7. Homotopy and the Fundamental Group

The group \( G \) will be called the fundamental group of the manifold \( V \).

J. Henri Poincaré, 1895

The properties of a topological space that we have developed so far have depended on the choice of topology, the collection of open sets. Taking a different tack, we introduce a different structure, algebraic in nature, associated to a space together with a choice of base point \((X, x_0)\). This structure will allow us to bring to bear the power of algebraic arguments. The fundamental group was introduced by Poincaré in his investigations of the action of a group on a manifold [64].

The first step in defining the fundamental group is to study more deeply the relation of homotopy between continuous functions \( f_0: X \to Y \) and \( f_1: X \to Y \). Recall that \( f_0 \) is homotopic to \( f_1 \), denoted \( f_0 \simeq f_1 \), if there is a continuous function \((a homotopy)

\[ H: X \times [0, 1] \to Y \]

with \( H(x, 0) = f_0(x) \) and \( H(x, 1) = f_1(x) \).

The choice of notation anticipates an interpretation of the homotopy—if we write \( H(x, t) = f_t(x) \), then a homotopy is a deformation of the mapping \( f_0 \) into the mapping \( f_1 \) through the family of mappings \( f_t \).

**Theorem 7.1.** The relation \( f \simeq g \) is an equivalence relation on the set, \( \text{Hom}(X, Y) \), of continuous mappings from \( X \) to \( Y \).

**Proof:** Let \( f: X \to Y \) be a given mapping. The homotopy \( H(x, t) = f(x) \) is a continuous mapping \( H: X \times [0, 1] \to Y \) and so \( f \simeq f \).

If \( f_0 \simeq f_1 \) and \( H: X \times [0, 1] \to Y \) is a homotopy between \( f_0 \) and \( f_1 \), then the mapping \( H': X \times [0, 1] \to Y \) given by \( H'(x, t) = H(x, 1 - t) \) is continuous and a homotopy between \( f_1 \) and \( f_0 \), that is, \( f_1 \simeq f_0 \).

Finally, for \( f_0 \simeq f_1 \) and \( f_1 \simeq f_2 \), suppose that \( H_1: X \times [0, 1] \to Y \) is a homotopy between \( f_0 \) and \( f_1 \), and \( H_2: X \times [0, 1] \to Y \) is a homotopy between \( f_1 \) and \( f_2 \). Define the homotopy \( H: X \times [0, 1] \to Y \) by

\[
H(x, t) = \begin{cases} 
H_1(x, 2t), & \text{if } 0 \leq t \leq 1/2, \\
H_2(x, 2t - 1), & \text{if } 1/2 \leq t \leq 1.
\end{cases}
\]

Since \( H_1(x, 1) = f_1(x) = H_2(x, 0) \), the piecewise definition of \( H \) gives a continuous function (Theorem 4.4). By definition, \( H(x, 0) = f_0(x) \) and \( H(x, 1) = f_2(x) \) and so \( f_0 \simeq f_2 \).

We denote the equivalence class under homotopy of a mapping \( f: X \to Y \) by \([f]\) and the set of homotopy classes of maps between \( X \) and \( Y \) by \([X, Y]\). If \( F: W \to X \) and \( G: Y \to Z \) are continuous mappings, then the sets \([X, Y]\), \([W, X]\) and \([Y, Z]\) are related.

**Proposition 7.2.** Continuous mappings \( F: W \to X \) and \( G: Y \to Z \) induce well-defined functions \( F^*: [X, Y] \to [W, Y] \) and \( G_*: [X, Y] \to [X, Z] \) by \( F^*([g]) = [h \circ F] \) and \( G_*([h]) = [G \circ h] \) for \([h] \in [X, Y]\).

**Proof:** We need to show that if \( h \simeq h' \), then \( h \circ F \simeq h' \circ F \) and \( G \circ h \simeq G \circ h' \). Fixing a homotopy \( H: X \times [0, 1] \to Y \) with \( H(x, 0) = h(x) \) and \( H(x, 1) = h'(x) \), then the desired homotopies are \( H_F(w, t) = H(F(w), t) \) and \( H_G(x, t) = G(H(x, t)) \).
To a space $X$ we associate a space particularly rich in structure, the \textit{mapping space of paths in $X$, $\text{map}([0, 1], X)$}. Recall that $\text{map}([0, 1], X)$ is the set of continuous mappings $\text{Hom}([0, 1], X)$ with the compact-open topology. The space $\text{map}([0, 1], X)$ has the following properties:

(1) $X$ embeds into $\text{map}([0, 1], X)$ by associating to each point $x \in X$ to the \textit{constant path}, $c_x(t) = x$ for all $t \in [0, 1]$.

(2) Given a path $\lambda: [0, 1] \to X$, we can \textit{reverse} the path by composing with $t \mapsto 1 - t$. Let $\lambda^{-1}(t) = \lambda(1 - t)$.

(3) Given a pair of paths $\lambda, \mu: [0, 1] \to X$ for which $\lambda(1) = \mu(0)$, we can \textit{compose} paths by

$$\lambda * \mu(t) = \begin{cases} \lambda(2t), & \text{if } 0 \leq t \leq 1/2, \\ \mu(2t - 1), & \text{if } 1/2 \leq t \leq 1. \end{cases}$$

Thus, for certain pairs of paths $\lambda$ and $\mu$, we obtain a new path $\lambda * \mu \in \text{map}([0, 1], X)$.

Composition of paths is always defined when we restrict to a certain subspace of $\text{map}([0, 1], X)$.

\textbf{Definition 7.3.} \textit{Suppose $X$ is a space and $x_0 \in X$ is a choice of base point in $X$. The \textbf{space of based loops} in $X$, denoted $\Omega(X, x_0)$, is the subspace of $\text{map}([0, 1], X)$,}

$$\Omega(X, x_0) = \{ \lambda \in \text{map}([0, 1], X) \mid \lambda(0) = \lambda(1) = x_0 \}. $$

\textit{Composition of loops determines a binary operation $*: \Omega(X, x_0) \times \Omega(X, x_0) \to \Omega(X, x_0)$}.

We restrict the notion of homotopy when applied to the space of based loops in $X$ in order to stay in that space during the deformation.

\textbf{Definition 7.4.} \textit{Given two based loops $\lambda$ and $\mu$, a \textbf{loop homotopy} between them is a homotopy of paths $H: [0, 1] \times [0, 1] \to X$ with $H(t, 0) = \lambda(t)$, $H(t, 1) = \mu(t)$ and $H(0, s) = H(1, s) = x_0$. That is, for each $s \in [0, 1]$, the path $t \mapsto H(t, s)$ is a loop at $x_0$.}

The relation of loop homotopy on $\Omega(X, x_0)$ is an equivalence relation; the proof follows the proof of Theorem 7.1. We denote the set of equivalence classes under loop homotopy by $\pi_1(X, x_0) = [\Omega(X, x_0)]$, a notation for the first of a family of such sets, to be explained later. As it turns out, $\pi_1(X, x_0)$ enjoys some remarkable properties:

\textbf{Theorem 7.5.} \textit{Composition of loops induces a group structure on $\pi_1(X, x_0)$ with identity element $[c_{x_0}(t)]$ and inverses given by $[\lambda]^{-1} = [\lambda^{-1}]$.}

\begin{align*}
\begin{tabular}{ccc}
$\lambda'$ & $\mu'$ & $H'(2t-1,s)$ \\
\hline
$H(t,s)$ & $H'(t,s)$ & $H(2t,s)$
\end{tabular}
\end{align*}

\textit{Proof:} We begin by showing that composition of loops induces a well-defined binary operation on the homotopy classes of loops. Given $[\lambda]$ and $[\mu]$, then we define $[\lambda] * [\mu] = [\lambda * \mu]$. Suppose that $[\lambda] = [\lambda']$ and $[\mu] = [\mu']$. We must show that $\lambda * \mu \simeq \lambda' * \mu'$. If
$H: [0, 1] \times [0, 1] \to X$ is a loop homotopy between $\lambda$ and $\lambda'$ and $H': [0, 1] \times [0, 1] \to X$ a loop homotopy between $\mu$ and $\mu'$, then form $H'': [0, 1] \times [0, 1] \to X$ defined by

$$H''(t, s) = \begin{cases} H(2t, s), & \text{if } 0 \leq t \leq 1/2, \\ H'(2t - 1, s), & \text{if } 1/2 \leq t \leq 1. \end{cases}$$

Since $H''(0, s) = H(0, s) = x_0$ and $H''(1, s) = H'(1, s) = x_0$, $H''$ is a loop homotopy. Also $H''(t, 0) = \lambda \ast \mu(t)$ and $H''(t, 1) = \lambda' \ast \mu'(t)$, and the binary operation is well-defined on equivalence classes of loops.

We next show that $\ast$ is associative. Notice that $(\lambda \ast \mu) \ast \nu = \lambda \ast (\mu \ast \nu)$; we only get $1/4$ of the interval for $\lambda$ in the first product and $1/2$ of the interval in the second product.

We define the explicit homotopy after its picture, which makes the point more clearly:

$$H(t, s) = \begin{cases} \lambda(4t/(1 + s)), & \text{if } 0 \leq t \leq (s + 1)/4, \\ \mu(4t - 1 - s), & \text{if } (s + 1)/4 \leq t \leq (s + 2)/4, \\ \nu \left(1 - \frac{4(1 - t)}{(2 - s)}\right), & \text{if } (s + 2)/4 \leq t \leq 1. \end{cases}$$

The class of the constant map, $e(t) = c_{x_0}(t) = x_0$ gives the identity for $\pi_1(X, x_0)$. To see this, we show, for all $\lambda \in \Omega(X, x_0)$, that $\lambda \ast e \simeq \lambda \ast \lambda$ via loop homotopies. This is accomplished in the case $\lambda \simeq e \ast \lambda$ by the homotopy:

$$F(t, s) = \begin{cases} x_0, & \text{if } 0 \leq t \leq s/2, \\ \lambda((2t - s)/(2 - s)), & \text{if } s/2 \leq t \leq 1. \end{cases}$$

The case $\lambda \simeq \lambda \ast e$ is similar. Finally, inverses are constructed by using the reverse loop $\lambda^{-1}(t) = \lambda(1 - t)$. To show that $\lambda \ast \lambda^{-1} \simeq e$ consider the homotopy:

$$G(t, s) = \begin{cases} \lambda(2t), & \text{if } 0 \leq t \leq s/2, \\ \lambda(s), & \text{if } s/2 \leq t \leq 1 - (s/2) \\ \lambda(2 - 2t), & \text{if } 1 - (s/2) \leq t \leq 1. \end{cases}$$

The homotopy resembles the loop, moving out for a while, waiting a little, and then shrinking back along itself. The proof that $\lambda^{-1} \ast \lambda \simeq e$ is similar.
Definition 7.6. The group \( \pi_1(X, x_0) \) is called the fundamental group of \( X \) at the base point \( x_0 \).

Suppose \( x_1 \) is another choice of basepoint for \( X \). If \( X \) is path-connected, there is

a path \( \gamma: [0, 1] \to X \) with \( \gamma(0) = x_0 \) and \( \gamma(1) = x_1 \). This path induces a mapping

\( u_\gamma: \pi_1(X, x_0) \to \pi_1(X, x_1) \) by \( [\lambda] \mapsto [\gamma^{-1} * \lambda * \gamma] \), that is, follow \( \gamma^{-1} \) from \( x_1 \) to \( x_0 \), then follow \( \lambda \) around and back to \( x_0 \), then follow \( \gamma \) back to \( x_1 \), all giving a loop based at \( x_1 \). Notice

\[
 u_\gamma([\lambda] * [\mu]) = u_\gamma([\lambda * \mu]) = [\gamma^{-1} * \lambda * \mu * \gamma] = [\gamma^{-1} * \lambda * \gamma^{-1} * \mu * \gamma] = [\gamma^{-1} * \lambda * \gamma] * [\gamma^{-1} * \mu * \gamma] = u_\gamma([\lambda]) * u_\gamma([\mu]).
\]

Thus \( u_\gamma \) is a homomorphism. The mapping \( u_\gamma^{-1}: \pi_1(X, x_1) \to \pi_1(X, x_0) \) is an inverse,

since \( [\gamma * (\gamma^{-1} * \lambda * \gamma) * \gamma^{-1}] = [\lambda] \). Thus \( \pi_1(X, x_0) \) is isomorphic to \( \pi_1(X, x_1) \) whenever \( x_0 \)

is joined to \( x_1 \) by a path. Though it is a bit of a lie, we write \( \pi_1(X) \) for a space \( X \) that is path-connected since any choice of basepoint gives an isomorphic group. In this case, \( \pi_1(X) \) denotes an isomorphism class of groups.

Following Proposition 7.2, a continuous function \( f: X \to Y \) induces a mapping

\[
f_*: \pi_1(X, x_0) \to \pi_1(Y, f(x_0)), \text{ given by } f_*([\lambda]) = [f * \lambda].
\]

In fact, \( f_* \) is a homomorphism of groups:

\[
f_*([\lambda] * [\mu]) = f_*([\lambda * \mu]) = [f * (\lambda * \mu)] = [(f * \lambda) * (f * \mu)] = [f * \lambda] * [f * \mu] = f_*([\lambda]) * f_*([\mu]).
\]

Furthermore, when we have continuous mappings \( f: X \to Y \) and \( g: Y \to Z \), we obtain \( f_*: \pi_1(X, x_0) \to \pi_1(Y, f(x_0)) \) and \( g_*: \pi_1(Y, f(x_0)) \to \pi_1(Z, g(f(x_0))) \). Observe that

\[
g_* \circ f_*([\lambda]) = g_*([f * \lambda]) = [g * f * \lambda] = (g * f)_*([\lambda]),
\]

so we have \( (g * f)_* = g_* \circ f_* \). It is evident that the identity mapping \( \text{id}: X \to X \) induces the identity homomorphism of groups \( \pi_1(X, x_0) \to \pi_1(X, x_0) \). We can summarize these observations by the (post-1945) remark that \( \pi_1 \) is a functor from pointed spaces and pointed maps to groups and group homomorphisms. Since we are focusing on classical notions in topology (pre-1935) and category theory was christened later, we will not use this language in what follows. For an introduction to this framework see [51].

The behavior of the induced homomorphisms under composition has the following consequence:

Corollary 7.7. The fundamental group is a topological invariant of a space. That is, if \( f: X \to Y \) is a homeomorphism, then the groups \( \pi_1(X, x_0) \) and \( \pi_1(Y, f(x_0)) \) are isomorphic.

Proof: Suppose \( f: X \to Y \) has continuous inverse \( g: Y \to X \). Then \( g * f = \text{id}_X \) and \( f * g = \text{id}_Y \). It follows that \( g_* \circ f_* = \text{id} \) and \( f_* \circ g_* = \text{id} \) on \( \pi_1(X, x_0) \) and \( \pi_1(Y, f(x_0)) \), respectively. Thus \( f_* \) and \( g_* \) are group isomorphisms. \( \diamond \)
Before we do some calculations we derive a few more formal properties of the fundamental group. In particular, what conditions imply \( \pi_1(X) = \{e\} \), and how does the fundamental group behave under the formation of subspaces, products, and quotients?

**Definition 7.8.** A subspace \( A \subset X \) is a retract of \( X \) if there is a continuous function, the retraction, \( r : X \rightarrow A \) for which \( r(a) = a \) for all \( a \in A \). The subset \( A \subset X \) is a deformation retraction if \( A \) is a retract of \( X \) and the composition \( i \circ r : X \rightarrow A \hookrightarrow X \) is homotopic to the identity on \( X \) via a homotopy that fixes \( A \), that is, there is a homotopy \( H : X \times [0,1] \rightarrow X \) with

\[
H(x,0) = x, H(x,1) = r(x) \text{ and } H(a,t) = a \text{ for all } a \in A, \text{ and all } t \in [0,1].
\]

**Proposition 7.9.** If \( A \subset X \) is a retract with retraction \( r : X \rightarrow A \), then the inclusion \( i : A \rightarrow X \) induces an injective homomorphism \( i_* : \pi_1(A,a) \rightarrow \pi_1(X,a) \) and the retraction induces a surjective homomorphism \( r_* : \pi_1(X,a) \rightarrow \pi_1(A,a) \).

**Proof:** The composite \( r \circ i : A \rightarrow X \rightarrow A \) is the identity mapping on \( A \) and so the composite \( r_* \circ i_* : \pi_1(A,a) \rightarrow \pi_1(X,a) \rightarrow \pi_1(A,a) \) is the identity on \( \pi_1(A,a) \). If \( i_*([\lambda]) = i_*([\lambda']) \), then \( [\lambda] = r_*i_*([\lambda]) = r_*i_*([\lambda']) = [\lambda'] \), and so the homomorphism \( i_* \) is injective. If \( [\lambda] \in \pi_1(A,a) \), then \( r_*i_*([\lambda]) = [\lambda] \) and so \( r_* \) is onto. 

**Examples:** Represent the Möbius band \( M \) by gluing the left and right edges of \([0,1] \times [0,1] \) with a twist (Chapter 4). Let \( A = \{(t, \frac{1}{2}) \mid 0 \leq t \leq 1\} \subset M \), be the circle in the middle of the band. After the identification, \( A \) is homeomorphic to \( S^1 \). Define the map \( r : M \rightarrow A \) by projecting straight down or up to this line, that is, \([t,s] \mapsto [t, \frac{1}{2}] \). It is easy to see that \( r \) is continuous and \( r|_A = \text{id}_A \) so we have a retract. Thus the composite \( r_* \circ i_* : \pi_1(S^1) \rightarrow \pi_1(M) \rightarrow \pi_1(S^1) \) is the identity on \( \pi(S^1) \).

![Diagram of Möbius band](image)

For any space \( X \), the inclusion followed by projection

\[
X \cong X \times \{0\} \hookrightarrow X \times [0,1] \rightarrow X,
\]

is the identity and so \( X \) is a retract of \( X \times [0,1] \). In fact, \( X \) is a deformation retraction via the deformation \( H : X \times [0,1] \times [0,1] \rightarrow X \times [0,1] \) given by \( H(x,t,s) = (x,ts) \): when \( s = 1 \), \( H(x,t,1) = (x,t) \) and for \( s = 0 \) we have \( H(x,t,0) = (x,0) \).

Recall that a subset \( K \) of \( \mathbb{R}^n \) is convex if whenever \( x \) and \( y \) are in \( K \), then for all \( t \in [0,1], tx + (1-t)y \in K \). If \( K \subset \mathbb{R}^n \) is convex, let \( x_0 \in K \), then \( K \) is a deformation retraction of the one-point subset \( \{x_0\} \) by the homotopy \( H(x,t) = tx_0 + (1-t)x \). When \( t = 0 \) we have \( H(x,0) = x \) and when \( t = 1 \), \( H(x,1) = x_0 \). The retraction \( K \rightarrow \{x_0\} \) is...
thus a deformation of the identity on \( K \). Examples of convex subsets of \( \mathbb{R}^n \) include \( \mathbb{R}^n \) itself, any open ball \( B(x, \epsilon) \) and the boxes \([a_1, b_1] \times \cdots \times [a_n, b_n] \).

More generally, there is always the retract \( \{x_0\} \hookrightarrow X \to \{x_0\}, \) which leads to the trivial homomorphisms of groups \( \{e\} \to \pi_1(X, x_0) \to \{e\} \). This retract is not always a deformation retract. We call a space \textbf{contractible} when it is a deformation retract of one of its points.

Deformation retracts give isomorphic fundamental groups.

\textbf{Theorem 7.10.} If \( A \) is a deformation retract of \( X \), then the inclusion \( i: A \to X \) induces an isomorphism \( i_*: \pi_1(A, a) \to \pi_1(X, a) \).

\textbf{Proof:} From the definition of a deformation retract, the composite \( i \circ r: X \to A \hookrightarrow X \) is homotopic to \( \text{id}_X \) via a homotopy fixing the points in \( A \), that is, there is a homotopy \( H: X \times [0, 1] \to X \) with \( H(x, 0) = i \circ r(x) \), \( H(x, 1) = x \), and \( H(a, t) = a \) for all \( t \in [0, 1] \). We show that \( i_* \circ r_*(\lambda) = [\lambda] \). In fact we show a little more:

\textbf{Lemma 7.11.} If \( f, g: (X, x_0) \to (Y, y_0) \) are continuous functions, homotopic through basepoint preserving maps, then \( f_* = g_*: \pi_1(X, x_0) \to \pi_1(Y, y_0) \).

\textbf{Proof:} Suppose there is a homotopy \( G: X \times [0, 1] \to Y \) with \( G(x, 0) = f(x) \), \( G(x, 1) = g(x) \) and \( G(x_0, t) = y_0 \) for all \( t \in [0, 1] \). Consider a loop based at \( x_0 \), \( \lambda: [0, 1] \to X \), and the compositions \( f \circ \lambda \), \( g \circ \lambda \) and \( G \circ (\lambda \times \text{id}): [0, 1] \times [0, 1] \to Y \):

\[ G(\lambda(s), 0) = f \circ \lambda(s) \]
\[ G(\lambda(s), 1) = g \circ \lambda(s) \]
\[ G(\lambda(0), t) = G(\lambda(1), t) = y_0 \text{ for all } t \in [0, 1]. \]

Thus \( f_*[\lambda] = [f \circ \lambda] = [g \circ \lambda] = g_*[\lambda] \). Hence \( f_* = g_*: \pi_1(X, x_0) \to \pi_1(Y, y_0) \).

A deformation retract gives a basepoint preserving homotopy between \( i \circ r \) and \( \text{id}_X \), so we have \( \text{id} = i_* \circ r_*: \pi_1(X, a) \to \pi_1(X, a) \). By Proposition 7.9, we already know \( i_* \) is injective; \( i_* \) is surjective because for \( [\lambda] \) any class in \( \pi_1(X, a) \), one has \( [\lambda] = i_*(r_*(\lambda)) \).

\textbf{Examples:} A convex subset of \( \mathbb{R}^n \) is a deformation retract of any point \( x_0 \) in the set. It follows from \( \pi_1(\{x_0\}) = \{e\} \), that for any convex subset \( K \subset \mathbb{R}^n \), \( \pi_1(K, x_0) = \{e\} \). Of course, this includes \( \pi_1(\mathbb{R}^n \setminus \{0\}) = \{e\} \). Next consider \( \mathbb{R}^n \setminus \{0\} \). The \((n - 1)\)-sphere \( S^{n-1} \subset \mathbb{R}^n \) is a deformation retract of \( \mathbb{R}^n \setminus \{0\} \) as follows: Let \( F: (\mathbb{R}^n \setminus \{0\}) \times [0, 1] \to \mathbb{R}^n \setminus \{0\} \) be given by

\[ F(x, t) = (1 - t)x + t \frac{x}{||x||} \]

Here \( F(x, 0) = x \) and \( F(x, 1) = x/||x|| \in S^{n-1} \). By the Theorem 7.10,

\[ \pi_1(\mathbb{R}^n \setminus \{0\}, x_0) \cong \pi_1(S^{n-1}, x_0/||x_0||). \]

A space \( X \) is said to be \textbf{simply-connected} (or \( 1 \)-\textit{connected}) if it is path-connected and \( \pi_1(X) = \{e\} \). Any convex subset of \( \mathbb{R}^n \), or more generally, any contractible space is simply-connected. Furthermore, simple connectivity is a topological property.

\textbf{Theorem 7.12.} Suppose \( X = U \cup V \) where \( U \) and \( V \) are open, simply-connected subspaces and \( U \cap V \) is path-connected; then \( X \) is simply-connected.
Proof: Choose a point \( x_0 \in U \cap V \) as basepoint. Let \( \lambda: [0,1] \to X \) be a loop based at \( x_0 \). Since \( \lambda \) is continuous, \( \{\lambda^{-1}(U), \lambda^{-1}(V)\} \) is an open cover of the compact space \([0,1]\). The Lebesgue Lemma gives points \( 0 = t_0 < t_1 < t_2 < \cdots < t_n = 1 \) with \( \lambda([t_{i-1}, t_i]) \subset U \) or \( V \).

We can join \( x_0 \) to \( \lambda(t_i) \) by a path \( \gamma_i \). Define for \( i \geq 1 \),

\[
\lambda_i(s) = \lambda((t_i - t_{i-1})s + t_{i-1}), \quad 0 \leq s \leq 1,
\]

for the path along \( \lambda \) joining \( \lambda(t_{i-1}) \) to \( \lambda(t_i) \).

![Diagram of \( U \) and \( V \) with loops \( \gamma_i \) and \( \lambda(t_i) \).]

Then \( \lambda \simeq \lambda_1 \ast \lambda_2 \ast \cdots \ast \lambda_n \) and furthermore,

\[
\lambda \simeq (\lambda_1 \ast \gamma_1^{-1}) \ast (\gamma_1 \ast \lambda_2 \ast \gamma_2^{-1}) \ast (\gamma_2 \ast \lambda_3 \ast \gamma_3^{-1}) \ast \cdots \ast (\gamma_{n-1} \ast \lambda_n).
\]

Each \( \gamma_i \ast \lambda_{i+1} \ast \gamma_{i+1}^{-1} \) lies in \( U \) or \( V \). Since \( U \) and \( V \) are simply-connected, each of these loops is homotopic to the constant map. Thus \( \lambda \simeq c_{x_0} \). It follows that \( \pi_1(X, x_0) \cong \{e\} \). \( \checkmark \)

**Corollary 7.13.** \( \pi_1(S^n) \cong \{e\} \) for \( n \geq 2 \).

**Proof:** We can decompose \( S^n \) as a union of \( U = \{(r_0, r_1, \ldots, r_n) \in S^n \mid r_n > -1/4 \} \) and \( V = \{(r_0, r_1, \ldots, r_n) \in S^n \mid r_n < 1/4 \} \). By stereographic projection from the each pole, we can establish that \( U \) and \( V \) are homeomorphic to an open disk in \( \mathbb{R}^n \), which is convex. The intersection \( U \cap V \) is homeomorphic to \( S^{n-1} \times (-1/4, 1/4) \), which is path-connected when \( n \geq 2 \). \( \checkmark \)

Since \( S^{n-1} \subset \mathbb{R}^n - \{0\} \) is a deformation retract, we have proven:

**Corollary 7.14.** \( \pi_1(\mathbb{R}^n - \{0\}) \cong \{e\} \), for \( n \geq 3 \).

In Chapter 8 we will consider the case \( \pi_1(S^1) \) in detail.

We next consider the fundamental group of a product \( X \times Y \).

**Theorem 7.15.** Let \( (X, x_0) \) and \( (Y, y_0) \) be pointed spaces. Then \( \pi_1(X \times Y, (x_0, y_0)) \) is isomorphic to \( \pi_1(X, x_0) \times \pi_1(Y, y_0) \), the direct product of these two groups.

Recall that if \( G \) and \( H \) are groups, the direct product \( G \times H \) has underlying set the cartesian product of \( G \) and \( H \) and binary operation \((g_1, h_1) \cdot (g_2, h_2) = (g_1g_2, h_1h_2)\).

**Proof:** Recall from Chapter 4 that a mapping \( \lambda: [0,1] \to X \times Y \) is continuous if and only if \( pr_1 \circ \lambda: [0,1] \to X \) and \( pr_2 \circ \lambda: [0,1] \to Y \) are continuous. If \( \lambda \) is a loop at \((x_0, y_0)\), then \( pr_1 \circ \lambda \) is a loop at \( x_0 \) and \( pr_2 \circ \lambda \) is a loop at \( y_0 \). We leave it to the reader to prove that

1) If \( \lambda \simeq \lambda': [0,1] \to X \times Y \), then \( pr_i \circ \lambda \simeq pr_i \circ \lambda' \) for \( i = 1, 2 \).
2) If we take \( \lambda \ast \lambda' : [0, 1] \to X \times Y \), then \( pr_1 \circ (\lambda \ast \lambda') = (pr_1 \circ \lambda) \ast (pr_1 \circ \lambda') \).

These facts allow us to define a homomorphism:

\[
p_{r_1 \ast} \times p_{r_2 \ast} : \pi_1(X \times Y, (x_0, y_0)) \to \pi_1(X, x_0) \times \pi_1(Y, y_0)
\]

by \( p_{r_1 \ast} \times p_{r_2 \ast}([\lambda]) = ([pr_1 \circ \lambda], [pr_2 \circ \lambda]) \). The inverse homomorphism is given by \( ([\lambda], [\mu]) \mapsto [(\lambda, \mu)(t)] \) where \( (\lambda, \mu)(t) = (\lambda(t), \mu(t)) \). Thus we have an isomorphism. \( \diamond \)

We can use such results to show that certain subspaces of a space are not deformation retracts. For example, if \( \pi_1(X, x_0) \) is a nontrivial group, then \( \pi_1(X \times X, (x_0, x_0)) \) is not isomorphic to \( \pi_1(X \times \{x_0\}, (x_0, x_0)) \). Although \( X \times \{x_0\} \) is a retract of \( X \times X \) via

\[
X \times \{x_0\} \hookrightarrow X \times X \to X \times \{x_0\},
\]

it is not a deformation retract of \( X \times X \).

Extra structure on a space can lead to more structure on the fundamental group. Recall (exercises of Chapter 4) that a topological group, \((G, e)\), is a Hausdorff topological space with basepoint \( e \in G \) together with a continuous function (the group operation) \( m : G \times G \to G \), satisfying \( m(g, e) = m(e, g) = g \) for all \( g \in G \), as well as another continuous function (the inverse) \( inv : G \to G \) with \( m(g, inv(g)) = e = m(inv(g), g) \) for all \( g \in G \).

Theorem 7.15 allows us to define a new binary operation on \( \pi_1(G, e) \), the composite of the isomorphism of the theorem with the homomorphism induced by \( m \):

\[
\mu_\ast : \pi_1(G, e) \times \pi_1(G, e) \to \pi_1(G \times G, (e, e)) \to \pi_1(G, e).
\]

We denote the binary operation by \( \mu_\ast([\lambda], [\mu]) = [\lambda \ast \mu] \). On the level of loops, this mapping is given explicitly by \( (\lambda, \mu) \mapsto \lambda \ast \mu \) where \( (\lambda \ast \mu)(t) = m(\lambda(t), \mu(t)) \). We next compare this binary operation with the usual multiplication of loops for the fundamental group.

**Theorem 7.16.** If \( G \) is a topological group, then \( \pi_1(G, e) \) is an abelian group.

**Proof:** We first show that \( \ast \) and the usual multiplication \( \ast \) on \( \pi_1(G, e) \) are actually the same binary operation! We argue as follows: Represent \( \lambda \ast \mu(t) \) by \( \lambda' \ast \mu'(t) \) where

\[
\begin{align*}
\lambda'(t) &= \begin{cases} 
\lambda(2t), & 0 \leq t \leq \frac{1}{2} \\
e, & \frac{1}{2} < t \leq 1
\end{cases} \\
\mu'(t) &= \begin{cases} 
e, & 0 \leq t \leq \frac{1}{2} \\
\mu(2t - 1), & \frac{1}{2} < t \leq 1
\end{cases}
\end{align*}
\]

Since \( \lambda(1) = e = \mu(0) \) and \( m(e, \mu'(t)) = \mu'(t) \), \( m(\lambda'(t), e) = \lambda'(t) \), we see \( \lambda \ast \mu(t) = m(\lambda'(t), \mu'(t)) \). We next show that \( \lambda \ast \mu \) is loop homotopic to \( \lambda \ast \mu \). Define two functions \( h_1, h_2 : [0, 1] \times [0, 1] \to [0, 1] \) by

\[
\begin{align*}
h_1(t, s) &= \begin{cases} 
2t/(2 - s), & 0 \leq t \leq 1 - (s/2) \\
1, & 1 - s/2 \leq t \leq 1
\end{cases} \\
h_2(t, s) &= \begin{cases} 
0, & 0 \leq t \leq s/2 \\
(2t - s)/(2 - s), & s/2 \leq t \leq 1
\end{cases}
\end{align*}
\]
Let \( F(t, s) = m(\lambda(h_1(t,s)), \mu(h_2(t,s))) \). Since it is a composition of continuous functions, \( F \) is continuous. Notice

\[
F(t, 0) = m(\lambda(h_1(t,0)), \mu(h_2(t,0))) = m(\lambda(t), \mu(t)) = \lambda \ast \mu(t)
\]

and \( F(t, 1) = m(\lambda(h_1(t,1)), \mu(h_2(t,1))) = m(\lambda'(t), \mu'(t)) = \lambda \ast \mu(t) \). Thus \( \lambda \ast \mu \) is loop homotopic to \( \lambda \ast \mu \) and we get the same binary operation.

**Corollary 7.17.** \( \pi_1(S^1, 1) \) is abelian.

**Exercises**

1. The unit sphere in \( \mathbb{R} \) is the set \( S^0 = \{-1, 1\} \). Show that the set of homotopy classes of basepoint preserving mappings \([S^0, -1, (X, x_0)]\), is the same set as \( \pi_0(X) \), the set of path components of \( X \).
2. Suppose that \( f: X \to S^2 \) is a continuous mapping that is not onto. Show that \( f \) is homotopic to a constant mapping.

3. If \( X \) is a space, recall that the cone on \( X \) is the quotient space \( CX = X \times [0, 1]/X \times \{1\} \). Suppose \( f: X \to Y \) is a continuous function and \( f \) is homotopic to a constant mapping \( c_y: X \to Y \) for some \( y \in Y \). Show that there is an extension of \( f \), \( \hat{f}: CX \to Y \) so that \( f = \hat{f} \circ i \) where \( i: X \to CX \) is the inclusion, \( i(x) = [(x, 0)] \).

4. Suppose that \( X \) is a path-connected space. When is it true that for any pair of points, \( p, q \in X \), all paths from \( p \) to \( q \) induce the same isomorphism between \( \pi_1(X, p) \) and \( \pi_1(X, q) \)?

5. Prove that a disk minus two points is a deformation retract of a figure 8 (that is, \( S^1 \vee S^1 \)).

6. A starlike space is a slightly weaker notion than a convex space—in a starlike space \( X \subset \mathbb{R}^n \), there is a point \( x_0 \in X \) so that for any other point \( y \in X \) and any \( t \in [0, 1] \) the point \( tx_0 + (1-t)y \) is in \( X \). Give an example of a starlike space that is not convex. Show that a starlike space is a deformation retract of a point.

7. If \( K = \alpha(S^1) \subset \mathbb{R}^3 \) is a knot, that is, a homeomorphic image of a circle in \( \mathbb{R}^3 \), then the complement of the knot, \( \mathbb{R}^3 - K \) has fundamental group \( \pi_1(\mathbb{R}^3 - K) \). In fact, this group is an invariant of the knot in a sense that can be made precise. Give a plausibility argument that \( \pi_1(\mathbb{R}^2 - K) \neq \{0\} \). See [66] for a thorough treatment of this important invariant of knots.