5. Connectedness

We begin our introduction to topology with the study of connectedness—traditionally the only topic studied in both analytic and algebraic topology.

C. T. C. Wall, 1972

The property at the heart of certain key results in analysis is connectedness. The definition, however, applies to any topological space.

Definition 5.1. *A space X is* **disconnected** *by a* separation $\{U, V\}$ *if U and V are open, non-empty, and disjoint ($U \cap V = \emptyset$) subsets of X with $X = U \cup V$. If no separation of the space X exists, then X is* **connected**.

Notice that $V = X - U$ is closed and likewise U is closed. A subset that is both open and closed is sometimes called *clopen*. Closure leads to an equivalent condition.

Theorem 5.2. *A space X is connected if and only if whenever $X = A \cup B$ with A, B, non-empty, then $A \cap (\text{cls } B) \neq \emptyset$ or $(\text{cls } A) \cap B \neq \emptyset$.*

Proof: If $A \cap (\text{cls } B) = \emptyset$ and $(\text{cls } A) \cap B = \emptyset$, then, since $A \cup B = X$, it will follow that $\{X - \text{cls } A, X - \text{cls } B\}$ is a separation of X. To see this, consider $x \in (X - \text{cls } A) \cap (X - \text{cls } B)$; then $x \notin \text{cls } A$ and $x \notin \text{cls } B$. But then $x \notin \text{cls } A \cup \text{cls } B = X$, a contradiction. Therefore $(X - \text{cls } A) \cap (X - \text{cls } B) = \emptyset$. Thus we have a separation.

Conversely, if $\{U, V\}$ is a separation of X, let $A = X - V = U$ and $B = X - U = V$. Since U and V are open, A and B are closed. Then $X = U \cup V = A \cup B$. However, $A \cap \text{cls } B = A \cap B = U \cap V = \emptyset$. ♢

Example: The canonical connected space is the unit interval $[0, 1] \subset (\mathbb{R}, \text{usual})$. To see this, suppose $\{U, V\}$ is a separation of $[0, 1]$. Suppose that $0 \in U$. Let $c = \sup\{0 \leq t \leq 1 \mid [0, t] \subset U\}$. If $c = 1$, then $V = \emptyset$, so suppose $c < 1$. Since $c \in [0, 1]$, $c \in U$ or $c \in V$. If $c \in U$, then there exists an $\epsilon > 0$, such that $(c - \epsilon, c + \epsilon) \subset U$ and there is a natural number $N > 1$ such that $c < c + (\epsilon/N) < 1$. But this contradicts c being a supremum since $c + (\epsilon/N) \in [0, 1]$. If $c \in V$, then there exists a $\delta > 0$, such that $(c - \delta, c + \delta) \subset V$. For some $N' > 1$, $c + (\delta/N') < 1$ and so $(c - (\delta/N'), c + (\delta/N'))$ does not meet U so c could not be a supremum. Since the set $\{0 \leq t \leq 1 \mid [0, t] \subset U\}$ is nonempty and bounded, it has a supremum. It follows that $c = 1$ and so $[0, 1]$ is connected. ♢

Is connectedness a topological property? In fact more is true:

Theorem 5.3. *If $f\colon X \to Y$ is continuous and X is connected, then $f(X)$, the image of X in Y, is connected.*

Proof: Suppose $f(X)$ has a separation. It would be of the form $\{U \cap f(X), V \cap f(X)\}$ with U and V open in Y. Consider the open sets $\{f^{-1}(U), f^{-1}(V)\}$. Since $U \cap f(X) \neq \emptyset$, we have $f^{-1}(U) \neq \emptyset$ and similarly $f^{-1}(V) \neq \emptyset$. Since $U \cap f(X) \cup V \cap f(X) = f(X)$, we have $f^{-1}(U) \cup f^{-1}(V) = X$. Finally, if $x \in f^{-1}(U) \cap f^{-1}(V)$, then $f(x) \in U \cap f(X)$ and $f(x) \in V \cap f(X)$. But $(U \cap f(X)) \cap (V \cap f(X)) = \emptyset$. Thus $f^{-1}(U) \cap f^{-1}(V) = \emptyset$ and X is disconnected. ♢

Corollary 5.4. *Connectedness is a topological property.*

Example: Suppose $a < b$, then there is a homeomorphism $h\colon [0,1] \to [a,b]$ given by $h(t) = a + (b-a)t$. Thus, every $[a,b]$ is connected.

A subspace A of a space X is disconnected when there are open sets U and V in X for which $A \cap U \neq \emptyset \neq A \cap V$, and $A \subset U \cup V$, and $A \cap U \cap V = \emptyset$. Notice that $U \cap V$ can be nonempty in X, but $A \cap U \cap V = \emptyset$.

LEMMA 5.5. *If $\{A_i \mid i \in J\}$ is a collection of connected subspaces of a space X with $\bigcap_{i\in J} A_i \neq \emptyset$, then $\bigcup_{i\in J} A_i$ is connected.*

Proof: Suppose U and V are open subsets of X with $\bigcup_{i\in J} A_i \subset U \cup V$ and $\bigcup_{i\in J} A_i \cap U \cap V = \emptyset$. Let $p \in \bigcap_{j\in J} A_j$, then $p \in A_j$ for all $j \in J$. Suppose that $p \in U$. Since U and V are open, $\{U \cap A_j, V \cap A_j\}$ would separate A_j if they were both non-empty. Since A_j is a connected subspace, this cannot happen, and so $A_j \subset U$. Since $j \in J$ was arbitrary, we can argue in this way to show $\bigcup A_i \subset U$ and hence, $\{U, V\}$ is not a separation. $\diamondsuit$

Example: Given an open interval $(a,b) \subset \mathbb{R}$, let $N > 2/(b-a)$. Then we can write $(a,b) = \bigcup_{n\geq N} [a + \frac{1}{n}, b - \frac{1}{n}]$, a union with nonempty intersection. It follows from the lemma that (a,b) is connected. Also $\mathbb{R} = \bigcup_{n>0} [-n, n]$ and so $\mathbb{R}$ is connected.

Let us review our constructions to see how they respect connectedness. A subset A of a space X is connected if it is connected in the subspace topology. Subspaces do not generally inherit connectedness; for example, $\mathbb{R}$ is connected but $[0,1] \cup (2,3) \subset \mathbb{R}$ is disconnected. A quotient of a connected space, however, is connected since it is the continuous image of the connected space. How about products?

PROPOSITION 5.6. *If X and Y are connected spaces, then $X \times Y$ is connected.*

Proof: Let x_0 and y_0 be points in X and Y, respectively. In the exercises of Chapter 4 we can prove that the inclusions $j_{x_0}\colon Y \to X \times Y$, given by $j_{x_0}(y) = (x_0, y)$ and $i_{y_0}\colon X \to X \times Y$, given by $i_{y_0}(x) = (x, y_0)$ are continuous; hence $j_{x_0}(Y)$ and $i_{y_0}(X)$ are connected in $X \times Y$. Furthermore, $j_{x_0}(Y) \cap i_{y_0}(X) = (x_0, y_0)$ so $i_{y_0}(X) \cup j_{x_0}(Y)$ is connected. We express $X \times Y$ as a union of similar connected subsets:

$$X \times Y = \bigcup_{x\in X} i_{y_0}(X) \cup j_x(Y),$$

a union with intersection given by $\bigcap_{x\in X} i_{y_0}(X) \cup j_x(Y) = i_{y_0}(X)$, which is connected. By Lemma 5.5, $X \times Y$ is connected. $\diamondsuit$

Example: By induction, $\mathbb{R}^n$ is connected for all n. Wrapping $\mathbb{R}$ onto S^1 by $w\colon \mathbb{R} \to S^1$, given by $w(\gamma) = (\cos(2\pi\gamma), \sin(2\pi\gamma))$, shows that S^1 is connected and so is the torus $S^1 \times S^1$. We can also prove this by arguing that $[0,1] \times [0,1]$ is connected and the torus is a quotient of $[0,1] \times [0,1]$. It also follows that S^2 is connected—$S^2 \cong \Sigma S^1$, a quotient of $S^1 \times [0,1]$. By induction and Theorem 4.19, S^n is connected for all $n \geq 1$.

A characterization of the connected subspaces of $\mathbb{R}$ has some interesting corollaries.

PROPOSITION 5.7. *If $W \subset (\mathbb{R}, \text{usual})$ is connected, then $W = (a,b)$, $[a,b)$, $(a,b]$, or $[a,b]$ for $-\infty \leq a \leq b \leq \infty$.*

Proof: Suppose $c, d \in W$ with $c < d$. We show $[c, d] \subset W$, that is, that W is convex. (In other words, if c, d are both in W, then $(1-t)c+td \in W$ for all $0 \le t \le 1$.) Otherwise there exists a value r_0, with $c < r_0 < d$ and $r_0 \notin W$. Then $U = (-\infty, r) \cap W$, $V = W \cap (r, \infty)$ is a separation of W. We leave it to the reader to show that a convex subset of $\mathbb{R}$ must be an open, closed, or half-open interval. $\diamondsuit$

Intermediate Value Theorem. *If $f\colon [a, b] \to \mathbb{R}$ is a continuous function and $f(a) < c < f(b)$ or $f(a) > c > f(b)$, then there is a value $x_0 \in [a, b]$ with $f(x_0) = c$.*

Proof: Since f is continuous, $f([a, b])$ is a connected subset of $\mathbb{R}$. Furthermore, this subset contains $f(a)$ and $f(b)$. By Proposition 5.7, the interval between $f(a)$ and $f(b)$, which includes c, lies in the image of $[a, b]$, and so there is a value $x_0 \in [a, b]$ with $f(x_0) = c$. $\diamondsuit$

Corollary 5.8. *Suppose $g\colon S^1 \to \mathbb{R}$ is continuous. Then there is a point $x_0 \in S^1$ with $g(x_0) = g(-x_0)$.*

Proof: Define $\tilde{g} : S^1 \to \mathbb{R}$ by $\tilde{g}(x) = g(x) - g(-x)$. Wrap $[0, 1]$ onto S^1 by $w(t) = (\cos(2\pi t), \sin(2\pi t))$. Then $w(0) = -w(1/2)$.

Let $F = \tilde{g} \circ w$. It follows that

$$\begin{aligned} F(0) = \tilde{g}(w(0)) &= g(w(0)) - g(-w(0)) \\ &= -[g(-w(0)) - g(w(0))] \\ &= -[g(w(1/2)) - g(-w(1/2))] \\ &= -F(1/2). \end{aligned}$$

If $F(0) > 0$, then $F(1/2) < 0$ and since F is continuous, it must take the value 0 for some t between 0 and 1/2. Similarly for $F(0) < 0$. If $F(t) = 0$, then let $x_0 = w(t)$ and $g(x_0) = g(-x_0)$. $\diamondsuit$

Here is a whimsical interpretation of this result: There are two antipodal points on the equator at which the temperatures are exactly the same. in later chapters we will generalize this result to continuous functions $S^n \to \mathbb{R}^n$.

It is the connectedness of the domain of a continuous real-valued function that leads to the Intermediate Value Theorem (IVT). Furthermore, the IVT can be used to prove that an odd-degree real polynomial has a real root (see the Exercises). Toward a proof of the Fundamental Theorem of Algebra, that every polynomial with complex coefficients has a complex root (see [Uspensky] and [Fine-Rosenberger]), we present an argument given by Gauss, in which connectedness plays a key role. Sadly, Gauss's argument is incomplete and another deep result is needed to complete the proof (see [Ostrowski]). Connectedness plays a prominent role in the argument, which illuminates the subtleness of Gauss's thinking. A complete proof of the Fundamental Theorem of Algebra, using the fundamental group, is presented in Chapter 8.

Let $p(z) = z^n + a_{n-1}z^{n-1} + \cdots + a_1 z + a_0$ be a complex monic polynomial of degree n. We begin with some estimates. We can write the complex numbers in polar form, $z = re^{i\theta}$ and $a_j = s_j e^{i\psi_j}$ and make the substitution

$$p(z) = r^n e^{ni\theta} + r^{n-1} s_{n-1} e^{(n-1)i\theta + i\psi_{n-1}} + \cdots + r s_1 e^{i\theta + i\psi_1} + s_0 e^{i\psi_0}.$$

Writing $e^{i\beta} = \cos(\beta) + i\sin(\beta)$ and $p(z) = T(z) + iU(z)$, we have

$$\begin{aligned} T(z) &= r^n \cos(n\theta) + r^{n-1} s_{n-1} \cos((n-1)\theta + \psi_{n-1}) + \cdots \\ &\qquad + r s_1 \cos(\theta + \psi_1) + s_0 \cos(\psi_0), \\ U(z) &= r^n \sin(n\theta) + r^{n-1} s_{n-1} \sin((n-1)\theta + \psi_{n-1}) + \cdots \\ &\qquad + r s_1 \sin(\theta + \psi_1) + s_0 \sin(\psi_0). \end{aligned}$$

Thus a root of $p(z)$ is a complex number $z_0 = re^{i\theta_0}$ with $T(z_0) = 0 = U(z_0)$.

Suppose $S = \max\{s_{n-1}, s_{n-2}, \ldots, s_0\}$ and $R = 1 + \sqrt{2}S$. Then if $r > R$, we can write

$$\begin{aligned} 0 < 1 - \frac{\sqrt{2}S}{r-1} &= 1 - \sqrt{2}S\left(\frac{1}{r} + \frac{1}{r^2} + \frac{1}{r^3} + \cdots\right) \\ &< 1 - \sqrt{2}S\left(\frac{1}{r} + \frac{1}{r^2} + \cdots + \frac{1}{r^n}\right). \end{aligned}$$

Multiplying through by r^n we deduce

$$0 < r^n - \sqrt{2}S(r^{n-1} + r^{n-2} + \cdots + r + 1) \le r^n - \sqrt{2}(s_{n-1}r^{n-1} + s_{n-2}r^{n-2} + \cdots + s_1 r + s_0).$$

The $\sqrt{2}$ factor is related to the trigonometric form of $T(z)$ and $U(z)$.

Fix a circle in the complex plane given by $z = re^{i\theta}$ for $r > R$. Denote points P_k on this circle with special values

$$P_k = r\left(\cos\left(\frac{(2k+1)\pi}{4n}\right) + i\sin\left(\frac{(2k+1)\pi}{4n}\right)\right).$$

When we evaluate $T(P_{2k})$, the leading term is $r^n \cos(n((4k+1)\pi/4n)) = (-1)^k r^n(\sqrt{2}/2)$. Thus we can write $(-1)^k T(P_{2k})$ as

$$\frac{r^n}{\sqrt{2}} + (-1)^k s_{n-1} r^{n-1} \cos((n-1)\left(\frac{(4k+1)\pi}{4n}\right) + \psi_{n-1}) + \cdots + (-1)^k s_0 \cos(\psi_0).$$

Since $(-1)^k \cos\alpha \ge -1$ for all α and $r > R$, we find that

$$(-1)^k T(P_{2k}) \ge \frac{r^n}{\sqrt{2}} - (s_{n-1}r^{n-1} + \cdots + s_1 r + s_0) > 0.$$

Similarly, in $T(P_{2k+1})$, the leading term is $(-1)^{k+1} r^n \sqrt{2}/2$ and the same estimate gives $(-1)^{k+1} T(P_{2k+1}) > 0$.

The estimates imply that the value of $T(z)$ alternates in sign at P_0, P_1, $\ldots$, P_{4n-1}. Since $T(re^{i\theta})$ varies continuously in θ, $T(z)$ has a zero between P_{2k} and P_{2k+1} for $k = 0$, 1, 2, $\ldots$, $2n-1$. We note that these are all of the zeroes of $T(z)$ on this circle. To see this, write

$$\cos\theta + i\sin\theta = \frac{1-\zeta^2}{1+\zeta^2} + i\frac{2\zeta}{1+\zeta^2}, \text{ where } \zeta = \tan(\theta/2).$$

Thus $T(z)$ can be written in the form

$$r^n\left(\frac{1-\zeta^2}{1+\zeta^2}\right)^n+s_{n-1}\cos(\psi_{n-1})r^{n-1}\left(\frac{1-\zeta^2}{1+\zeta^2}\right)^{n-1}+\cdots+s_1\cos(\psi_1)r\left(\frac{1-\zeta^2}{1+\zeta^2}\right)+s_0\cos(\psi_0),$$

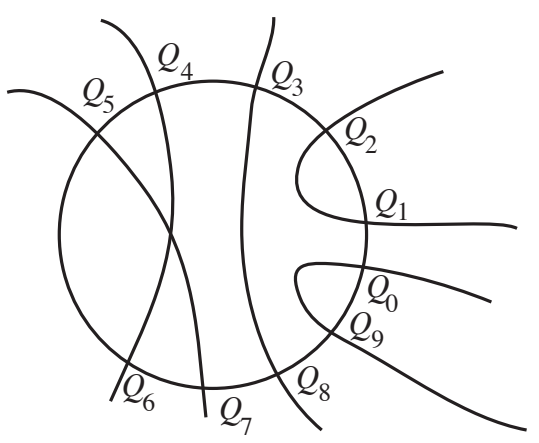
that is, $T(z)=f(\zeta)/(1+\zeta^2)^n$, where $f(\zeta)$ is a polynomial of degree less than or equal to $2n$. Since $T(z)$ has $2n$ zeroes, $f(\zeta)$ has degree $2n$ and has exactly $2n$ roots. Thus we can name the zeroes of $T(z)$ on the circle of radius r by Q_0, Q_1, $\ldots$, Q_{2n-1} with Q_k between P_{2k} and P_{2k+1}.

Let $Q_k=re^{i\phi_k}$. Then $n\phi_k$ lies between $\frac{\pi}{4}+k\pi$ and $\frac{3\pi}{4}+k\pi$. It follows from properties of the sine function that $(-1)^k\sin(n\phi_k)\geq\sqrt{2}/2$. From this estimate we find that

$$(-1)^kU(Q_k)\geq(-1)^kr^n\sin(n\phi_k)-s_{n-1}r^{n-1}-\cdots-s_0\geq\frac{r^n}{\sqrt{2}}-s_{n-1}r^{n-1}-\cdots-s_0>0.$$

Then $U(z)$ is positive at Q_{2k} and negative at Q_{2k+1} for $0\leq k\leq n-1$, and by continuity, $U(z)$ is zero between consecutive pairs of Q_j. This gives us points q_i, for $i=0,1,\ldots,2n-1$ with q_i between Q_i and Q_{i+1} and $U(q_i)=0$.

The game is clear now—a zero of $p(z)$ is a value z_0 with $T(z_0)=0=U(z_0)$. Gauss argued that, as the radius of the circle varied, the distinguished points Q_j and q_k would form curves. As the radius grew smaller, these curves determine regions whose boundary is where $T(z)=0$. The curve of q_j, where $U(z)=0$, must cross some curve of Q_j's, and so give us a root of $p(z)$. The geometric properties of curves of the type given by $T(z)=0$ and $U(z)=0$ are needed to complete this part of the argument, and require more analysis than is appropriate here. The identification of the curves and reducing the existence of a root to the necessary intersection of curves are served up by connectedness.



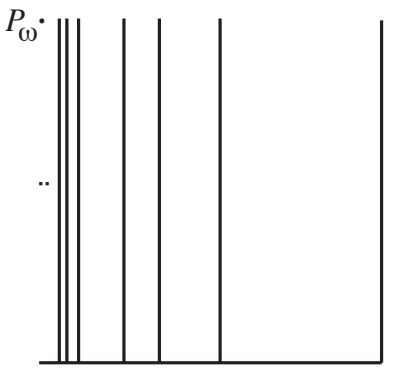
Connectedness is related to the intuitive geometric ideas of Chapter 3 by the following result.

PROPOSITION 5.9. *If A is a connected subspace of a space X, and $A\subset B\subset \operatorname{cls} A$, then B is connected.*

Proof: Suppose B has a separation $\{U\cap B, V\cap B\}$ with U, V open subsets of X. Since A is connected, either $A\subset U$ or $A\subset V$. Suppose $A\subset U$ and $x\in V$. Since V is open, and

$x \in V$, because $x \in B \subset \text{cls } A$, we have that x is a limit point of A. Hence there is a point of A in $U \cap V$ and so $x \in B \cap U \cap V$. This contradicts the assumption that $\{U \cap B, V \cap B\}$ is a separation. Thus B is connected. $\diamondsuit$

Some wild connected spaces can be constructed from this proposition.



Let $P_\omega = (0,1) \in \mathbb{R}^2$ and let X be the subspace of $\mathbb{R}^2$ given by

$$X = \{P_\omega\} \cup \Big((0,1] \times \{0\}\Big) \cup \left(\bigcup_{n=1}^{\infty} \left\{\frac{1}{n}\right\} \times [0,1]\right).$$

We call X the *deleted comb space* [Munkres]. The spokes together with the base form a connected subspace of X. The stray point P_ω is the limit point of the sequence given by the tops of the spokes, $\{(1/n, 1)\}$. So X lies between the connected space of the spokes and base and its closure. Hence X is connected.

Connectedness determines an equivalence relation on a space X: $x \sim y$ if there is a connected subset A of X with $x, y \in A$. (Can you prove that this is an equivalence relations?) An equivalence class $[x]$ under this relation is called a **connected component** of X. The equivalence classes satisfy the property that if $x \in [x]$, then $[x]$ is the union of all connected subsets of X containing x and so it follows from Lemma 5.5 that $[x]$ is the largest connected subset containing x. Since $[x] \subseteq \text{cls } [x]$, it follows from Proposition 5.9 that cls $[x]$ is also connected and hence $[x] = \text{cls } [x]$ and connected components are closed.

Because the connected components partition a space, and each is closed, then each is also open if there are only finitely many connected components. By way of contrast with the case of finitely many components, the connected components of $\mathbb{Q} \subset \mathbb{R}$ are the points themselves—closed but not open.

PROPOSITION 5.10. *The cardinality of the set of connected components of a space X is a topological invariant.*

Proof: We show that if $[x]$ is a component of X, and $h: X \to Y$ a homeomorphism, then $h([x])$ is a component of Y. By Theorem 5.3, $h([x])$ is connected and $h([x]) \subset [h(x)]$. By a symmetric argument, $h^{-1}([h(x)]) \subset [x]$. Thus $[h(x)] \subset h([x])$ and so $h([x]) = [h(x)]$. Since h maps components to components, h induces a one-one correspondence between connected components. $\diamondsuit$

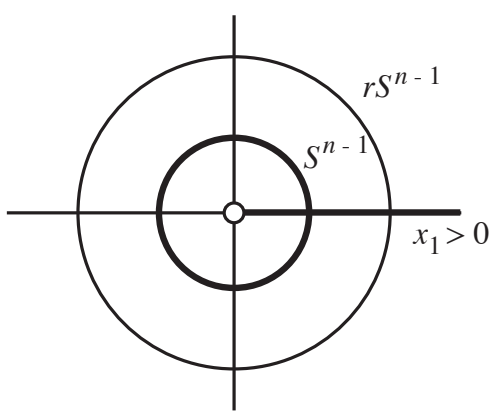
We have developed enough topology to handle a case of our main goal. Connectedness allows us to distinguish between $\mathbb{R}$ and $\mathbb{R}^n$ for $n > 2$.

Invariance of dimension for $(1, n)$: $\mathbb{R}$ is not homeomorphic to $\mathbb{R}^n$, for $n > 1$.

We first make a useful observation.

Lemma 5.11. *If $f: X \to Y$ is a homeomorphism and $x \in X$, then f induces a homeomorphism between $X - \{x\}$ and $Y - \{f(x)\}$.*

Proof: The restriction $f|: X - \{x\} \to Y - \{f(x)\}$ of f to $X - \{x\}$ is a one-one correspondence between $X - \{x\}$ and $Y - \{f(x)\}$. Each subset is endowed with the subspace topology and $f|$ is continuous because an open set in $Y - \{f(x)\}$ is the intersection of an open set V in Y with the complement of $\{f(x)\}$. The inverse image is the intersection of $f^{-1}(V)$ and the complement of $\{x\}$, an open set in $X - \{x\}$. The inverse of $f|$ is similarly seen to be continuous. ♢



Proof of this case of Invariance of Dimension: Suppose we had a homeomorphism $h: \mathbb{R} \to \mathbb{R}^n$. By composing with a translation we arrange that $h(0) = \mathbf{0} = (0, 0, \ldots, 0) \in \mathbb{R}^n$. By Lemma 5.11, we consider the homeomorphism $h|: \mathbb{R} - \{0\} \to \mathbb{R}^n - \{\mathbf{0}\}$. But $\mathbb{R} - \{0\}$ has two connected components. To demonstrate invariance of dimension in this case we show for $n > 1$ that $\mathbb{R}^n - \{\mathbf{0}\}$ has only one component. Fix the connected subset of $\mathbb{R}^n - \{\mathbf{0}\}$ given by

$$Y = \{(x_1, 0, \ldots, 0) \mid x_1 > 0\}.$$

This is an open ray, which we know to be connected. We can express $\mathbb{R}^n - \{\mathbf{0}\}$ as a union:

$$\mathbb{R}^n - \{\mathbf{0}\} = \bigcup_{r>0} rS^{n-1} \cup Y,$$

where $rS^{n-1} = \{(a_1, \ldots, a_n) \in \mathbb{R}^n \mid a_1^2 + \cdots + a_n^2 = r^2\}$. Each subset in the union is connected being the union of a homeomorphic copy of S^{n-1} and Y with nonempty intersection. The intersection of all of the sets in the union is Y and so, by Lemma 5.5, $\mathbb{R}^n - \{\mathbf{0}\}$ is connected and thus has only one component. ♢

Path-connectedness

A more natural formulation of connection is given by the following notion.

Definition 5.12. *A space X is* **path-connected** *if, for any $x, y \in X$, there is a continuous function $\lambda: [0, 1] \longrightarrow X$ with $\lambda(0) = x$, $\lambda(1) = y$. Such a function λ is called a* **path** *joining x to y in X.*

The connectedness of $[0,1]$ plays a role in relating connectedness with path-connectedness.

PROPOSITION 5.13. *If X is path-connected, then it is connected.*

Proof: Suppose X is disconnected and $\{U,V\}$ is a separation. Since $U \neq \emptyset \neq V$, there are points $x \in U$ and $y \in V$. If X is path-connected, there is a path $\lambda\colon [0,1] \to X$ with $\lambda(0) = x$, $\lambda(1) = y$, and λ continuous. But then $\{\lambda^{-1}(U), \lambda^{-1}(V)\}$ would separate $[0,1]$, a connected space. This contradiction implies that X is connected. $\diamondsuit$

Connectedness and path-connectedness are not equivalent. We saw that the *deleted comb space* is connected but it is not path-connected. Suppose there is a path $\lambda\colon [0,1] \to X$ with $\lambda(0) = (1,0)$ and $\lambda(1) = (0,1) = P_\omega$. The subset $\lambda^{-1}(\{P_\omega\})$ is closed in $[0,1]$ because X is Hausdorff and λ is continuous. We will show that it is also open. Consider $V = B(P_\omega, \epsilon) \cap X$ for $\epsilon = 1/k > 0$ and $k > 1$. Then $\lambda^{-1}(V)$ is nonempty and open in $[0,1]$, so for $x_0 \in \lambda^{-1}(V)$, there exists $\delta > 0$ with $(x_0 - \delta, x_0 + \delta) \cap [0,1] \subset \lambda^{-1}(V)$. I claim that $(x_0 - \delta, x_0 + \delta) \subset \lambda^{-1}(\{P_\omega\})$. Suppose not and T is such that $\lambda(T) = (\frac{1}{n}, s)$ for some $n > k$. Let $W_1 = (-\infty, r) \times \mathbb{R}$, $W_2 = (r, \infty) \times \mathbb{R}$, for $1/(n+1) < r < 1/n$. Then $\{W_1 \cap \lambda((x_0 - \delta, x_0 + \delta)), W_2 \cap \lambda((x_0 - \delta, x, +\delta))\}$ separates the image $\lambda((x_0 - \delta, x_0 + \delta))$ of a connected space under a continuous mapping, and this is a contradiction. It follows that no such value of T exists. Since $\lambda^{-1}(B(P_\omega, \epsilon) \cap X)$ is both open and closed, λ is a constant path with image P_ω.

By analogy with the property of connectedness, we have the following results.

THEOREM 5.14. *If X is path-connected and $f\colon X \to Y$ continuous, then $f(X) \subset Y$ is path connected.*

Proof: Let $f(x), f(y) \in f(X)$. There is a path $\lambda\colon [0,1] \to X$ joining x, and y. Then $f \circ \lambda$ is a path joining $f(x)$ and $f(y)$. $\diamondsuit$

COROLLARY 5.15. *Path-connectedness is a topological property.*

LEMMA 5.16. *If $\{A_i \mid i \in J\}$ is a collection of path-connected subsets of a space X and $\bigcap_{i \in J} A_i \neq \emptyset$, then $\bigcup_{i \in J} A_i$ is path-connected.*

Proof: Suppose $x, y \in \bigcup_{i \in J} A_i$ and $z \in \bigcap_{i \in J} A_i$. Then, for some i_1 and $i_2 \in J$, we have $x \in A_{i_1}$, $y \in A_{i_2}$, both subsets path-connected. There are paths then $\lambda_1\colon [0,1] \to A_{i_1}$ with $\lambda_1(0) = x$, $\lambda_1(1) = z$, and $\lambda_2\colon [0,1] \to A_{i_2}$ with $\lambda_2(0) = z$, $\lambda_2(1) = y$. Define the path $\lambda_1 * \lambda_2$ by

$$\lambda_1 * \lambda_2(t) = \begin{cases} \lambda_1(2t), & 0 \leq t \leq \frac{1}{2}, \\ \lambda_2(2t-1), & \frac{1}{2} \leq t \leq 1. \end{cases}$$

By Theorem 4.4, the path $\lambda_1 * \lambda_2$ is continuous. Furthermore, $\lambda_1 * \lambda_2$ joins x to y and so $\bigcup_{i \in J} A_i$ is path-connected. $\diamondsuit$

By Proposition 5.7, the connected subsets of $\mathbb{R}$ are intervals. If $r, s \in (a,b)$, then the path $t \mapsto (1-t)r + ts$ joins r to s in (a,b). Thus, the connected subspaces of $\mathbb{R}$ are path-connected.

As is the case for connectedness, path-connectedness of subspaces of a path-connected space is unpredictable. However, by Theorem 5.14 quotients of path-connected spaces are connected. We consider products.

Proposition 5.17. *If X and Y are path-connected, then so is $X \times Y$.*

Proof: Let (x, y) and (x', y') be points in $X \times Y$. Since X and Y are path-connected there are paths $\lambda\colon [0,1] \to X$ and $\lambda'\colon [0,1] \to Y$ with $\lambda(0) = x$, $\lambda(1) = x'$, $\lambda'(0) = y$, and $\lambda'(1) = y'$. Consider $\lambda \times \lambda'\colon [0,1] \to X \times Y$ given by

$$(\lambda \times \lambda')(t) = (\lambda(t), \lambda'(t)).$$

By Proposition 4.10, $\lambda \times \lambda'$ is continuous with $\lambda \times \lambda'(0) = (x, y)$ and $\lambda \times \lambda'(1) = (x', y')$ as required. So $X \times Y$ is path-connected. $\diamondsuit$

This shows, by induction, that $\mathbb{R}^n$ is path-connected for all n. Together with the remark about quotients, spaces such as S^{n-1}, $S^1 \times S^1$ and $\mathbb{R}P^2$ are all path-connected.

Paths lead to another relation on a space X: we write $x \approx y$ if there is a path $\lambda\colon [0,1] \to X$ with $\lambda(0) = x$ and $\lambda(1) = y$. The constant path $c_{x_0}\colon [0,1] \to X$, given by $c_{x_0}(t) = x_0$ is continuous and so, for all $x_0 \in X$, $x_0 \approx x_0$. If $x \approx y$, then there is a path λ joining x to y. Consider the mapping $\lambda^{-1}(t) = \lambda(1-t)$. Then λ^{-1} is continuous and determines a path joining y to x. Thus $y \approx x$. Finally, if $x \approx y$ and $y \approx z$, then if λ_1 joins x to y and λ_2 joins y to z, then $\lambda_1 * \lambda_2$ joins x to z, and so the relation $\approx$ is an equivalence relation.

We define a **path component** to be an equivalence class under the relation $\approx$. A space is path-connected if and only if it has only one path component. Since each path component $[x]$ is path-connected we know that for $f\colon X \to Y$ a continuous function, $f([x]) \subset [f(x)]$, since the image of a path-connected subspace is path-connected. We extend this fact a little further as follows.

Definition 5.18. *The* **set of path components** $\pi_0(X)$ *is the set of equivalence classes under the relation $\approx$. If $f\colon X \to Y$ is a continuous function, then f induces a well-defined mapping $\pi_0(f)\colon \pi_0(X) \to \pi_0(Y)$, given by $\pi_0(f)([x]) = [f(x)]$.*

We note that the association $X \mapsto \pi_0(X)$ and $f \mapsto \pi_0(f)$ satisfies the following basic properties: (1) If $\mathrm{id}\colon X \to X$ is the identity mapping, then $\pi_0(\mathrm{id})\colon \pi_0(X) \to \pi_0(X)$ is the identity mapping; (2) If $f\colon X \to Y$ and $g\colon Y \to Z$ are continuous mappings, then $\pi_0(g \circ f) = \pi_0(g) \circ \pi_0(f)\colon \pi_0(X) \to \pi_0(Z)$. These properties are shared with several constructions to come and they came to be identified as the *functoriality* of π_0 [Eilenberg-Mac Lane]. The alert reader will recognize functoriality at work in later chapters.

As with connected components, we ask when path components are open or closed. The deleted comb space, however, indicates that we cannot expect much of closure.

Definition 5.19. *A space X is* **locally path-connected** *if, for every $x \in X$, and $x \in U$ an open set in X, there is an open set $V \subset X$ with $x \in V \subset U$ and V path-connected.*

Proposition 5.20. *If X is locally path-connected, then path components of X are open.*

Proof: Let $y \in [x]$, a path component of X. Take any open set containing y and there is a path-connected open set V_y with $y \in V_y$. Since every point in V_y is related to y and y is related to x, we get that $V_y \subset [x]$. Thus $[x] = \bigcup_{y \in [x]} V_y$ and $[x]$ is open. $\diamondsuit$

We see how this can work together with connectedness to obtain path-connectedness.

Corollary 5.21. *If X is connected and locally path-connected, then it is path-connected.*

Proof: Suppose X has more than one path component. Choose one component $[x] = U$, which is open in X. The union of the rest of the components we denote by V, which is also open in X. Then $U \cup V = X$, and $U \cap V = \emptyset$ and so X is disconnected, a contradiction. Hence X has only one path component. $\diamondsuit$

It follows that deleted comb space is not even locally path-connected. (This can also be proved directly.)

Exercises

1. Prove that any infinite set X with the finite-complement topology is connected. Is the space $(\mathbb{R}, \text{half-open})$ connected?

2. A subset $K \subset \mathbb{R}$ is convex if for any $c, d \in K$, the set $[c, d] = \{c(1-t)+dt \mid 0 \leq t \leq 1\}$ is contained in K. Show that a convex subset of $\mathbb{R}$ is an open, closed, or half-open interval.

3. *..., the hip bone's connected to the thigh bone, and the thigh bone's connected to the knee bone, and the ....* Let's prove a proposition that shows that the skeleton should be connected as in the song. Suppose we have a sequence of connected subspaces $\{X_i \mid i = 1, 2, 3, \ldots\}$ of a given space X. Suppose further that $X_i \cap X_{i+1} \neq \emptyset$ for all i. Show that the union $\bigcup_{i=1}^{\infty} X_i$ is connected. (Hint: consider the sequence of subspaces $Y_j = X_1 \cup X_2 \cup \cdots \cup X_j$ for $j \geq 1$. Are these connected? What is their intersection? What is their union?)

4. Suppose we have a collection of non-empty connected spaces, $\{X_j \mid j \in J\}$. Does it follow that the product $\displaystyle\prod_{j \in J} X_j$ is connected?

5. One of the easier parts of the Fundamental Theorem of Algebra is the fact that an odd degree polynomial $p(x)$ has at least one real root. Notice that such a polynomial is a continuous function $p\colon \mathbb{R} \to \mathbb{R}$. The theorem follows by showing that there is a real number b with $p(b) > 0$ and $p(-b) < 0$, and using the Intermediate Value Theorem.

   Let $p(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ with n odd. Write $p(x) = x^n q(x)$ for the function $q(x)$ that will be the sum of the coefficients of $p(x)$ over powers of x. Estimate $|q(x)-1|$ and show that it is less than or equal to $A/|x|$ where $A = |a_{n-1}| + \cdots |a_1| + |a_0|$ for $|x| \geq 1$. Letting $|b| > \max\{1, 2A\}$ we get $|q(b) - 1| < \frac{1}{2}$ or $q(b) > 0$ and $q(-b) > 0$. Show that this implies that there is a zero of $p(x)$ between $-|b|$ and $|b|$.

6. Suppose that the space X can be written as a product $X = Y_1 \times Y_2$. Determine the relationship between $\pi_0(X)$ and $\pi_0(Y_1)$ and $\pi_0(Y_2)$. Suppose that G is a topological group. Show that $\pi_0(G)$ is also a group.