on aspects of Lie theory.

The study of homogeneous spaces has its origins in the Erlangen Program of Klein. Transformation groups are often Lie groups and homogeneous spaces enjoy the action of such a transformation group. The topological properties of Lie groups were the focus of a series of seminal papers by É. Cartan in the 1920's, culminating in the book [8]. In these papers, he built upon the observation of Poincaré that differential forms and exterior differentiation on a manifold depended upon its topology, and hence could be used to determine topological invariants. This led Cartan to conjecture the values of the Betti numbers of compact Lie groups from the rank of a basis of left invariant forms. G. de Rham (1903–1990) established this conjecture in 1931 [46] by proving the more general relation between chains and forms on a manifold given by integration—the Betti numbers could be obtained by what amounts to a duality argument.

In his 1933 Ph.D. thesis written under Cartan, Ehresmann computed the Betti numbers of the Grassmann manifolds, $G_{n,m} = O(n)/O(m) \times O(n-m)$, using the fact that $G_{n,m}$ is both a manifold and an algebraic variety. The data provided by this computation was crucial for the classification problem for fibre spaces.

In the late 1930's, Ehresmann, at Strasbourg, began work with his first doctoral student J. Feldbau (1914–1945) on generalizing fibre spaces to include Lie groups as examples. Given a closed subgroup H of a Lie group G, the left cosets of H give a decomposition of G as a union of homeomorphic spaces, $G = \bigcup_{g \in G} gH$. In 1939, Feldbau gave a definition of fibre space similar to Seifert's and emphasizing the local product structure. To generalize from sphere spaces, Feldbau added a structure group, here given as a family of homeomorphisms $H(x)\colon F_x \to F$ from the fibre over a point x, F_x, to some fixed fibre F. He required that the $H(x)$ be continuous and, on intersections of neighborhoods, the composite $\phi_{ij}(x) = H_i^{-1}(x) \circ H_j(x)$ lie in a given group of automorphisms of F. For example, if $F = S^{n-1}$, then the group $SO(n)$ may be chosen for an oriented sphere space á la Whitney. This definition was sufficient to prove the theorem that a fibre space with a contractible base space is homotopy equivalent to a product. Using this result and a double induction, Feldbau gave a complete classification up to homotopy equivalence of fibre spaces with base space a sphere S^n: they are in one-to-one correspondence with homotopy classes of mappings $S^{n-1} \to G$, the group of automorphisms of the fibre.

The step into classification up to homotopy is crucial in Feldbau's work. At the end of the note he observed the corollary of his classification that $\pi_{2n-1}(O(2n)) \neq \{0\}$ by virtue of the classical fact that S^{2n} does not admit an everywhere nonzero vector field. Reversing the argument, the computation of Alexandroff-Hopf that $\pi_2(O(n)) = \{0\}$ implies the triviality of every fibre space with fibre S^3 and base S^r. Thus his new definition of fibre space was important for the study of general fibre spaces and placed the problem of computing homotopy groups of certain spaces at the center of this geometric question. In a 1958 handwritten report on the work of Feldbau [14], Ehresmann described the unfinished doctoral thesis of Feldbau. In it, the first chapter concerned the then current state of computations of higher homotopy groups.

The outbreak of World War II led to the closing of the Université de Strasbourg and the removal of university activities to Clermont-Ferrand in unoccupied France. From there Ehresmann and Feldbau continued their work on fibre spaces. Feldbau published a second note in the Bulletin Soc. Math. France [17] in which he analyzed Stiefel's work on parallelizability in his context. The note was published under the alias of *Jacques Laboureur*. Feldbau, a Jew, feared the consequences of the appearance of his name in a journal published in Paris and so chose the pseudonym Laboureur (ploughman), a French form of Feldbau (ploughing). He was arrested in 1943 and died in Auschwitz on April 22, 1945 ([68]).

8. Independent unifications

The Moscow conference of 1935 is an historical moment for the community of topologists as it brought together an energetic and developing group from which new problems, methods, and collaboration was possible. World War II brought isolation to national groups of researchers by the delay or suppression[6] of publication and communication. In 1941, a 'miraculous year' for the theory of fibre spaces, three papers appeared, each arriving at the same key lemma as a consequence of their independent definitions.

Ehresmann and Feldbau [13] extended the definition of Feldbau [16], which applied to manifolds, to spaces giving the notion of a general fibre space associated to a group G of automorphisms of the fibre. The definition enabled them to prove their *Lemme de déformation* by a subdivision argument. This lemma is the covering homotopy property that gives the present definition of a fibration:

A mapping $p\colon E \to B$ has the **covering homotopy property** if, for any mapping $\Phi_0\colon K \to E$ that has as projection $p \circ \Phi_0 = \phi_0\colon K \to B$ where $\phi_0(x) = \phi(0, x)$ for a homotopy $\phi\colon [0,1]\times K \to B$, then there is a continuous mapping $\Phi\colon [0,1]\times K \to E$ with $\Phi(0, y) = \Phi_0(y)$ and $p \circ \Phi = \phi$.

The lemma leads directly to the relations among the homotopy groups of a fibre space E, its base space B, and fibre F. In the modern parlance (not available at that time), there is a long exact sequence:

$$\cdots \pi_n(F) \to \pi_n(E) \to \pi_n(B) \to \pi_{n-1}(F) \to \cdots.$$

[6] A case in point is the interruption of the publication of *Compositio Mathematica* and the subsequent appearance of Hopf's paper *Über die Topologie der Gruppenmannigfaltigkeiten und ihre Verallgemeinerungen* in the *Annals of Mathematics*.

The language of exact sequences had not yet appeared, however, and Ehresmann and Feldbau gave explicit isomorphisms of certain subgroups of the homotopy groups of each space involved in terms of the others. From their main theorem they computed the homotopy groups of complex projective spaces in terms of the homotopy groups of spheres and recovered results announced by Hurewicz in [25].

In the United States, Hurewicz and Steenrod published an account of fibre spaces for which the point-set topological properties of the spaces involved were given the principal focus. Hurewicz had introduced homotopy groups based on his point-set investigations of mapping spaces, and Steenrod had an early interest in point-set questions from his introduction to topology under Wilder. The defining property of a fibre space, $p: E \to B$, in [26] is the existence of a *slicing function*, $\phi: A \to E$, where the base space B is a metric space and A is a subspace of $E \times B$ given by $\{(x, b) \mid d_B(p(x), b) < \epsilon_0\}$ for $\epsilon_0 > 0$, some given real number. The slicing function satisfies the additional properties that $p \circ \phi(x, b) = b$ for all $(x, b) \in A$ and that $\phi(x, p(x)) = x$ for any $x \in E$. They give examples including covering spaces, a nonsingular mapping of one manifold onto another (this includes the Hopf mappings), and the quotient mappings $G \to G/H$ for H a closed subgroup of a Lie group G. The main theorem is the covering homotopy property, proved in this case by subdividing the interval $[0, 1]$ into small enough pieces to apply the slicing function. From this theorem they deduce the homotopy equivalence of all fibres, the nontriviality of the Hopf mappings, and the exactness relations among the higher homotopy groups of the fibre space, base space and fibre. They apply these relations amid homotopy groups to the Hopf fibrations and to the fibre space $p: SO(n) \to S^{n-1} = SO(n)/SO(n-1)$.

R. H. Fox (1913–1973) objected to the definition of Hurewicz and Steenrod [18]: The assumption of a metric space was not topological. He attempted to remove the assumption using another definition of slicing function. In the case of interest to Hurewicz and Steenrod, Fox's definition agreed, and in the case of a metrizable space, Fox extended the definition using Urysohn's lemma. In fact, there is a hole in Fox's proof, pointed out in [68], that can be filled using the notion of a partition of unity introduced later.

Steenrod continued his work on fibre spaces during the war. He introduced operations on the tangent bundle of a manifold [53] that give the dual bundle and tensor powers of such bundles. Thus sections of such bundles are tensor functions on the base manifold. In an important paper [54] he took up the problem of classifying sphere spaces. He distinguished between fibre spaces (a homotopy notion) and fibre bundles, for which the key property is a local product structure and a structure group that determines the glueing together of the local products on intersections. He remarked in the paper, "The concept of fibre bundle is somewhat complicated." He also described the analogy of fibre bundles with group extensions as a digression. For sphere bundles (the sphere spaces of Whitney) he studied the homotopy theory of the spaces $M_l^k = SO(k + l + 2)/SO(k + 1) \times SO(l + 1)$, the Grassmannian manifold, and its two-sheeted cover $\tilde{M}_l^k$. This space can be identified with the great k spheres of the $(k + l + 1)$ sphere. Steenrod showed that the problem of classifying k sphere bundles over a base space B is equivalent to the computation of the set of homotopy classes of mappings from B to M_l^k for any $l \geq \dim B$. Thus, as in the work of Feldbau, a geometric problem, the classification of sphere bundles, is shown to be equivalent to a problem in homotopy theory.

The final entry among the competing definitions of fibre space is due to B. Eckmann (1917–) [9] in his doctoral thesis written under Hopf. The hypothesis of choice for Eckmann's fibre spaces is the existence of a *retrahierbare Zerlegungen* (a retracting decomposition): A mapping $p\colon E \to B$ is a fibre space if every point $b \in B$ has a neighborhood $U(b)$ such that there is a retraction $R(x, b)$ of $p^{-1}(U(b))$ onto $p^{-1}(\{b\})$ depending on b such that $R(x, b) = x$ if $x \in p^{-1}(\{b\})$. For Eckmann the main examples had base space given by a compact metric space and so the retracting decomposition could be given by lifting arguments similar to those of Hurewicz and Steenrod. The main application of the definition is, once again, the covering homotopy property, and the consequences for relations among the homotopy groups of the base space, total space, and fibre. The main examples for Eckmann were the Hopf fibrations and certain homogeneous spaces for which he computed the homotopy groups of spheres and compact Lie groups from the relations. Corollaries of these computations include non-existence theorems for sections of fibres spaces with geometric interpretations. In papers that followed, Eckmann applied these methods to Lie groups and to the problem of the existence of solutions to systems of linear equations with continuous functions as coefficients [10]. The reduction of these geometric problems to homotopy theory questions lent importance to the internally conceived problems in the emerging subdiscipline of homotopy theory and forged ties between this new topic and accepted areas of mathematics.

9. Summary

In the first textbook treatment of fibre spaces and fibre bundles [55], Steenrod writes that the subject of fibre bundles "marks the return of algebraic topology to its origin; and after many years of introspective development, a revitalization of the subject from its roots in the study of classical manifolds." The origins of fibre spaces indeed lie in the study of manifolds, in particular, in the quest for examples and invariants that might settle the Poincaré conjecture (which remains open at the time of this writing). The failure of this program to decide the conjecture did not marginalize the ideas—new contexts were found for the idea of a fibre space, in particular, to address problems of the existence of vector fields on manifolds by Stiefel and to study more general vector bundles by Whitney.

The problem of classification remained primary for the new objects—in the case of Seifert, for three-manifolds, and in the case of Whitney and Stiefel, for new objects. The tools for the study of fibre spaces were being forged out of new algebraic invariants, introduced on an international stage in 1935, namely, cohomology rings and homotopy groups. The key computation of Hurewicz relating the higher homotopy groups of the fibre space $G \to G/H$ provided a basic example for an unusual serendipity, brought about by the isolation of research groups during World War II, in which three groups found the same property, the covering homotopy property, as the key tool to unlocking the applications of fibre spaces.

The main problem had changed, however, and the focus turned to the question of computation of homotopy groups of spaces. The homology theory of fibre spaces was being developed by others during the war, particularly by students of Hopf, and by J. Leray (1906–1998) while in a prisoner-of-war camp (see [32]). By 1950, the time of Steenrod's textbook, fibre spaces were well established as basic tools in algebraic topology. The next big event in the story is the doctoral thesis of J.-P. Serre (1926–) who revised the definition of fibre space further by placing the covering homotopy property at the center [52]. The homological methods developed by Serre infused homotopy theory with a set of tools and examples that deepened the subject considerably. The introspective development of the subject mentioned by Steenrod bore fruit, and new connections of the subject to other fields, such as differential geometry, were added. A mature subject emerged in the following decade. The fifty years of development between Poincaré and Serre was particularly vital, led by hares like Whitney and tortoises like Stiefel, that is, via a blend of generality and well-grounded examples. The story of the development of fibre spaces—of the examples and definitions—is one of "introspective development" and "revitalization," an instance of the usual development of modern mathematics.

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