9. The Jordan Curve Theorem

It is established then that every continuous (closed) curve divides the plane into two regions, one exterior, one interior, ...

Camille Jordan, 1882

In his 1882 *Cours d'analyse* [Jordan], Camille Jordan (1838–1922) stated a classical theorem, topological in nature and inadequately proved by Jordan. The theorem concerns separation and connectedness in the plane on one hand, and the topological properties of simple, closed curves on the other.

The Jordan Curve Theorem. *If $\mathcal{C}$ is a simple, closed curve in the plane $\mathbb{R}^2$, that is, $\mathcal{C} \subset \mathbb{R}^2$ and $\mathcal{C}$ is homeomorphic to S^1, then $\mathbb{R}^2 - \mathcal{C}$, the complement of $\mathcal{C}$, has two components, each sharing $\mathcal{C}$ as boundary.*

The statement of the theorem borders on the obvious—few would doubt it to be true. However, mathematicians of the nineteenth century had developed a healthy respect for the monstrous possibilities that their new researches into analysis revealed. Furthermore, a proof using rigorous and appropriate tools of a fact that seemed obvious meant that the obvious was a solid point of departure for generalization.

The proof that follows is an amalgam of two celebrated proofs—the principal part is based on work of Brouwer in which the notion of the *index* of a point relative to a curve plays a key role. Brouwer's proof was simplified by Erhard Schmidt (1876–1959) (see [Schmidt] and [Alexandroff]). The second proof, due to J. W. Alexander (1888–1971) is based on the combinatorial and algebraic notion of a *grating* (see [Newman]). Although each proof can be developed independently, the main ideas of combinatorial approximation and an index provide a point of departure for generalizations that will be the focus of the final chapters.

A **Jordan curve**, or *simple, closed curve*, is a subset $\mathcal{C}$ of $\mathbb{R}^2$ that is homeomorphic to a circle. A **Jordan arc**, or *simple arc*, is a subset of $\mathbb{R}^2$ homeomorphic to a closed line segment in $\mathbb{R}$. A choice of homeomorphism gives a parameterization of the Jordan curve or arc, $\alpha\colon [0,1] \to \mathbb{R}^2$, as the composite of the homeomorphism $f\colon S^1 \to \mathcal{C} \subset \mathbb{R}^2$ with $w\colon [0,1] \to S^1$, given by $w(t) = (\cos 2\pi t, \sin 2\pi t)$. A Jordan curve will have many choices of parameterization α and so relevant properties of the curve $\mathcal{C}$ must be shown to be independent of the choice of α. Notice that the parameterization $\alpha\colon [0,1] \to \mathbb{R}^2$ is one-one on $[0,1)$ and $\alpha(0) = \alpha(1)$.

Gratings and arcs

We begin by analyzing the separation properties of Jordan arcs. Choose a homeomorphism $\lambda\colon [0,1] \to \Lambda \subset \mathbb{R}^2$, which parameterizes an arc. Notice that $\Lambda = \lambda([0,1])$ is compact and closed in $\mathbb{R}^2$ and so $\mathbb{R}^2 - \Lambda$ is open.

Separation Theorem for Jordan arcs. *A Jordan arc Λ does not separate the plane, that is, $\mathbb{R}^2 - \Lambda$ is connected.*

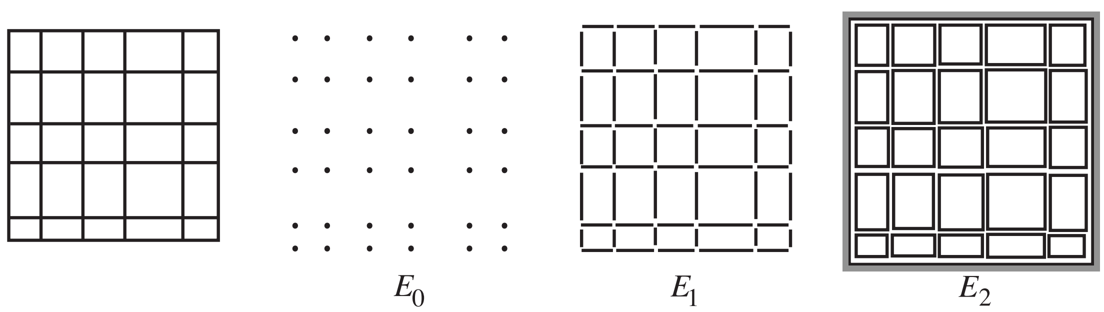
Since $\mathbb{R}^2$ is locally path-connected, the complement of Λ is connected if and only if it is path-connected. An intuitive argument to establish the separation theorem begins with a pair of points P and Q in $\mathbb{R}^2 - \Lambda$. We can join P and Q by a path in $\mathbb{R}^2$, and then try to show that the path can be modified to a path that avoids Λ. It may be the case that Λ is very complicated, and a general proof requires great care to show that you can always find such a path.

Toward a rigorous argument we introduce a combinatorial structure that will allow us to make the modifications of paths in a methodical manner and so turn intuition into proof. The combinatorial structure is interesting in its own right—it combines approximation and algebraic manipulation, features that will be generalized to spaces in the remaining chapters. It is the interplay between the topological and combinatorial that makes this structure so useful. I have followed the classic text of Newman [Newman] in this section.

A *square region* in the plane is a subset $S = [a, a+s] \times [b, b+s] \subset \mathbb{R}^2$ where $a, b \in \mathbb{R}$ and $s > 0$. The region may be subdivided into rectangles by choosing values

$$a = a_0 < a_1 < a_2 < \cdots < a_{n-1} < a_n = a+s, \quad b = b_0 < b_1 < b_2 < \cdots < b_{m-1} < b_m = b+s,$$

with subrectangles given by $[a_i, a_{i+1}] \times [b_j, b_{j+1}]$ for $0 \leq i < n$ and $0 \leq j < m$. Such a subdivision is called a **grating**, introduced by Alexander in [Alexander]. We denote a grating by $\mathcal{G} = (S, \{a_i\}, \{b_j\})$.



To a grating $\mathcal{G}$ we associate the following combinatorial data:

i) $E_0(\mathcal{G}) = \{(a_i, b_j) \in \mathbb{R}^2 \mid 0 \leq i \leq n, 0 \leq j \leq m\}$, its set of *vertices or* 0*-cells*;
ii) $E_1(\mathcal{G}) = \{PQ \mid P = (a_i, b_j) \text{ and } Q = (a_{i+1}, b_j) \text{ or } (a_i, b_{j+1})\}$, its set of *edges or* 1*-cells*, and
iii) $E_2(\mathcal{G}) = \{[a_i, a_{i+1}] \times [b_j, b_{j+1}] \subset \mathbb{R}^2 \mid 0 \leq i < n, 0 \leq j < m\} \cup \{\mathcal{O}\}$, its set of *faces or* 2*-cells*, where the 'outside face' $\mathcal{O}$ is the face that is exterior to the grating, that is, $\mathcal{O} = \mathbb{R}^2 - \operatorname{int} S$.

Including the 'outside face' $\mathcal{O}$ simplifies the statement of later results.

To emphasis the difference between the combinatorics and the topology, we introduce the **locus** of an i-cell, denoted $|u|$ for $u \in E_i(\mathcal{G})$, defined to be the subset of $\mathbb{R}^2$ that underlies u. For example, if $PQ \in E_1(\mathcal{G})$, then $|PQ| = \{(1-t)P + tQ \mid t \in [0,1]\} \subset \mathbb{R}^2$ when $P = (a_i, b_j)$ and $Q = (a_{i+1}, b_j)$ or $P = (a_i, b_j)$ and $Q = (a_i, b_{j+1})$. Define the following subspaces of $\mathbb{R}^2$,

$$sk_0(\mathcal{G}) = \bigcup\nolimits_{u \in E_0} |u| = E_0(\mathcal{G}); \quad sk_1(\mathcal{G}) = \bigcup\nolimits_{u \in E_1} |u|; \text{ and } sk_2(\mathcal{G}) = \bigcup\nolimits_{u \in E_2} |u| = \mathbb{R}^2.$$

The subspace $sk_0(\mathcal{G})$ is a discrete set and $sk_1(\mathcal{G})$ is a union of line segments. For topological constructions with vertices or edges, such as finding boundaries or interiors, we restrict to these subspaces of $\mathbb{R}^2$.

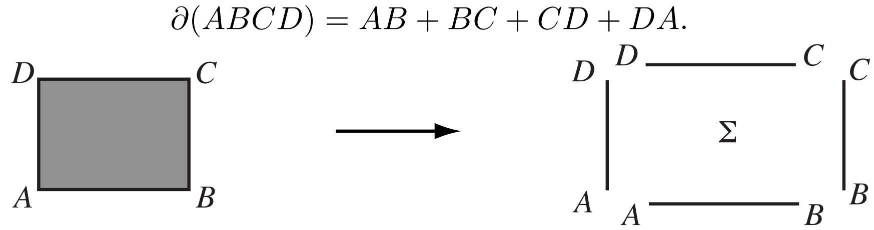
Suppose we have two elements $u,\ v \in E_1(\mathcal{G})$ with $u = PQ$ and $v = QR$. The boundaries in the subspace $sk_1(\mathcal{G})$ of $|u|$ and $|v|$ are given by $\operatorname{bdy}_{sk_1}|u| = \operatorname{bdy}_{sk_1} PQ = \{P, Q\}$ and $\operatorname{bdy}_{sk_1}|v| = \{Q, R\}$. The union $|u| \cup |v| = PQ \cup QR$ has boundary $\{P, R\}$, because Q has become an interior point in the subspace topology on $sk_1(\mathcal{G})$, as in the picture:

We can encode this topological fact in an algebraic manner by associating a union to an addition of cells and boundary to a linear mapping between sums.

Definition 9.1. *Let $\mathbb{F}_2$ denote the field with two elements, that is, $\mathbb{F}_2 = \mathbb{Z}/2\mathbb{Z}$. Let the (vector)* **space of i-chains** *on $\mathcal{G}$ be defined by $C_i(\mathcal{G}) = \mathbb{F}_2[E_i(\mathcal{G})]$, the vector space over $\mathbb{F}_2$ with basis the set $E_i(\mathcal{G})$ for $i = 0, 1, 2$. The* **boundary operator** *on chains is the linear transformation $\partial\colon C_i(\mathcal{G}) \to C_{i-1}(\mathcal{G})$, for $i = 1, 2$, defined on the basis by $\partial(u) = \sum_l e_l^{i-1}$, where $\partial(u)$ is the sum of the $i-1$-cells in $C_{i-1}(\mathcal{G})$ that make up the boundary of the i-cell u, that is, the sum is over $i-1$-cells that satisfy $|e_l^{i-1}| \subset \operatorname{bdy}_{sk_i}|u|$.*

For example, the boundary operator on a face $ABCD \in E_2(\mathcal{G})$ is given by

$$\partial(ABCD) = AB + BC + CD + DA.$$

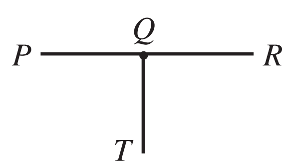


Elements of $C_i(\mathcal{G})$ take the form $\sum_{k=1}^n e_k^i$ where $e_k^i \in E_i(\mathcal{G})$ and n is finite. The boundary operator is extended to sums by linearity, $\partial(\sum_{k=1}^n e_k^i) = \sum_{k=1}^n \partial(e_k^i) \in C_{i-1}(\mathcal{G})$.

The manner in which the combinatorial structure mirrors the topological situation is evident when we compare the formulas:

$$\partial(PQ + QR) = P + Q + Q + R = P + 2Q + R = P + R; \quad \operatorname{bdy}_{sk_1}|PQ| \cup |QR| = \{P, R\}.$$

Because $2 = 0$ in $\mathbb{F}_2$, we can drop the term $2Q$. One must be cautious in using these parallel notions—for example,



$$\partial(PQ + QR + QT) = P + Q + R + T; \text{ while } \operatorname{bdy}_{sk_1}|PQ| \cup |QR| \cup |QT| = \{P, R, T\}.$$

To compare chains and their underlying sets, we extend the notion of locus to chains. If $c = \sum_{l=1}^n e_l^i$, then the *locus of c* is

$$|c| = \left| \sum_{l=1}^n e_l^i \right| = \bigcup_{l=1}^n |e_l^i|.$$

The addition of chains is related to their locus by a straightforward topological condition.

LEMMA 9.2. *If c_1 and c_2 are i-chains, then $|c_1+c_2| \subset |c_1| \cup |c_2|$, and $|c_1+c_2| = |c_1| \cup |c_2|$ if and only if $\mathrm{int}_{sk_i}|c_1| \cap \mathrm{int}_{sk_i}|c_2| = \emptyset$.*

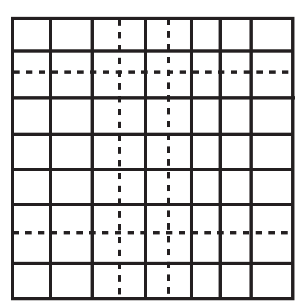
Proof: Since the locus of an i-cell e^i is a subset of $|c_j|$ $(j = 1, 2)$ whenever the cell occurs in the sum c_j, the union $|c_1| \cup |c_2|$ contains the locus of every cell that appears in either c_1 or c_2. It is possible for a cell to vanish from the algebraic sum if it occurs once in both chains. Thus $|c_1+c_2| \subset |c_1| \cup |c_2|$. For equality, we need that no i-cell in the sum c_1 appear in c_2. The topological condition on the interiors of cells is equivalent to this condition. $\diamondsuit$

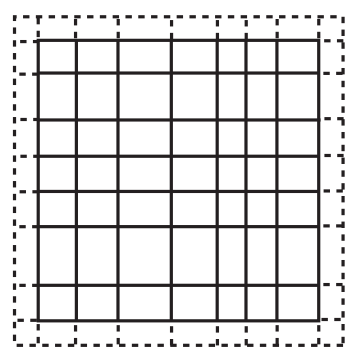
Another relation between the combinatorial and the topological holds for 2-chains.

PROPOSITION 9.3. *If $w \in C_2(\mathcal{G})$, then* bdy $|w| = |\partial(w)|$.

Proof: Observe that every edge in $E_1(\mathcal{G})$ is contained in two faces (for this, you need the outside face $\mathcal{O}$ counted among faces). So, if PQ is an edge in $\partial(w)$, then PQ appears only once among the boundaries of faces in w. If x is any point of $|PQ|$, then any open ball centered at x meets the interior of the face w and the exterior of the set $|w|$ and so x is in bdy $|w|$. Conversely, if $x \in$ bdy $|w|$, then x is an element of the locus $|PQ|$ which is an edge PQ in the boundary of a face e^2 in the sum determined by w. Since any open ball centered at x meets points outside $|w|$, the face sharing PQ with e^2 is not in w and so PQ is an edge in $\partial(w)$. $\diamondsuit$

A grating can be *refined* by adding vertical and horizontal lines. We could also expand the square region, adding cells that extend the given grating.





We leave it to the reader to give an expression for the partition of the square region that determines a refinement from the data for a grating. By adding lines we can subdivide the rectangles to have any chosen maximum diameter, no matter how small. We use such an approximation procedure to avoid certain subsets of the plane.

LEMMA 9.4. *Let K_1 and K_2 be disjoint compact subsets of $\mathbb{R}^2$ and S a square region with $K_1 \cup K_2 \subset S$. Then any grating $\mathcal{G}$ of S can be refined to a grating $\mathcal{G}^*$ so that no cell of $\mathcal{G}^*$ meets both K_1 and K_2.*

Proof: Since K_1 and K_2 are disjoint and compact, there is a distance $\epsilon > 0$ such that, for any $x \in K_1$ and $y \in K_2$, $d(x, y) \geq \epsilon$. Given the grating $\mathcal{G}$, subdivide the square further so that the diameter of any rectangle is less than $\epsilon/2$. If the locus of a cell contains points x and y, then $d(x, y) < \epsilon/2$ and so it cannot be that $x \in K_1$ and $y \in K_2$. $\diamondsuit$

We next consider how the combinatorial data are affected by refinement. Of course, certain vertices will be added, edges subdivided and added, and faces subdivided. If $\mathcal{G}$ is refined to a grating $\mathcal{G}^*$ and $c \in C_i(\mathcal{G})$, then we write $c^* \in C_i(\mathcal{G}^*)$ for the i-chain consisting of the the i-cells involved in the subdivision of the i-cells in c. For example, if a 2-cell $ABCD$ is refined by adding a horizontal and a vertical line, then AB is subdivided as AMB, BC as BNC, CD as $CM'D$ and DA as $DN'A$, and we add the vertex P where MM' meets NN'. Then $c^* = AMPN' + MBNP + NCM'P + M'DN'P$. Refinement does not change the locus of an i-cell, that is, $|c| = |c^*|$.

LEMMA 9.5. *If c_1 and c_2 are i-chains in $C_i(\mathcal{G})$, then $(c_1+c_2)^* = c_1^*+c_2^*$ and $(\partial c_1)^* = \partial(c_1^*)$.*

Proof: When we subdivide an i-cell, the number of times (once or not at all) it appears in an i-chain is the same for the parts that constitute its subdivision. Thus the number of times the i-cell appears in the sum will be the same as the number of times the parts appear in the sum of the refined chains and $(c_1 + c_2)^* = c_1^* + c_2^*$.

As for the boundary operator, for 1-chains, subdivision introduces a new intermediate vertex, shared by the 1-cells of the subdivided edge. Thus the new vertices do not appear in $\partial(c^*)$; since refinement does not affect the 0-cells of $\mathcal{G}$, we have $\partial(c^*) = \partial(c) = (\partial c)^*$. For 2-chains,

$$|\partial(c_1^*)| = \mathrm{bdy}\,|c_1^*| = \mathrm{bdy}\,|c_1| = |\partial(c_1)| = |(\partial c_1)^*|.$$

Since $\partial(c_1^*)$ and $(\partial c_1)^*$ are 1-chains in $\mathcal{G}^*$ with the same loci, they are the same 1-chains.$\diamondsuit$

The combinatorial data provided by chains can be used to study the connectedness of subsets of $\mathbb{R}^2$.

DEFINITION 9.6. *The* **components** *of an i-chain $c \in C_i(\mathcal{G})$ are the components of its locus, $|c| \subset \mathbb{R}^2$. We say that two vertices P and Q in a grating $\mathcal{G}$ can be* **connected** *if there is a 1-chain $\lambda \in C_1(\mathcal{G})$ with $\partial(\lambda) = P + Q$. A subset $A \subset \mathbb{R}^2$* **separates** *the vertices P and Q in $\mathbb{R}^2 - A$ if any 1-chain λ connecting P to Q meets A (that is, $|\lambda| \cap A \neq \emptyset$).*

We investigate how these combinatorial notions of component and connectedness compare with the usual topological notions.

PROPOSITION 9.7. *Suppose $\mathcal{G}$ is a grating. If c is an i-chain and $c = c_1 + \cdots + c_n$ where each c_j is a maximally connected chain, then the components of $|c|$ are the loci $|c_j|$.*

Proof: If c_j is a maximally connected chain in c, then its locus is connected and $|c_j|$ does not meet the loci of the other chains c_m, $j \neq m$, because if $|c_j| \cap |c_m| \neq \emptyset$, then the chains share an edge ($i = 2$) or a vertex ($i = 1, 2$). In this case, $|c_j| \cup |c_m|$ is connected and $c_j + c_m$ is a connected chain larger than c_j or c_m and hence they are not maximal, a contradiction. Thus the components of c are the maximally connected chains in the sum determined by the chain c. $\diamondsuit$

PROPOSITION 9.8. *If $A \subset \mathbb{R}^2$ is compact and P and Q are points in $\mathbb{R}^2 - A$, then there is a path in $\mathbb{R}^2 - A$ connecting P to Q if and only if there is a grating $\mathcal{G}$ for which P and Q are vertices, and there is a 1-chain λ with $P + Q = \partial(\lambda)$.*

Proof: Suppose we are given a grating $\mathcal{G}$. If ω is a 1-chain, then we first show that the boundary $\partial(\omega)$ has an even number of vertices. We prove this by induction on the number of 1-cells in the 1-chain. If ω has only one 1-cell, then $\omega = PQ$ and $\partial(\omega) = P + Q$, two vertices. Suppose $\omega = \sum_{i=1}^{n} e_i^1$. Then $\partial(\omega) = \partial(e_1^1) + \sum_{i=2}^{n} \partial(e_i^1)$. By induction, $\sum_{i=2}^{n} \partial(e_i^1)$ is a sum of an even number of vertices. We add to this sum $\partial(e_1^1)$ which consists of two vertices. If either vertex appears in $\sum_{i=2}^{n} \partial(e_i^1)$, then the pair cancels and parity is preserved. Thus $\partial(\omega)$ is a sum of an even number of vertices.

Suppose that $P+Q = \partial(\lambda)$ for some 1-chain $\lambda \in C_1(\mathcal{G})$. If $\lambda = \lambda_1 + \cdots + \lambda_n$ with each λ_i a maximally connected 1-chain in λ, then $\partial(\lambda) = \partial(\lambda_1) + \cdots \partial(\lambda_n) = P+Q$. Since P and Q must be part of the sum, we can assume that $P + \text{stuff}_1 = \partial(\lambda_1)$ and $Q + \text{stuff}_2 = \partial(\lambda_n)$. Since all the extra stuff must cancel to give $\partial(\lambda) = P + Q$, any vertex appearing in stuff_1 joins λ_1 to another component and so such components were not maximal. Arguing in this manner, we can join P to Q by a connected 1-chain. One can then parameterize the locus of that 1-chain giving a path joining P to Q.

Finally, suppose P and Q are in $\mathbb{R}^2 - A$, an open set. If we can join P to Q by a continuous mapping in $\mathbb{R}^2 - A$, then the image of that path is compact and so some distance $\epsilon > 0$ away from A. Working in the open balls of radius $\epsilon/2$ around points along the curve joining P to Q, we can substitute the path with a path made up of vertical and horizontal line segments. After finding such a path, we extend the line segments to a grating in which the polygonal path is the locus of a 1-chain λ with $\partial(\lambda) = P + Q$. $\diamondsuit$

Since a grating $\mathcal{G} = (S, \{a_i\}, \{b_j\})$ is described by finite sets, we can develop some of the purely combinatorial properties of these sets. In particular, the sets $E_i(\mathcal{G})$ are finite, and so we can form the sum of all i-cells into a special i-chain, the *total i-chain*, denoted

$$\Theta^i = \sum\nolimits_{e^i \in E_i(\mathcal{G})} e^i.$$

Notice that $\partial\Theta^2 = 0$. This follows from the fact that every edge is contained in exactly two cells.

The classes Θ^i give an algebraic expression for the **complement** of an i-chain c, which is denoted by Cc, and defined to be $Cc = \sum_l e_l^i$, where the sum is over all i-cells e_l^i that do *not* appear in the sum c. This sum is easily recovered by observing

$$Cc = c + \Theta^i.$$

Any i-cell appearing in the sum c is cancelled by itself in Θ^i, leaving only the i-cells that did not appear in c.

It is an immediate consequence of the formulas $Cc = c + \Theta^i$ and $\partial\Theta^2 = 0$ that if a 1-chain λ is the boundary of a 2-chain, then $\lambda = \partial(w) = \partial(Cw)$, and so it is the boundary of two complementary 2-chains. This follows from the algebraic version of the complement

$$\partial(Cw) = \partial(w + \Theta^2) = \partial(w) + \partial(\Theta^2) = \lambda.$$

The complement operation leads to a combinatorial version of the Jordan Curve Theorem.

Definition 9.9. *An i-chain $c \in C_i(\mathcal{G})$ is an i-**cycle** if $\partial(c) = 0$.*

Theorem 9.10. *Every 1-cycle on a grating $\mathcal{G}$ is the boundary of exactly two 2-chains.*

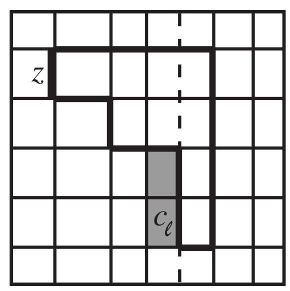
Proof: First observe that the only 2-cycles are 0 and Θ^2. This follows from Proposition 9.3 that $|\partial(c)| = \operatorname{bdy}|c|$ for 2-chains. Any nonzero 2-chain c, with $c \neq \Theta^2$, has a nonempty boundary and so is not a 2-cycle.

We prove the theorem by induction on the number of lines involved in the grating. The minimal grating has only the boundary lines of the square region S as edges. The only 2-cells are $ABCD$ and $\mathcal{O}$ both of which satisfy

$$\partial(\mathcal{O}) = \partial(ABCD) = AB + BC + CD + DA.$$

Furthermore, the only nonzero 1-cycle for this grating is $AB + BC + CD + DA = \Theta^1$, so the theorem holds.

Suppose that the theorem holds for a grating $\mathcal{G}$ and we refine $\mathcal{G}$ to $\mathcal{G}^*$ by adding a single vertical line ℓ. (The argument for adding a single horizontal line is analogous.) Suppose that z is a 1-cycle in $C_1(\mathcal{G}^*)$. Define c_ℓ to be the 2-chain which is the sum of all 2-cells with right edges that are on ℓ *and* in the sum z.



By cancellation, $z + \partial(c_\ell)$ has no edges on ℓ and so we can consider $z + \partial(c_\ell)$ as a 1-chain on $\mathcal{G}$. Furthermore, $\partial(z + \partial(c_\ell)) = \partial(z) + \partial\partial(c_\ell) = 0$, so $z + \partial(c_\ell)$ is a 1-cycle. Since the theorem holds for $\mathcal{G}$, $z + \partial(c_\ell) = \partial(c)$ for some $c \in C_2(\mathcal{G})$. The 2-chain $c^* + c_\ell \in C_2(\mathcal{G})$ has boundary given by

$$\partial(c^* + c_\ell) = (z + \partial(c_\ell))^* + \partial(c_\ell) = (z + \partial(c_\ell)) + \partial(c_\ell) = z.$$

Thus z is the boundary of $c^* + c_\ell$. It is also the boundary of the complement of this 2-chain, $C(c^* + c_\ell) = c^* + c_\ell + \Theta^2 \in C_2(\mathcal{G}^*)$. This follows from $\partial\Theta^2 = 0$.

Finally, we check that no other 2-cell has z as boundary. Suppose b, $b' \in C_2(\mathcal{G})$. If $\partial(b) = \partial(b')$, then $\partial(b + b') = 0$ and so $b + b' = 0$ or $b + b' = \Theta^2$. Then $b = b'$ or $b = Cb'$. Thus, at most two 2-cells can have z as boundary. ♢

On a grating, a 1-cycle that is simple (connected without crossings) is a Jordan curve. Theorem 9.10 is a combinatorial version of the Jordan Curve Theorem. The next theorem uses what we have developed so far to establish a general result about separation. It is the key lemma in the proof of the Separation Theorem for Jordan arcs.

THE ALEXANDER THEOREM. *Suppose K and L are compact subsets of $\mathbb{R}^2$ and $\mathcal{G}$ a grating of a square S with $K \cup L \subset S$. If $P + Q = \partial(\lambda_1)$ in $\mathbb{R}^2 - K$ and $P + Q = \partial(\lambda_2)$ in $\mathbb{R}^2 - L$, and $\lambda_1 + \lambda_2 = \partial(w)$ with $|w| \cap K \cap L = \emptyset$, then P is connected to Q by a path that does not meet $K \cup L$.*

Proof: Since $|w| \cap K \cap L = \emptyset$, the compact sets $|w| \cap K$ and $|w| \cap L$ are disjoint. By Lemma 9.3 there is a refinement $\mathcal{G}^*$ of the grating for which no 2-cell meets both $|w| \cap K$

and $|w| \cap L$. Let $w_K = \sum_i e_i^2 \in C_2(\mathcal{G}^*)$ where the sum is over the set $\{e_i^2 \mid e_i^2$ is a 2-cell in w^* and $|e_i^2| \cap K \neq \emptyset\}$. Define the 1-chain $\lambda_0 = \lambda_2^* + \partial(w_K)$. It follows immediately that

$$\partial(\lambda_0) = \partial(\lambda_2^* + \partial(w_K)) = \partial(\lambda_2^*) = P + Q.$$

We know that λ_2^* does not meet L. Since none of the faces of $\mathcal{G}^*$ meet both $|w| \cap K$ and $|w| \cap L$, w_K does not meet L.

To prove the theorem we show that λ_0 does not meet K. Consider the loci:

$$|\lambda_0| = |\lambda_2^* + \partial(w_K)| = |\lambda_1^* + (\lambda_1^* + \lambda_2^* + \partial(w_K))| = |\lambda_1^* + \partial(w^* + w_K)| \subset |\lambda_1| \cup \mathrm{bdy}\, |w^* + w_K|.$$

By assumption, λ_1 does not meet K and so λ_1^* does not meet K. In the sum $w^* + w_K$, any 2-cells of w^* that meet K are cancelled by w_K and so $w^* + w_K$ does not meet K. Therefore, $|\lambda_0| \cap K = \emptyset$. Since λ_0 joins P and Q and does not meet $K \cup L$, the theorem is proved. $\diamondsuit$

Corollary 9.11. *Suppose Λ is a Jordan arc and $\lambda\colon [0,1] \to \Lambda \subset \mathbb{R}^2$ is a parameterization. Let $L_1 = \lambda([0,1/2])$ and $L_2 = \lambda([1/2,1])$. If P is connected to Q in $\mathbb{R}^2 - L_1$ and in $\mathbb{R}^2 - L_2$, then P is connected to Q in $\mathbb{R}^2 - \Lambda$.*

To prove the corollary, simply choose paths that avoid $\lambda(1/2) = L_1 \cap L_2$.

We deduce immediately that if Λ separates P from Q, then one of L_1 or L_2 separates P from Q. From this observation we can give a proof of the Separation Theorem for Jordan arcs. Suppose a Jordan arc Λ separates P from Q, then one of the subsets L_1 or L_2 separates P from Q. Say it is L_1. Then $L_1 = \lambda([0,1/4]) \cup \lambda([1/4,1/2])$ and one of these subsets must separate P from Q by Corollary 9.11. We write L_{1i_2} for a choice of subset that separates P from Q. Halving the relevant subset of $[0,1/2]$ again we can write $L_{1i_2} = L_{1i_21} \cup L_{1i_22}$ and one of these subsets must separate P from Q. Continuing in this manner we get a sequence of nested compact subsets:

$$\cdots \subset L_{1i_2\cdots i_{n-1}i_n} \subset L_{1i_2\cdots i_{n-1}} \subset \cdots L_{1i_2} \subset L_1$$

with the property that each subset separates P from Q. By the intersection property of nested compact sets (Exercise 6.3), $\bigcap_n L_{1i_2\cdots i_n} = R$, a point on Λ. Since the endpoints of the $L_{1i_2\cdots i_n}$ constitute a series that converges to R, given an $\epsilon > 0$, there is a natural number N for which $L_{1i_2\cdots i_n} \subset B(R,\epsilon)$ for $n \geq N$. By choosing a grating $\mathcal{G}$ to contain P and Q as vertices and for which the subset $B(R,\epsilon) \subset \mathrm{int}|w|$ for some $w \in E_2(\mathcal{G})$, we can join P to Q without meeting $L_{1i_2\cdots i_N}$, a contradiction. It follows that Λ does not separate P from Q and the theorem is proved. $\diamondsuit$

From this point it is possible to give a proof of the Jordan curve theorem using the methods developed so far. Such a proof is outlined in the exercises (or see [Newman]). We instead use the fundamental group to introduce an integer-valued index whose properties lead to a proof of the Jordan Curve Theorem.

THE INDEX OF A POINT NOT ON A JORDAN CURVE

Suppose that $\Omega \in \mathbb{R}^2 - \mathcal{C}$ is a point in $\mathbb{R}^2$ not on a Jordan curve $\mathcal{C}$. To the choice of Ω and a parametrization of $\mathcal{C}$, $\alpha\colon [0,1] \to \mathcal{C} \subset \mathbb{R}^2$, we associate

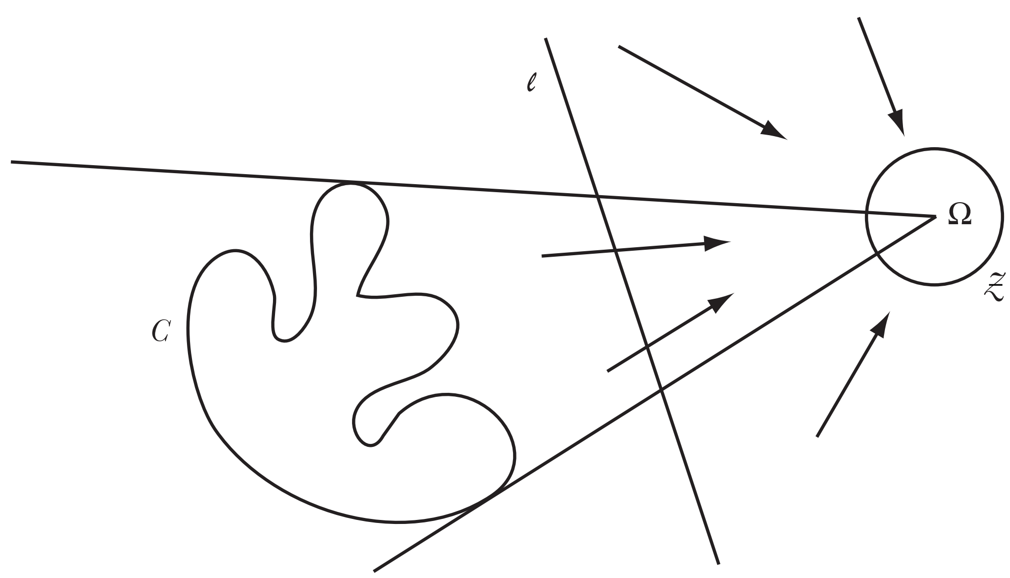
$$\mathrm{ind}_\Omega(\alpha) = [\alpha] \in \pi_1(\mathbb{R}^2 - \{\Omega\}, \alpha(0)),$$

that is, $\mathrm{ind}_\Omega(\alpha)$ is the homotopy class of the closed curve α in the fundamental group of $\mathbb{R}^2 - \{\Omega\}$ based at $\alpha(0)$. Since the plane with a point removed has the homotopy type of a circle, $\mathrm{ind}_\Omega(\alpha)$ determines an integer via a choice of an isomorphism $\pi_1(\mathbb{R}^2 - \{\Omega\}, \alpha(0)) \cong \mathbb{Z}$. The integer is determined up to a choice of sign and so we write $\mathrm{ind}_\Omega(\alpha) = \pm k \in \mathbb{Z}$ when convenient. We call the choice of integer $\mathrm{ind}_\Omega(\alpha)$ the *index of* Ω *with respect to* α.

Example: Suppose $\triangle ABC$ is a triangle in the plane and Ω is an interior point. Since $\triangle ABC \simeq S^1$ and Ω may be chosen as a center of S^1, $\mathrm{ind}_\Omega(\triangle ABC) = \pm 1$.

We develop the properties of the index from the basic properties of the fundamental group (Chapters 7 and 8).

LEMMA 9.12. *If ℓ is a line in the plane that does not meet $\mathcal{C}$, and Ω and $\mathcal{C}$ lie on opposite sides of ℓ, then* $\mathrm{ind}_\Omega(\alpha) = 0$ *for any parameterization of $\mathcal{C}$.*



Proof: Let $\mathcal{Z}$ be a small circle centered at Ω entirely in the half-plane determined by ℓ and Ω. We can take $\mathcal{Z}$ as the copy of S^1 which generates $\pi_1(\mathbb{R}^2 - \{\Omega\})$. Since $\mathcal{C}$ is compact and lies on the side of ℓ opposite Ω, all of $\mathcal{C}$ lies in an angle with vertex Ω that is less than two right angles. In the deformation retraction of $\mathbb{R}^2 - \{\Omega\}$ to $\mathcal{Z}$, $\mathcal{C}$ will be taken to a part of $\mathcal{Z}$ where it can be deformed to a point. Thus $\mathrm{ind}_\Omega(\alpha) = 0$ for any choice of parameterization of $\mathcal{C}$. $\diamondsuit$

The next result takes its name from the shape of the Greek letter θ. Suppose that $\mathcal{C}$ is parameterized in two parts as $\alpha * \gamma\colon [0,1] \to \mathcal{C} \subset \mathbb{R}^2$, where $\alpha(t)$ parameterizes part of the curve, and then $\gamma(t)$ takes over to end at $\gamma(1) = \alpha(0)$. Recall that

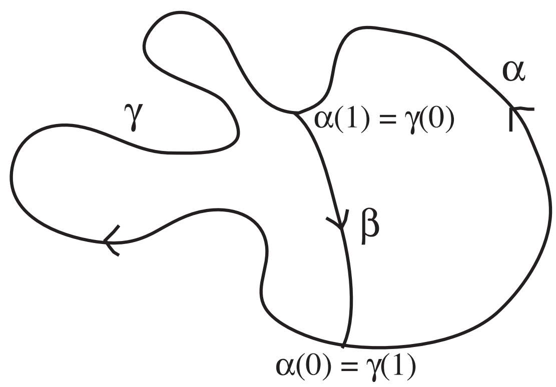
$$\alpha * \gamma(t) = \begin{cases} \alpha(2t), & 0 \le t \le 1/2, \\ \gamma(2t-1), & 1/2 \le t \le 1. \end{cases}$$

Suppose that there is a Jordan arc, parameterized by $\beta\colon [0,1] \to \mathbb{R}^2$, joining $\alpha(1) = \beta(0)$ to $\alpha(0) = \beta(1)$, for which $\beta(t) \notin \mathcal{C}$ for $0 < t < 1$. Then we have three loops beginning at $\alpha(0)$, namely,

$$\omega_0 = \alpha * \gamma, \quad \omega_1 = \alpha * \beta, \text{ and } \omega_2 = \beta^{-1} * \gamma,$$

where $\beta^{-1}(t) = \beta(1-t)$. The index of a point Ω that does not lie on $\mathcal{C}$ or on β can be computed for ω_0, ω_1 and ω_2. The next result relates these values.

THE THETA LEMMA. *In* $\pi_1(\mathbb{R}^2 - \{\Omega\}, \alpha(0))$, *we have* $\mathrm{ind}_\Omega(\omega_0) = \mathrm{ind}_\Omega(\omega_1) + \mathrm{ind}_\Omega(\omega_2)$.



Proof: The binary operation on $\pi_1(\mathbb{R}^2 - \{\Omega\}, \alpha(0))$ is path composition, $*$, which we write as $+$ since $\pi_1(\mathbb{R}^2 - \{\Omega\}, \alpha(0)) \cong \mathbb{Z}$. The lemma follows from the fact that $\beta * \beta^{-1} \simeq c_{\alpha(0)}$, the constant loop at $\alpha(0)$, which is the identity element in the fundamental group:

$$\mathrm{ind}_\Omega(\omega_0) = \mathrm{ind}_\Omega(\alpha*\gamma) = [\alpha*\gamma] = [\alpha*\beta*\beta^{-1}*\gamma] = [\alpha*\beta]+[\beta^{-1}*\gamma] = \mathrm{ind}_\Omega(\omega_1)+\mathrm{ind}_\Omega(\omega_2).\diamondsuit$$

The next property of the index is crucial to the proof of the Jordan curve theorem.

LEMMA 9.13. *If* Ω *and* Ω' *lie in the same path component of* $\mathbb{R}^2 - \mathcal{C}$, *then* $\mathrm{ind}_\Omega(\alpha) = \mathrm{ind}_{\Omega'}(\alpha)$ *for any parameterization of* $\mathcal{C}$.

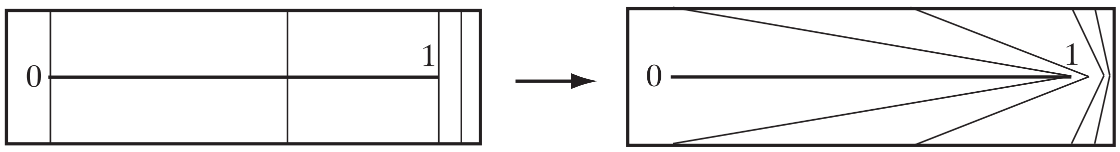
Proof: Suppose $\lambda\colon [0,1] \to \mathbb{R}^2 - \mathcal{C}$ is a piecewise linear curve joining $\Omega = \lambda(0)$ to $\Omega' = \lambda(1)$. Because $\mathbb{R}^2$ is locally path-connected, and $\mathbb{R}^2 - \mathcal{C}$ is an open set, if Ω and Ω' are in the same path component, then it is possible to join them by a piecewise linear curve. We first assume that λ is, in fact, the line segment $\Omega\Omega'$. In the general case, λ will be a finite sequence of line segments connected at endpoints. An induction on the number of such segments completes the argument.

Since the line segment determined by $\Omega\Omega'$ and $\mathcal{C}$ are compact, there is some distance $\epsilon > 0$ between the sets and using this distance we can find a closed rectangle around $\Omega\Omega'$ with the line segment in the center and which is homeomorphic to $[\epsilon, 1+\epsilon] \times [-\epsilon, \epsilon]$. We use this closed rectangle, contained in $\mathbb{R}^2 - \mathcal{C}$, to construct a homeomorphism $F\colon \mathbb{R}^2 - \{\Omega\} \to \mathbb{R}^2 - \{\Omega'\}$ that leaves $\mathcal{C}$ fixed and so induces an isomorphism

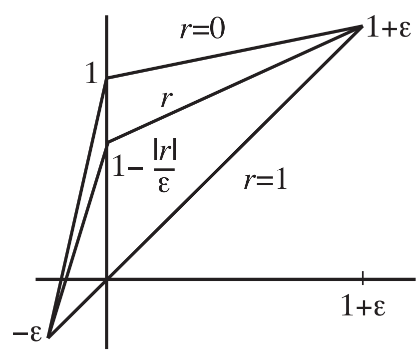
$$F_*\colon \pi_1(\mathbb{R}^2 - \{\Omega\}, \alpha(0)) \longrightarrow \pi_1(\mathbb{R}^2 - \{\Omega'\}, \alpha(0)),$$

that sends $[\alpha] \mapsto [\alpha]$. We construct the homeomorphism on the rectangle by first fixing a nice orientation preserving homeomorphism of $[\epsilon, 1+\epsilon] \times [-\epsilon, \epsilon]$ to the rectangle that takes

$[0, 1] \times \{0\}$ to $\Omega\Omega'$. Then make the desired homeomorphism on $[\epsilon, 1 + \epsilon] \times [-\epsilon, \epsilon]$. It is easier to picture the stretching map that will take 0 to 1.



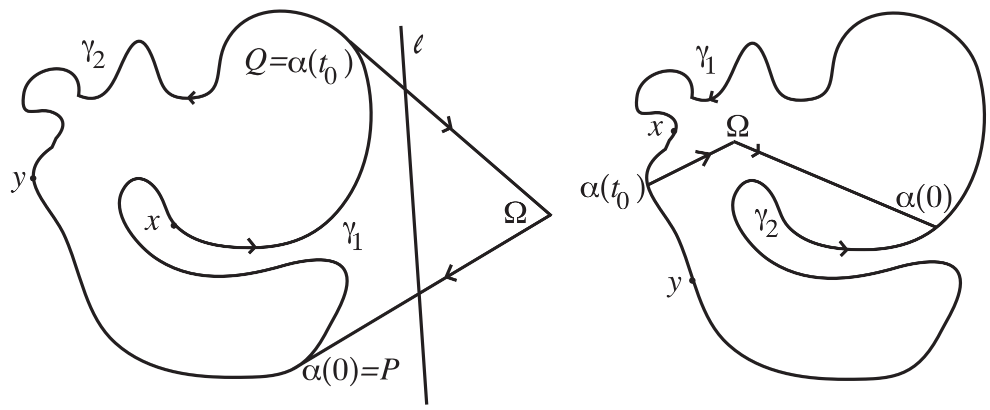
The second parameter, $r \in [-\epsilon, \epsilon]$, is a scaling factor and along each horizontal line segment $[-\epsilon, 1+\epsilon] \times \{r\}$, we stretch toward the right, pushing $[-\epsilon, 0]$ onto $[-\epsilon, 1 - \frac{|r|}{\epsilon}]$ and $[0, 1+\epsilon]$ onto $[1 - \frac{|r|}{\epsilon}, 1 + \epsilon]$. The stretch is the identity along the boundary of the rectangle. The graph of the stretch for various r is shown here:



Pasting this change, suitably scaled and rotated, into $\mathbb{R}^2 - \mathcal{C}$ is possible because the stretch is the identity at the boundary. So we can cut out the first closed rectangle and sew in the stretched one to get the desired homeomorphism.

Finally, orienting the boundary of the closed rectangle, we can take its homotopy class as the loop that generates the fundamental group of both spaces $\mathbb{R}^2 - \{\Omega\}$ and $\mathbb{R}^2 - \{\Omega'\}$. Thus, the induced isomorphism F_* takes $[\alpha]$ to $[\alpha]$ and so $\mathrm{ind}_\Omega(\alpha) = \mathrm{ind}_{\Omega'}(\alpha)$ via the isomorphism. $\diamondsuit$

The constancy of index along a path and the Theta Lemma have the following important consequence. Suppose that ℓ is a line not passing through $\mathcal{C}$, and Ω a point in the half-plane determined by ℓ opposite $\mathcal{C}$. Choose points P and Q on the curve such that the line segments $P\Omega$ and $Q\Omega$ do not meet $\mathcal{C}$ except at the endpoints. Parameterize $\mathcal{C}$ by $\alpha\colon [0, 1] \to \mathcal{C} \subset \mathbb{R}^2$ with $\alpha(0) = P$ and $\alpha(t_0) = Q$. Let $\gamma_1 = \alpha \circ f_1$ where $f_1\colon [0, 1] \to [0, t_0]$ is given by $f_1(s) = t_0 s$. Let $\gamma_2 = \alpha \circ f_2$ where $f_2\colon [0, 1] \to [t_0, 1]$ is given by $f_2(s) = (1 - t_0)s + t_0$. Then $\alpha \simeq \gamma_1 * \gamma_2$. Finally, let $l_1\colon [0, 1] \to \mathbb{R}^2$ and $l_2\colon [0, 1] \to \mathbb{R}^2$ be the line segments, $l_1(t) = (1 - t)\alpha(t_0) + t\Omega$, and $l_2(t) = (1 - t)\Omega + t\alpha(0)$, for $t \in [0, 1]$. These data give the hypotheses for the Theta Lemma with $\omega_0 = \gamma_1 * \gamma_2$, $\omega_1 = \gamma_1 * (l_1 * l_2)$ and $\omega_2 = (l_1 * l_2)^{-1} * \gamma_2$.



Suppose x that lies on γ_1 and y lies on γ_2. Then we can compute the integers $\pm k_1 = \mathrm{ind}_y(\omega_1)$ and $\pm k_2 = \mathrm{ind}_x(\omega_2)$.

Lemma 9.14. *Suppose that* R *is a point in* $\mathbb{R}^2 - (\mathcal{C} \cup P\Omega \cup Q\Omega)$ *and suppose that* $\mathrm{ind}_R(\omega_1) \neq \mathrm{ind}_y(\omega_1)$ *or* $\mathrm{ind}_R(\omega_2) \neq \mathrm{ind}_x(\omega_2)$. *Then* R *can be joined to* Ω *by a path in* $\mathbb{R}^2 - \mathcal{C}$.

Proof: Suppose that $\mathrm{ind}_R(\omega_1) \neq \pm k_1$. Since γ_1 does not separate the plane, there is a path joining R to Ω that does not meet γ_1. Suppose $\zeta\colon [0,1] \to \mathbb{R}^2$ is such a path with $\zeta(0) = R$ and $\zeta(1) = \Omega$, and $\mathrm{im}\ \zeta \cap \mathrm{im}\ \gamma_1 = \emptyset$. Suppose t_1 is the first value in $[0,1]$ with $\zeta(t_1)$ on $l_1 * l_2$, that is, on either line segment $P\Omega$ or ΩQ. Then for $0 \leq t < t_1$, $\mathrm{ind}_{\zeta(t)}(\omega_1)$ is constant. If $\zeta(t)$ meets γ_2 for some $0 \leq t < t_1$, then

$$k_1 \neq \mathrm{ind}_R(\omega_1) = \mathrm{ind}_{\zeta(t)}(\omega_1) = \mathrm{ind}_y(\omega_1) = k_1,$$

a contradiction. Thus ζ on $[0, t_1)$ does not meet γ_1 or γ_2 and so joining ζ restricted to $[0, t_1]$ to the line segment $\zeta(t_1)\Omega$ gives a path from R to Ω. ♢

A proof of the Jordan Curve Theorem

To complete a proof of the Jordan Curve Theorem, consider the following subsets of $\mathbb{R}^2 - \mathcal{C}$:

$$U = \{\Omega \in \mathbb{R}^2 - \mathcal{C} \mid \mathrm{ind}_\Omega(\alpha) = 0\}, \quad V = \{R \in \mathbb{R}^2 - \mathcal{C} \mid \mathrm{ind}_R(\alpha) \neq 0\}.$$

For a pair of points, $\Omega \in U$ and $R \in V$, there is no path joining them because their indices do not agree. It is clear that $U \neq \emptyset$ because $\mathcal{C}$ is compact and there are lines in the plane that separate the curve from points of index zero. We first prove that $V \neq \emptyset$ and then show that U and V are path-connected.

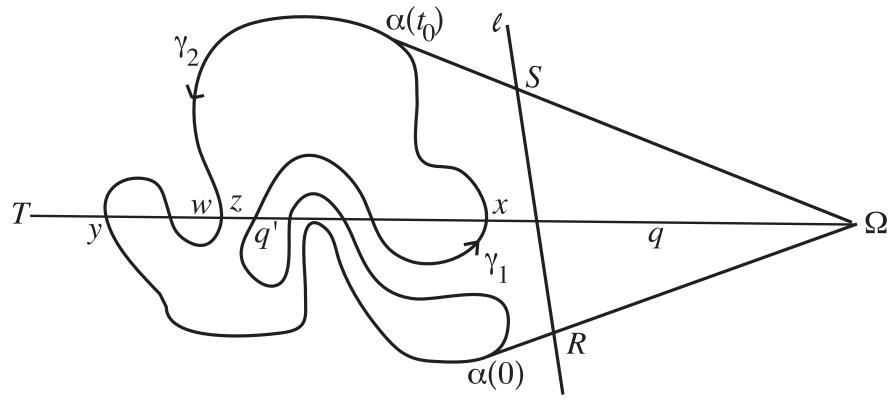
Let ℓ be a line that does not pass through $\mathcal{C}$. Let Ω lie on the side of ℓ opposite $\mathcal{C}$. Introduce the lines ΩP and ΩQ meeting $\mathcal{C}$ at points $P = \alpha(0)$ and $Q = \alpha(t_0)$, respectively, for some parameterization $\alpha\colon [0,1] \to \mathcal{C} \subset \mathbb{R}^2$. Introduce the curves $\gamma_1 = \alpha \circ f_1\colon [0,1] \to \mathbb{R}^2$ with $f_1(s) = t_0 s$, and $\gamma_2 = \alpha \circ f_2\colon [0,1] \to \mathbb{R}^2$ with $f_2(s) = (1-t_0)s + t_0$. Thus $\alpha \simeq \gamma_1 * \gamma_2 = \omega_0$. As in the proof of Lemma 9.14, let $l_1(t) = (1-t)Q + t\Omega$ and $l_2(t) = (1-t)\Omega + tP$ for $t \in [0,1]$. Form the curve $\omega_1 = \gamma_1 * (l_1 * l_2)$, which travels from $\alpha(0)$ along $\mathcal{C}$ to $\alpha(t_0) = Q$, follows $Q\Omega$ to Ω, then ΩP to $P = \alpha(0)$, and $\omega_2 = (l_1 * l_2)^{-1} * \gamma_2$, which first travels from P along $P\Omega Q$, then follows γ_2 around back to P.

We introduce some other curves in this situation. Let ℓ meet ΩP at R and ΩQ at S. If $l_3(t) = (1-t)S + tR$, $l_4(t) = (1-t)Q + tS$ and $l_5(t) = (1-t)R + tP$, then the curve

$\omega_3 = l_5 * \gamma_1 * l_4 * l_3$ together with the triangle $\triangle RS\Omega$ satisfy the conditions for the Theta Lemma. Parametrize the triangle as $l_3^{-1} * l_1' * l_2' = \triangle$, l_1' and l_2' being l_1 and l_2 from S and to R, respectively. The full curve in the Theta Lemma is $\omega_1' \simeq \triangle * \omega_3$ where ω_1' is ω_1 reparameterized to begin and end at R. Suppose that q is a point in the interior of the triangle $\triangle RS\Omega$. Then we know that $\mathrm{ind}_q(\triangle) = \pm 1$. We apply the Theta Lemma to compute

$$\mathrm{ind}_q(\omega_1) = \mathrm{ind}_q(\omega_1') = \mathrm{ind}_q(\triangle) + \mathrm{ind}_q(\omega_3) = \pm 1 + 0 = \pm 1.$$

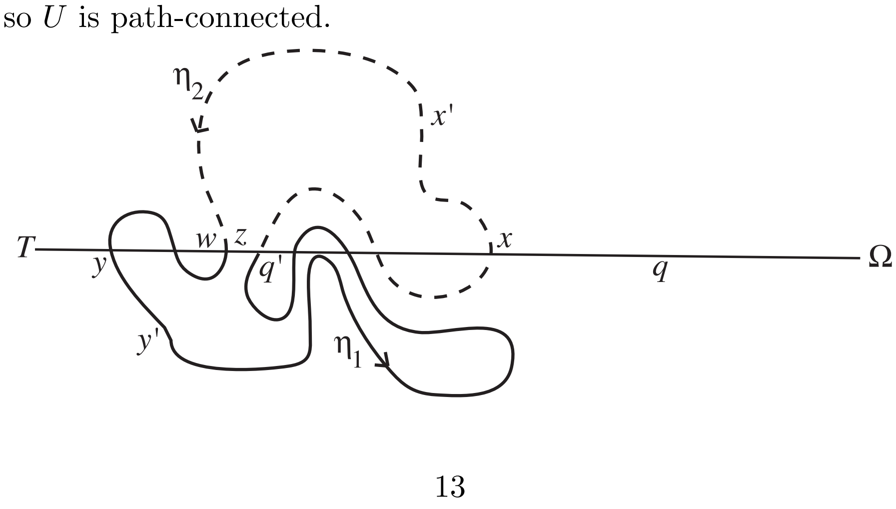
We know that $\mathrm{ind}_q(\omega_3) = 0$ since we can separate q from ω_3 by a line parallel to ℓ but close to ℓ.



Since $\mathrm{ind}_q(\omega_0) = 0$, we find $\mathrm{ind}_q(\omega_2) = \mp 1$ because $\mathrm{ind}_q(\omega_0) = \mathrm{ind}_q(\omega_1) + \mathrm{ind}_q(\omega_2)$. Extend the ray $\overrightarrow{\Omega q}$ to meet γ_1 first at x, to meet γ_2 last at y. We can compute the indices $\pm k_1 = \mathrm{ind}_y(\omega_1)$ and $\pm k_2 = \mathrm{ind}_x(\omega_2)$ from these points. If T lies on $\overrightarrow{\Omega q}$ far from $\mathcal{C}$, then $\mathrm{ind}_T(\omega_1) = 0$. Since $\overrightarrow{\Omega q}$ meets γ_2 last, by Lemma 9.13, $\mathrm{ind}_T(\omega_1) = \mathrm{ind}_y(\omega_1) = 0 = k_1$. Since $\overrightarrow{\Omega q}$ meets γ_1 first at x, $\mathrm{ind}_x(\omega_2) = \mathrm{ind}_q(\omega_2) = \mp 1 = \pm k_2$.

Suppose $\overrightarrow{\Omega q}$ meets γ_1 last at q' and the next meeting with $\mathcal{C}$ is at w. Let z lie on $\overrightarrow{\Omega q}$ between q' and w. Then $\mathrm{ind}_z(\omega_1) = \mathrm{ind}_w(\omega_1) = \mathrm{ind}_y(\omega_1) = 0$. We also have $\mathrm{ind}_z(\omega_2) = \mathrm{ind}_{q'}(\omega_2) = \mathrm{ind}_x(\omega_2) = \mp 1$. Since $\mathrm{ind}_z(\alpha) = \mathrm{ind}_z(\omega_0) = \mathrm{ind}_z(\omega_1) + \mathrm{ind}_z(\omega_2) = 0 + \mp 1 \neq 0$, we have found $z \in V$ and so $V \neq \emptyset$. Thus $\mathbb{R}^2 - \mathcal{C}$ has at least two components.

We next show that U and V are path-connected. The main tool is Lemma 9.14. Suppose $\Omega' \in U$, that is, $\Omega' \in \mathbb{R}^2 - \mathcal{C}$ and $\mathrm{ind}_{\Omega'}(\alpha) = 0$. Since $\mathrm{ind}_{\Omega'}(\alpha) = \mathrm{ind}_{\Omega'}(\omega_1) + \mathrm{ind}_{\Omega'}(\omega_2)$, and $\mathrm{ind}_{\Omega'}(\alpha) = 0$, either both $\mathrm{ind}_{\Omega'}(\omega_i)$ are zero or both nonzero. In both cases, the values do not agree with $k_1 = 0$ and $k_2 = \mp 1$. By Lemma 9.14, there is a path joining Ω' to Ω and so U is path-connected.



Suppose that M is a point in V. We have shown that the point z constructed from the intersection of the ray $\overrightarrow{\Omega q}$ with $\mathcal{C}$ is also in V. It suffices to show that there is a path joining M to z. We apply Lemma 9.14 again. Reparameterize $\mathcal{C}$, $\beta\colon [0,1] \to \mathcal{C} \subset \mathbb{R}^2$, with $q' = \beta(0)$ and $w = \beta(t_0)$. Let $\eta_1 = \beta \circ f_1$ and $\eta_2 = \beta \circ f_2$ with f_1 and f_2 as before. The curve $\mathcal{C}$ is now parameterized with $\beta \simeq \eta_1 * \eta_2$. Also $\mathrm{ind}_M(\beta) = \mathrm{ind}_M(\alpha) \neq 0$. Let $L_1(t) = (1-t)w + tz$, $L_2(t) = (1-t)z + tq'$. Then

$$\eta_1 * \eta_2 \simeq (\eta_1 * L_1 * L_2) * (L_2^{-1} * L_1^{-1} * \eta_2).$$

Let $\eta_1 * L_1 * L_2 = \overline{\omega}_1$ and $L_2^{-1} * L_1^{-1} * \eta_2 = \overline{\omega}_2$, Take x' on η_1, y' on η_2, not lying on the line ΩT. Since Ω and T are far from the curves, $\mathrm{ind}_\Omega(\overline{\omega}_i) = 0 = \mathrm{ind}_T(\overline{\omega}_i)$. Recall that x and q' were on γ_1, the same parameter range of $\mathcal{C}$, and so $x \in \eta_1$. It follows that $\pm k_2 = \mathrm{ind}_{x'}(\overline{\omega}_2) = \mathrm{ind}_x(\overline{\omega}_2) = \mathrm{ind}_\Omega(\overline{\omega}_2) = 0$. Similarly, $\pm k_1 = \mathrm{ind}_{y'}(\overline{\omega}_1) = \mathrm{ind}_T(\overline{\omega}_1) = 0$.

We can now apply Lemma 9.14, this time with $k_1 = k_2 = 0$. Since $\mathrm{ind}_M(\alpha) \neq 0$, there is a path joining M to z. Thus V is path-connected and we have proved the Jordan Curve Theorem. ♢

Although we have developed some sophisticated notions to prove so intuitively simple an assertion, the proof has the virtues of being rigorous and that it features some ideas that we can develop, namely, the combinatorial and algebraic object given by a grating and the association of an integer or group-valued index to topological objects with nice properties. In the following chapters these ideas take center stage.

Exercises

1. Suppose that X and Y are points in $\mathbb{R}^2$ and $\mathcal{G}$ is a grating with X and Y lying in the interior of two faces in $\mathcal{G}$. A 1-cycle λ is *non-bounding* if any 2-chain w with $\partial(w) = \lambda$ must contain one of the faces containing X or Y. Show that the sum of two non-bounding 1-cycles is not non-bounding.

2. Using the previous exercise, prove that a Jordan curve separates the plane into at most two components. (Hint: Suppose x, y and z are vertices of a grating $\mathcal{G}$ that contains $\mathcal{C}$. Split the curve into two parts, $\mathcal{C} = \alpha([0,1/2]) \cup \alpha([1/2,1])$, that do not separate the points and join them by 1-chains. The subsequent sums are 1-cycles that are non-bounding in the complement of $\{\alpha(0), \alpha(1/2)\}$.)

3. Prove that $\mathbb{R}^2 - \mathcal{C}$ has at least two components using exercise 1.

4. Give an alternate proof of the Separation Theorem for Jordan arcs along the following lines: If Λ is parameterized by $\lambda\colon [0,1] \to \Lambda \subset \mathbb{R}^2$, then consider the subset $\mathcal{R} = \{r \in [0,1] \mid [0,r]$ does not separate the plane$\}$. Show that $\mathcal{R}$ is nonempty, open and closed.

5. Suppose that $\alpha\colon [0,1] \to \mathcal{C} \subset \mathbb{R}^2$ and $\beta\colon [0,1] \to \mathcal{C} \subset \mathbb{R}^2$ are parameterizations of a Jordan curve $\mathcal{C}$ and Ω is a point in $\mathbb{R}^2 - \mathcal{C}$. Show that $\mathrm{ind}_\Omega(\alpha) = \pm\mathrm{ind}_\Omega(\beta)$. Show by example that the sign can change with the parameterization.

6. Suppose K is a subset of $\mathbb{R}^2$ that is homeomorphic to a figure eight (the one-point union of two circles). Generalize the Jordan Curve Theorem to prove that $\mathbb{R}^2 - K$ has three components.