10. Simplicial Complexes

The upshot was that he (Poincaré) introduced an entirely new approach to algebraic topology: the concept of complex *and the highly elastic algebra going so naturally with it.*

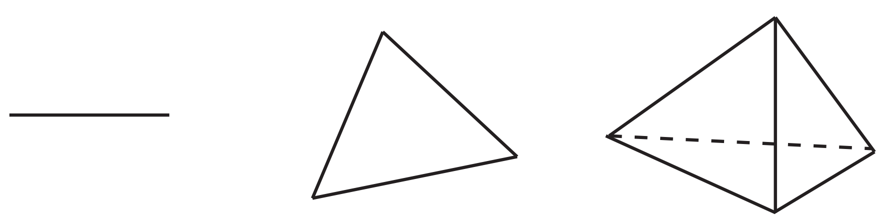
Solomon Lefschetz, 1970

The gratings of the previous chapter have two nice features—they provide approximations to compact spaces that can be refined to any degree of necessity, and they enjoy a combinatorial and algebraic calculus. These aspects are greatly extended in this chapter and the next. We replace a grating of a square in the plane with a simplicial complex, a particular sort of topological space defined by combinatorial data. Continuous mappings between simplicial complexes can be defined using the combinatorial data. By refining simplicial complexes, we can approximate arbitrary continuous mappings by these combinatorial ones. Approximations are related by homotopies between mappings, giving the homotopy relation further importance. In the next chapter, we will introduce the algebraic structures associated to the combinatorial data. We begin with the basic building blocks.

Definition 10.1. *A set of vectors $S = \{\mathbf{v}_0, \ldots, \mathbf{v}_n\}$ in $\mathbb{R}^N$ for N large is in* **general position** *if the set $\{\mathbf{v}_0 - \mathbf{v}_n, \mathbf{v}_1 - \mathbf{v}_n, \ldots, \mathbf{v}_{n-1} - \mathbf{v}_n\}$ is linearly independent. A set $S = \{\mathbf{v}_0, \ldots, \mathbf{v}_n\}$ in general position is called an n-***simplex** *or a simplex of* **dimension** *n and it determines a subset of $\mathbb{R}^N$ defined by*

$$\begin{aligned}\Delta^n[S] &= \{t_0\mathbf{v}_0 + t_1\mathbf{v}_1 + \cdots + t_n\mathbf{v}_n \in \mathbb{R}^N \mid t_i \geq 0, t_0 + \cdots + t_n = 1\} \\ &= \text{convex hull}(\{\mathbf{v}_0, \ldots, \mathbf{v}_n\}).\end{aligned}$$

If the set $S = \{\mathbf{v}_0, \ldots, \mathbf{v}_n\}$ is not in general position, then we say that the n-simplex determined by S is **degenerate**.



$\Delta^1[\{\mathbf{v}_0,\mathbf{v}_1\}]$ $\Delta^2[\{\mathbf{v}_0,\mathbf{v}_1,\mathbf{v}_2\}]$ $\Delta^3[\{\mathbf{v}_0,\mathbf{v}_1,\mathbf{v}_2,\mathbf{v}_3\}]$

For example, a triple $\{\mathbf{v}_0, \mathbf{v}_1, \mathbf{v}_2\}$ is in general position if the points are not collinear. A 0-simplex $\Delta^0[\{\mathbf{v}_0\}]$ is simply the point $\mathbf{v}_0 \in \mathbb{R}^N$. A 1-simplex $\{\mathbf{v}_0, \mathbf{v}_1\}$ determines a line segment $\Delta^1[\{\mathbf{v}_0, \mathbf{v}_1\}]$; $\Delta^2[\{\mathbf{v}_0, \mathbf{v}_1, \mathbf{v}_2\}]$ is a triangle (with its interior) and $\Delta^3[\{\mathbf{v}_0, \mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}]$ is a solid tetrahedron. In general we write $\Delta^n = \Delta^n[S]$ when there is no need to be specific about vertices. When a vertex is repeated, the simplex is degenerate. Degenerate simplices will be important when discussing mappings between simplicial complexes.

In what follows, the combinatorics of sets of vertices play the principal role. We will assume that the vertices determining a simplex are ordered. This assumption is for convenience; in fact, coherent orderings around a simplicial complex determine a useful topological property, orientability (see [Croom], [Giblin]), an extra bit of structure to be developed another day.

A point $\mathbf{p} \in \Delta^n$ may be specified uniquely by the coefficients $(t_0, t_1, \ldots, t_n)$. To see this suppose

$$t_0\mathbf{v}_0 + t_1\mathbf{v}_1 + \cdots + t_n\mathbf{v}_n = t_0'\mathbf{v}_0 + t_1'\mathbf{v}_1 + \cdots + t_n'\mathbf{v}_n.$$

Then $(t_0 - t_0')\mathbf{v}_0 + \cdots + (t_n - t_n')\mathbf{v}_n = \mathbf{0}$. Since $\sum_{i=0}^n t_i = \sum_{i=0}^n t_i' = 1$, it follows that $\sum_{i=0}^n (t_i - t_i') = 0$, and so $t_n - t_n' = \sum_{i=0}^{n-1} -(t_i - t_i')$. In particular,

$$(t_0 - t_0')\mathbf{v}_0 + \cdots + (t_n - t_n')\mathbf{v}_n = (t_0 - t_0')(\mathbf{v}_0 - \mathbf{v}_n) + \cdots + (t_{n-1} - t_{n-1}')(\mathbf{v}_{n-1} - \mathbf{v}_n) = \mathbf{0}.$$

Because the set $\{\mathbf{v}_0 - \mathbf{v}_n, \mathbf{v}_1 - \mathbf{v}_n, \ldots, \mathbf{v}_{n-1} - \mathbf{v}_n\}$ is linearly independent, we deduce that $t_i = t_i'$ for all i and so the coefficients are uniquely determined by $\mathbf{p}$. The list of coefficients $(t_0, t_1, \ldots, t_n)$ is called the **barycentric coordinates** of $\mathbf{p} \in \Delta^n$.

Although $\Delta^n[\{\mathbf{v}_0, \ldots, \mathbf{v}_n\}]$ is a subspace of $\mathbb{R}^N$, as a topological space, it is determined by the barycentric coordinates.

PROPOSITION 10.2. *Let $\boldsymbol{\Delta}^n$ denote the subspace of $\mathbb{R}^{n+1}$ given by $\boldsymbol{\Delta}^n = \{(t_0, \ldots, t_n) \in \mathbb{R}^{n+1} \mid t_0 + \cdots + t_n = 1, t_i \geq 0\}$. If $S = \{\mathbf{v}_0, \ldots, \mathbf{v}_n\}$ is a set of vectors in general position in $\mathbb{R}^N$, then $\Delta^n[S]$ is homeomorphic to $\boldsymbol{\Delta}^{\mathbf{n}}$.*

Proof: The mapping $\phi\colon \boldsymbol{\Delta}^n \to \Delta^n[S]$ given by $\phi(t_0, \ldots, t_n) = t_0\mathbf{v}_0 + \cdots + t_n\mathbf{v}_n$ is a bijection by the uniqueness of barycentric coordinates. The mapping ϕ is given by matrix multiplication and so is continuous. The inverse of ϕ is given by projections on a subspace, and so it too is continuous. $\diamondsuit$

The topological properties of $\boldsymbol{\Delta}^n$ are shared with $\Delta^n[S]$ for any other n-simplex. For example, as a subspace of $\mathbb{R}^N$, $\Delta^n[S]$ is compact because $\boldsymbol{\Delta}^n$ is closed and bounded in $\mathbb{R}^{n+1}$.

PROPOSITION 10.3. *The points $\mathbf{p} \in \Delta^n[S]$ with barycentric coordinates that satisfy $t_i > 0$ for all i form an open subset of $\Delta^n[S]$ (as a subspace of $\mathbb{R}^N$); $\mathbf{p}$ is in the boundary of $\Delta^n[S]$ if and only if $t_i = 0$ for some i.*

Proof: In $\boldsymbol{\Delta}^n \subset \mathbb{R}^{n+1}$, the subset of points with barycentric coordinates $t_i > 0$ is the intersection of the open subsets $U_i = \{(t_0, \ldots, t_n) \in \mathbb{R}^{n+1} \mid t_i > 0\}$ with $\boldsymbol{\Delta}^n$ and so it is an open subset of $\boldsymbol{\Delta}^n$. Its homeomorphic image in $\Delta^n[S]$ is also open in $\Delta^n[S]$.

We can extend the mapping $\phi\colon \boldsymbol{\Delta}^n \to \Delta^n[S]$ to the subspace Π of $\mathbb{R}^{n+1}$, where

$$\Pi = \{(t_0, \ldots, t_n) \in \mathbb{R}^{n+1} \mid t_0 + \cdots + t_n = 1\},$$

the hyperplane containing $\boldsymbol{\Delta}^n$ in $\mathbb{R}^{n+1}$. The mapping $\widehat{\phi}\colon \Pi \to \mathbb{R}^N$, given by $\widehat{\phi}(t_0, \ldots, t_n) = t_0\mathbf{v}_0 + \cdots + t_n\mathbf{v}_n$, takes points on the boundary of $\boldsymbol{\Delta}^n$ to points on the boundary of $\Delta^n[S]$. The points on the boundary have some $t_i = 0$ because open sets in $\mathbb{R}^{n+1}$ containing such points must contain points with $t_i < 0$ which map by $\widehat{\phi}$ to points outside $\Delta^n[S]$. Conversely, if a point $\mathbf{p}$ is on the boundary of $\Delta^n[S]$, any open set containing $\mathbf{p}$ meets the complement of $\Delta^n[S]$ and, by a distance argument, points in the image of Π under $\widehat{\phi}$ with negative coordinates. This implies some $t_j = 0$. $\diamondsuit$

Notice that a 0-simplex is also its own interior—the topology is discrete on a one-point space. Interesting subsets of a simplex, like the boundary or interior, have nice combinatorial expressions. Define the *face opposite a vertex* $\mathbf{v}_i$ as the subset

$$\partial_i\{\mathbf{v}_0, \ldots, \mathbf{v}_n\} = \{\mathbf{v}_0, \ldots, \widehat{\mathbf{v}_i}, \ldots, \mathbf{v}_n\} = \{\mathbf{v}_0, \ldots, \mathbf{v}_{i-1}, \mathbf{v}_{i+1}, \ldots, \mathbf{v}_n\},$$

where the hat over a vertex means that it is omitted. Any subset of $S = \{\mathbf{v}_0, \ldots, \mathbf{v}_n\}$ determines a *subsimplex* of S, and so a subspace of $\Delta^n[S]$; for example, the subset $T = \{\mathbf{v}_{j_0}, \ldots, \mathbf{v}_{j_k}\}$ determines $\Delta^k[T] = \Delta^k[\{\mathbf{v}_{j_0}, \ldots, \mathbf{v}_{j_k}\}] \subset \Delta^n[S]$. The inclusion is based on the fact that $\sum_i t_{j_i}\mathbf{v}_{j_i} = \sum_{l=0}^n t_l \mathbf{v}_l$ where $t_l = 0$ if $l \neq j_i$.

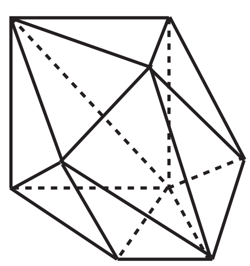
When $S = \{\mathbf{v}_0, \ldots, \mathbf{v}_n\}$ and $T \subset S$, we denote the inclusion of the subsimplex by $T \prec S$. If $j_0 < j_1 < \cdots < j_k$, then each such face can be obtained by iterating the operation of taking the face opposite some vertex. The combinatorics of the face opposite operators encodes the lower dimensional subsimplices (or faces) of $\Delta^n[S]$. By Proposition 10.3, the geometric boundary of $\Delta^n[S]$ can be expressed combinatorially:

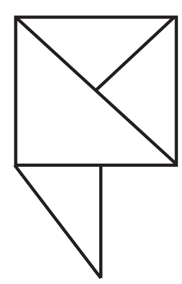
$$\operatorname{bdy} \Delta^n[S] = \Delta^{n-1}[\partial_0 S] \cup \cdots \cup \Delta^{n-1}[\partial_n S] \subset \Delta^n[S].$$

Given any point $\mathbf{p} \in \Delta^n[S]$, writing $\mathbf{p} = t_0\mathbf{v}_0 + \cdots + t_n\mathbf{v}_n$, we can eliminate the summands with $t_i = 0$ to write $\mathbf{p} = t_{i_0}\mathbf{v}_{i_0} + \cdots + t_{i_m}\mathbf{v}_{i_m}$ with $\sum t_{i_j} = 1$ and $t_{i_j} > 0$ for all j. Thus $\mathbf{p}$ is in the interior of $\Delta^m[\{\mathbf{v}_{i_0}, \ldots, \mathbf{v}_{i_m}\}]$. Because barycentric coordinates are unique, every point in $\Delta^n[S]$ is contained in the interior of a unique subsimplex, $\Delta^m[\{\mathbf{v}_{i_0}, \ldots, \mathbf{v}_{i_m}\}] \subset \Delta^n[S]$.

The simplices $\Delta^n[S]$ form the building blocks of an important class of spaces.

Definition 10.4. *A (geometric)* **simplicial complex** *is a finite collection K of simplices in $\mathbb{R}^N$ satisfying* 1) *if $S = \{\mathbf{v}_0, \ldots, \mathbf{v}_n\}$ is in K and $T \prec S$ (T is a subset of S), then T is also in K;* 2) *for S and T in K, if $\Delta^n[S] \cap \Delta^m[T] \neq \emptyset$, then $\Delta^n[S] \cap \Delta^m[T] = \Delta^k[U]$ for some U in K, that is, if simplices of K intersect, then they do so along a common face. The* **dimension** *of a geometric simplicial complex,* $\dim K$, *is the largest n for which there is an n-simplex in K.*





Two collections of triangles in $\mathbb{R}^3$ are shown in the picture. The one on the left represents a simplicial complex, while on the right we have just a union of triangles—this is because the intersections fail to satisfy condition 2) in the definition.

Since n-simplices are homeomorphic to one another for fixed n, it is the collection K of simplices that determines a simplicial complex. We distinguish between the combinatorial data K, collections of sets of vertices, and the topological space determined by the union of the simplices $\Delta^n[S]$ as a subspsace of $\mathbb{R}^N$,

$$|K| = \bigcup\nolimits_{S \in K} \Delta^n[S].$$

The space $|K|$ is called the **realization** of K; $|K|$ is also referred to as the *underlying space* of K [Giblin], the *geometric carrier* of K [Croom], or the *polyhedron* determined by K [Hilton-Wylie].

By separating the combinatorial data from the topological data for a simplicial complex, this definition frees us to introduce an abstraction of geometric simplicial complexes.

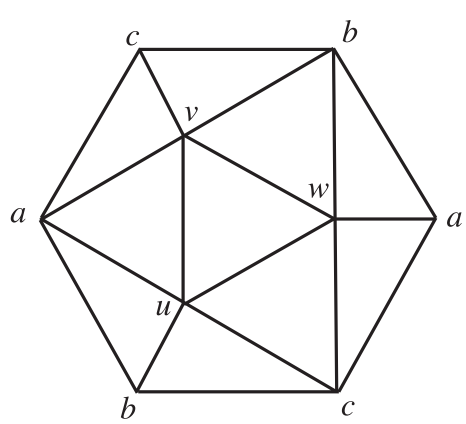
Definition 10.5. *A finite collection of sets $L = \{S_\alpha \mid S_\alpha = \{v_{\alpha 0}, \ldots, v_{\alpha n_\alpha}\}, 1 \leq \alpha \leq N\}$ is an* **abstract simplicial complex** *if whenever $T = \{v_{j_0}, \ldots, v_{j_k}\}$ is a subset of S and S is in L, then T is also in L.*

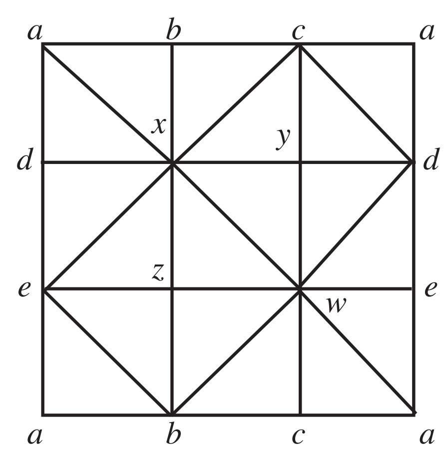
In its simplicity there is a gain in flexibility with the notion of an abstract simplicial complex. We can define all sorts of combinatorial objects in this manner (see, for example, [Björner]). To maintain the connection to topology, we ask if it is possible to associate to every vertex v in an abstract simplicial complex L a point $\mathbf{v}$ in $\mathbb{R}^N$ in such a way that L determines a geometric simplicial complex. The answer is yes, and the proof is an exercise in linear algebra (sketched in the exercises) in which we associate a list of vectors in $\mathbb{R}^N$ in general position to each set S in L. In fact, if the abstract simplicial complex contains a set of cardinality at most $m+1$, then there is a geometric simplicial complex L' with corresponding sets consisting of vectors in $\mathbb{R}^{2m+1}$ in general position.

Another way to connect with topology is to use the combinatorial data given by an abstract simplicial complex and construct a topological space by gluing simplices together: If $L = \{S \mid S = \{v_0, \ldots, v_n\}\}$, then the set of equivalence classes, $|L| = \left[\bigcup_{S \in L} \mathbf{\Delta}^n_S\right]$, associated to the equivalence relation given by $\mathbf{p} \sim \mathbf{q}$ for $\mathbf{p} \in \mathbf{\Delta}^n_S$ and $\mathbf{q} \in \mathbf{\Delta}^m_T$ if there is a shared face $U \prec S$, $U \prec T$ and $\mathbf{p} = \mathbf{q}$ in $\mathbf{\Delta}^k_U \subset \mathbf{\Delta}^n_S$ and $\mathbf{\Delta}^k_U \subset \mathbf{\Delta}^m_T$, that is, we glue the simplices S and T along their shared subsimplex U. We give this space the quotient topology as a quotient of the disjoint union of the simplices $\mathbf{\Delta}^n_S$. The reader should check that this quotient construction determines a space homeomorphic to the realization of a geometric simplicial complex built out of vertices in $\mathbb{R}^N$.

The general class of topological spaces modeled by simplicial complexes is the class of the triangulable spaces.

Definition 10.6. *A topological space X is said to be* **triangulable** *if there is an abstract simplicial complex K and a homeomorphism $f\colon X \to |K|$.*





Examples: 1) We can describe triangulable spaces by giving the triangulation explicitly, not as a collection of sets of vectors, but as a collection of simplices with clear gluing data. For example, the diagrams above show how $\mathbb{R}P(2)$ and the torus $S^1 \times S^1$ are triangulable spaces. Notice how the simplices abu and abw in $\mathbb{R}P(2)$ and the simplices abx and abe in the torus share the side ab, encoding the gluing data by the identification of the simplices as shown.

2) The sphere $S^n \subset \mathbb{R}^{n+1}$ is triangulable in a particularly nice way. Consider the n-simplex $\mathbf{\Delta}^n \subset \mathbb{R}^{n+1}$ for which the vertices are $\mathbf{e}_0, \mathbf{e}_1, \ldots, \mathbf{e}_n$ with $\mathbf{e}_i = (0, 0, \ldots, 0, 1, 0, \ldots, 0)$ where the one is in the $(i+1)$-st place. Consider the point

$$\beta_n = \sum\nolimits_{i=0}^{n} \frac{1}{n+1} \mathbf{e}_i = (1/(n+1), 1/(n+1), \ldots, 1/(n+1)).$$

This point is the **barycenter** of $\mathbf{\Delta^n}$, and it can be defined for any simplex as the center of gravity of the vertices. We use the barycenter to move the hyperplane in which $\mathbf{\Delta^n}$ lies to pass through the origin. Since $\mathbf{\Delta^n}$ lies in the hyperplane $\Pi = \{(t_0, \ldots, t_n) \mid t_0 + \cdots + t_n = 1\}$, the translated hyperplane through the origin is $\Pi - \beta_n = \{(s_0, \ldots, s_n) \mid s_0 + \cdots + s_n = 0\}$. We identify a copy of S^{n-1} with the intersection of S^n and $\Pi - \beta_n$, that is, elements of $\mathbf{x} \in \mathbb{R}^{n+1}$ satisfying $x_0^2 + x_1^2 + \cdots + x_n^2 = 1$ and $x_0 + x_1 + \cdots + x_n = 0$.

Define the following mapping

$$\Psi \colon \mathrm{bdy}\, \mathbf{\Delta^n} \to S^{n-1}, \quad \Psi(\mathbf{x}) = \frac{\mathbf{x} - \beta_n}{\|\mathbf{x} - \beta_n\|}.$$

Since the sum of the coordinates of $\mathbf{x}$ is 1, $\mathbf{x} - \beta_n$ lies in $\Pi - \beta_n$ and hence $\Psi(\mathbf{x})$ is in S^{n-1}. Furthermore, Ψ is defined by translation followed by normalization and so Ψ is continuous. Since $\mathrm{bdy}\, \mathbf{\Delta^n}$ is given by $\partial_0 \mathbf{\Delta^n} \cup \cdots \cup \partial_n \mathbf{\Delta^n}$, $\mathrm{bdy}\, \mathbf{\Delta^n}$ is compact. To see that Ψ is a homeomorphism, it suffices, by Proposition 6.9, to show that Ψ has an inverse.

Suppose $\mathbf{s} = (s_0, \ldots, s_n)$ is an element of $S^{n-1} = S^n \cap (\Pi - \beta_n)$, then there is an entry s_k for which $s_k \leq s_i$ for all $0 \leq i \leq n$. Furthermore, since $\sum_i s_i = 0$ and $\sum_i s_i^2 = 1$, we must have $s_k < 0$. Define

$$\Phi \colon S^{n-1} = S^n \cap (\Pi - \beta_n) \to \mathrm{bdy}\, \mathbf{\Delta^n}, \quad \Phi(\mathbf{s}) = \frac{-1}{s_k(n+1)} \mathbf{s} + \beta_n.$$

To see that $\Phi \circ \Psi$ is the identity, let $\mathbf{x} \in \mathrm{bdy}\, \mathbf{\Delta^n}$. Then for some $0 \leq k \leq n$, there is an entry $x_k = 0$ in $\mathbf{x}$. It follows that $\mathbf{s} = \Psi(\mathbf{x})$ has entry $s_k = \dfrac{-1}{(n+1)\|\mathbf{x} - \beta_n\|}$. Furthermore, since $x_i \geq 0$ for all i, s_k is the least entry in $\mathbf{s}$ and so the composite $\Phi \circ \Psi$ gives

$$\Phi \circ \Psi(\mathbf{x}) = \Phi\left(\frac{\mathbf{x} - \beta_n}{\|\mathbf{x} - \beta_n\|}\right) = \frac{-1}{(n+1)(-1/((n+1)\|\mathbf{x} - \beta_n\|))} \left(\frac{\mathbf{x} - \beta_n}{\|\mathbf{x} - \beta_n\|}\right) + \beta_n = \mathbf{x}.$$

The opposite composite $\Psi \circ \Phi$ gives the identity on S^{n-1}: because $\|\mathbf{s}\| = 1$ and $s_k < 0$,

$$\Psi \circ \Phi(\mathbf{s}) = \Psi\left(\frac{-1}{(n+1)s_k} \mathbf{s} + \beta_n\right) = \frac{(-1/(n+1)s_k)\, \mathbf{s} + \beta_n - \beta_n}{\|(-1/(n+1)s_k)\, \mathbf{s} + \beta_n - \beta_n\|} = \mathbf{s}.$$

It follows that bdy $\mathbf{\Delta^n}$ is homeomorphic to S^{n-1}. Since the boundary of $\mathbf{\Delta^n}$ is given as a simplicial complex by the union $\partial_0\mathbf{\Delta^n}\cup\cdots\cup\partial_n\mathbf{\Delta^n}$, the sphere S^{n-1} is triangulable. This fact will prove useful in Chapter 11.

As with spaces we can apply set-theoretic constructions to simplicial complexes to produce new ones.

Definition 10.7. *If K is an abstract simplicial complex and L is a subset of simplices in K, then L is a* **subcomplex** *of K if whenever $S \prec T$ and $T \in L$, then $S \in L$.*

In example 2) above we have shown that $\bigcup_{i=0}^{n}\partial_i\Delta^n = \operatorname{bdy}\Delta^n$ is a subcomplex of Δ^n. In the torus triangulation, notice that the set of simplices $\{dex, xez, xzw, xyw, dyw, dew\}$ together with all the associated subsimplices forms a subcomplex of the torus, whose realization is a cylinder. In the projective plane the subcomplex generated by the collection of 2-simplices $\{abu, auv, uvw, vbw, abw\}$ determines a triangulation of the Möbius band.

Simplicial mappings and barycentric subdivision

How do we compare simplicial complexes? Mappings between simplicial complexes are based on their combinatorial structure.

Definition 10.8. *Let K and L be two simplicial complexes. A* **simplicial mapping** *is function $\phi\colon K \to L$ satisfying, for all $n \geq 0$, if $S = \{v_0,\ldots,v_n\}$ is an n-simplex in K, then $\{\phi(v_0),\ldots,\phi(v_n)\}$ is a (possibly degenerate) simplex in L. Two simplicial complexes are* **isomorphic** *if there are simplicial mappings $\phi\colon K \to L$ and $\gamma\colon L \to K$ with $\phi\circ\gamma = \mathrm{id}_L$ and $\gamma\circ\phi = \mathrm{id}_K$. A simplicial mapping $\phi\colon K \to L$ determines a continuous mapping of the associated realizations $|\phi|\colon |K| \to |L|$: If $\phi\colon K \to L$ is a simplicial mapping, then $\mathbf{p} = \sum_{i=0}^{n} t_i\mathbf{v}_i \in |K|$ maps to $|\phi|(\mathbf{p}) = \sum_{i=0}^{n} t_i\phi(\mathbf{v}_i) \in |L|$.*

Given a subcomplex $L \subset K$ of a simplicial complex, then the inclusion map, $i\colon L \to K$ is a simplicial mapping. Also, a composite of simplicial mappings $K \xrightarrow{\phi} L \xrightarrow{\gamma} M$ is a simplicial mapping.

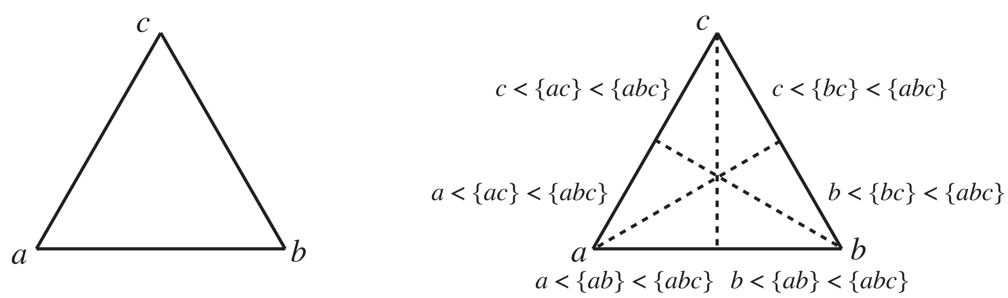
Since the mapping $|\phi|\colon |K| \to |L|$ associated to a simplicial mapping is linear on each simplex, it is continuous. Notice that there are only finitely many continuous mappings $|K| \to |L|$ that are realized in this manner. Because there are only finitely many 0-simplices in K and L, there are only finitely many vertex mappings, of which the simplicial mappings are a subset. In what follows, we construct more simplicial mappings between $|K|$ and $|L|$. To do so, we refine a simplicial complex in order to make approximations. A refinement of a grating in Chapter 9 was accomplished by the addition of line segments, subdividing the rectangles into smaller cells. To refine a simplicial complex, we subdivide the simplices.

Definition 10.9. *Let K be a simplicial complex. The* **barycentric subdivision** *of K, denoted $\operatorname{sd} K$, is the simplicial complex whose simplices are given by*

$$\{\beta(S_0),\beta(S_1),\ldots,\beta(S_r)\},\ \text{where } S_i \in K,\ \text{and } S_0 \prec S_1 \prec \cdots \prec S_r.$$

Here $\beta(S) = \beta(\{\mathbf{v}_0,\ldots,\mathbf{v}_n\}) = \sum_{i=0}^{n}\frac{1}{n+1}\mathbf{v}_i$ is the barycenter of $\Delta^n[S]$ for S in K. If $\phi\colon K \to L$ is a simplicial mapping, then the barycentric subdivision of ϕ is the simplicial mapping $\operatorname{sd}\phi\colon \operatorname{sd} K \to \operatorname{sd} L$ given on vertices by $\operatorname{sd}\phi(\beta(S)) = \beta(\phi(S))$.

The operation $K \mapsto \operatorname{sd} K$ may be summarized: First find the barycenters of every simplex in K, then subdivide the simplices of K into new simplices organized by the subset ordering of simplices, $S \prec T$. For example, a one-simplex $\{a, b\}$ is realized by the line segment ab. The barycenter is the midpoint of ab and the barycentric subdivision $\operatorname{sd}\{a, b\}$ has two one-simplices $\{a, \beta_1\}$ and $\{\beta_1, b\}$ corresponding to $\{a\} \prec \{a, b\}$ and $\{b\} \prec \{a, b\}$. The barycentric subdivision of a two-simplex, $\Delta^2[\{a, b, c\}]$ has six two-simplices as in the picture:



The effect of barycentric subdivision on a simplicial mapping is to send the new barycenters of simplices in K to the corresponding barycenters of the image simplices in L.

To understand the kind of approximation the barycentric subdivision provides, we introduce the *diameter of a simplex*: Let K be a simplicial complex, realized in $\mathbb{R}^N$. Then

$$\operatorname{diam} S = \max\{\|\mathbf{v}_i - \mathbf{v}_j\| \mid i \neq j,\ S = \{\mathbf{v}_0, \ldots, \mathbf{v}_q\}\}.$$

The diameter depends on the embedding of $|K|$ in $\mathbb{R}^N$, but this dependence will not affect the combinatorial use of subdivision.

Proposition 10.10. *If S is a q-simplex in K, a geometric simplicial complex, then for any simplex $T \in \operatorname{sd} K$ with $\Delta^p[T] \subset \Delta^q[S]$, we have* $\operatorname{diam} T \leq \frac{q}{q+1} \operatorname{diam} S$.

Proof: We proceed by induction on q. If $q = 1$, then $\Delta^1[S]$ is a line segment and the simplices of the barycentric subdivision are halves of the segment with diameter equal to $1/2$ the length of the segment. Assume the result for simplices of dimension less than $q \geq 2$.

A p-simplex $T \in \operatorname{sd} K$ can be written as

$$T = \left\{\mathbf{v}_{\sigma(0)}, \frac{\mathbf{v}_{\sigma(0)} + \mathbf{v}_{\sigma(1)}}{2}, \frac{\mathbf{v}_{\sigma(0)} + \mathbf{v}_{\sigma(1)} + \mathbf{v}_{\sigma(2)}}{3}, \ldots, \frac{\mathbf{v}_{\sigma(0)} + \mathbf{v}_{\sigma(1)} + \cdots + \mathbf{v}_{\sigma(p)}}{p+1}\right\},$$

where σ is some permutation of $(0, 1, \ldots, q)$. If $p < q$, then we are done because T is a simplex in the barycentric subdivision of a face of S. When $p = q$, write the vertices of T as $T = \{\mathbf{w}_0, \mathbf{w}_1, \ldots, \mathbf{w}_q\}$. The diameter of T is given by $\|\mathbf{w}_{i_0} - \mathbf{w}_{j_0}\| = \max\{\|\mathbf{w}_i - \mathbf{w}_j\| \mid \mathbf{w}_i, \mathbf{w}_j \in T\}$. If i_0 and j_0 are less than q, then the diameter of T is achieved on the face $\partial_q T$ and we deduce

$$\|\mathbf{w}_{i_0} - \mathbf{w}_{j_0}\| \leq \frac{q-1}{q} \operatorname{diam} \partial_q S \leq \frac{q}{q+1} \operatorname{diam} S.$$

If one of i_0 or j_0 is q, then we first observe the following estimate:

$$\left\| \mathbf{v}_i - \frac{\mathbf{v}_{\sigma(0)} + \mathbf{v}_{\sigma(1)} + \cdots + \mathbf{v}_{\sigma(q)}}{q+1} \right\| = \left\| \sum\nolimits_{j=0}^{q} \frac{1}{q+1} (\mathbf{v}_i - \mathbf{v}_j) \right\|$$
$$\leq \sum\nolimits_{j=0}^{q} \frac{1}{q+1} \|\mathbf{v}_i - \mathbf{v}_j\| \leq \frac{q}{q+1} \max\{\|\mathbf{v}_i - \mathbf{v}_j\|\} = \frac{q}{q+1} \operatorname{diam} S.$$

This proves the proposition. ♢

We define a measure of the refinement of a simplicial complex by taking the maximum of the diameters of the constituent simplices, the *mesh* of K,

$$\operatorname{mesh}(K) = \max\{\operatorname{diam} S \mid S \in K\}.$$

Corollary 10.11. *If K has dimension q, then* $\operatorname{mesh}(\operatorname{sd} K) \leq \dfrac{q}{q+1} \operatorname{mesh}(K)$.

By iterating barycentric subdivision, we can make the simplices in $\operatorname{sd}^N K$ as small as we like: For any $\epsilon > 0$, there is an N with $\operatorname{mesh}(\operatorname{sd}^N K) \leq \left(\dfrac{q}{q+1}\right)^N \operatorname{mesh}(K) < \epsilon$.

How does barycentric subdivision affect the topological space $|K|$?

Theorem 10.12. *If K is a geometric simplicial complex, then $|\operatorname{sd} K| = |K|$.*

Proof: Suppose that $\mathbf{p} \in |K|$. Then we can write $\mathbf{p} = \sum_{i=0}^{q} t_i \mathbf{v}_i \in \Delta^q[S]$ with $S = \{\mathbf{v}_0, \ldots, \mathbf{v}_q\}$. Permute the values $\{t_i\}$ to bring them into descending order

$$t_{\sigma(0)} \geq t_{\sigma(1)} \geq \cdots \geq t_{\sigma(q)} \geq 0.$$

Next solve the matrix equation:

$$\begin{pmatrix} 1 & \frac{1}{2} & \frac{1}{3} & \cdots & \frac{1}{q+1} \\ 0 & \frac{1}{2} & \frac{1}{3} & \cdots & \frac{1}{q+1} \\ 0 & 0 & \frac{1}{3} & \cdots & \frac{1}{q+1} \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & \cdots & \frac{1}{q+1} \end{pmatrix} \begin{pmatrix} s_0 \\ s_1 \\ s_2 \\ \vdots \\ s_q \end{pmatrix} = \begin{pmatrix} t_{\sigma(0)} \\ t_{\sigma(1)} \\ t_{\sigma(2)} \\ \vdots \\ t_{\sigma(q)} \end{pmatrix}.$$

The solution exists and is unique. Furthermore, by solving from the bottom up, the solution satisfies $s_q = (q+1)t_{\sigma(q)}$ and $s_{j-1} = j(t_{\sigma(j-1)} - t_{\sigma(j)}) \geq 0$. Summing the values of s_j we get

$$\begin{aligned} \sum_{j=0}^{q} s_j &= s_0 + 2((1/2)s_1) + 3((1/3)s_2) + \cdots + (q+1)((1/(q+1))s_q) \\ &= (s_0 + (1/2)s_1 + (1/3)s_2 + \cdots + (1/(q+1))s_q) \\ &\quad + ((1/2)s_1 + (1/3)s_2 + \cdots + (1/(q+1))s_q) + \cdots + (1/(q+1))s_q \\ &= t_{\sigma(0)} + t_{\sigma(1)} + \cdots + t_{\sigma(q)} = t_0 + \cdots + t_q = 1. \end{aligned}$$

Thus $(s_0, \ldots, s_q)$ are the barycentric coordinates of $\mathbf{p}$ in the simplex with

$$\begin{aligned}\mathbf{p} = s_0\mathbf{v}_{\sigma(0)} + s_1\left(\frac{\mathbf{v}_{\sigma(0)} + \mathbf{v}_{\sigma(1)}}{2}\right) + s_2\left(\frac{\mathbf{v}_{\sigma(0)} + \mathbf{v}_{\sigma(1)} + \mathbf{v}_{\sigma(2)}}{3}\right)\\ + \cdots + s_q\left(\frac{\mathbf{v}_{\sigma(0)} + \mathbf{v}_{\sigma(1)} + \cdots + \mathbf{v}_{\sigma(q)}}{q+1}\right).\end{aligned}$$

Thus $\mathbf{p}$ lies in the q-simplex $\Delta^q[T]$ where $T \in \operatorname{sd} K$ is given by

$$T = \{\beta(\{\mathbf{v}_{\sigma(0)}\}), \beta(\{\mathbf{v}_{\sigma(0)}, \mathbf{v}_{\sigma(1)}\}), \beta(\{\mathbf{v}_{\sigma(0)}, \mathbf{v}_{\sigma(1)}, \mathbf{v}_{\sigma(2)}\}), \ldots, \beta(\{\mathbf{v}_{\sigma(0)}, \mathbf{v}_{\sigma(1)}, \ldots, \mathbf{v}_{\sigma(q)}\}\}.$$

This proves that $|K| \subset |\operatorname{sd} K|$. The inclusion $|\operatorname{sd} K| \subset |K|$ follows by rewriting the expression for a point in the barycentric coordinates of $\operatorname{sd} K$ in terms of the contributing vertices of K by rearranging terms. $\diamondsuit$

Barycentric subdivision leads to a notion of approximation. Given a continuous mapping $f\colon |K| \to |L|$, we seek a simplicial mapping $\phi\colon K \to L$ that *approximates* f in some sense. Since we can replace $|K|$ with $|\operatorname{sd}^n K|$ where $\operatorname{sd}^n K$ denotes the iterated barycentric subdivision of K, $\operatorname{sd}^0 K = K$, and $\operatorname{sd}^n K = \operatorname{sd}(\operatorname{sd}^{n-1} K)$, then we can approximate f by using simplicial mappings between subdivisions of the complexes involved. To make precise what we mean by an approximation, we introduce a point-set notion.

Definition 10.13. *If v is a vertex in a simplicial complex K, then the* **star** *of v,* $\operatorname{star}_K(v)$, *is the collection of all simplices in K for which v is a vertex. The* **open star** *of v, $O_K(v)$, is the union of the interiors of simplices in K with v as a vertex,*

$$\operatorname{star}_K(v) = \bigcup\nolimits_{\{v\}\prec S} \Delta^n[S], \quad O_K(v) = \bigcup\nolimits_{\{v\}\prec S} \operatorname{int}\, \Delta^n[S].$$

The stars of vertices can be used to recognize simplices in a simplicial complex.

Lemma 10.14. *Suppose v_0, v_1, $\ldots$, v_n are vertices in a simplicial complex K. Then $\{v_0, \ldots, v_q\}$ is a simplex in K if and only if $\bigcap_{i=0}^q O_K(v_i) \neq \emptyset$. If $\mathbf{p} \in |K|$, then $\mathbf{p} \in O_K(v)$ if and only if $\mathbf{p} = \sum_{i=0}^q t_i v_i$ with $v = v_j$ for some $0 \leq j \leq q$ and $t_j \neq 0$.*

Proof: If $S = \{v_0, \ldots, v_q\}$ is a q-simplex in K, then $\operatorname{int}\, \Delta^q[S] \subset O_K(v_i)$ for $i = 0, \ldots, q$. Hence $\bigcap_{i=0}^q O_K(v_i) \neq \emptyset$.

Suppose $\mathbf{p} \in \bigcap_{i=0}^q O_K(v_i) \neq \emptyset$. then $\mathbf{p} = \sum t_j w_j \in \Delta^r[S]$ with $\{v_0, \ldots, v_q\} \subset \{w_0, \ldots, w_r\}$. Furthermore, if $w_{m_i} = v_i$, then $t_{m_i} > 0$. Thus all of the v_i appear in the barycentric coordinates of $\mathbf{p}$ and so the subset of S, $\{v_0, \ldots, v_q\}$, is a simplex in K. $\diamondsuit$

To approximate a continuous mapping $f\colon |K| \to |L|$ by a simplicial mapping $\phi\colon K \to L$, we expect that points in $f(\Delta^q[S])$ are 'close' to points in $|\phi|(\Delta^q[S])$.

Definition 10.15. *If K and L are simplicial complexes and $f\colon |K| \to |L|$ a continuous function, then a simplicial mapping $\phi\colon K \to L$ is a* **simplicial approximation** *to f if whenever $\mathbf{p} \in |K|$, then $f(\mathbf{p}) \in \Delta^q[T]$ for $T \in L$ implies $|\phi|(\mathbf{p}) \in \Delta^q[T]$.*

This definition can be difficult to establish, but there is a more convenient condition for our purposes that works in a manner analogous to the way open sets simplify continuity arguments when compared with the classical ϵ-δ arguments.

PROPOSITION 10.16. *A simplicial mapping $\phi\colon K \to L$ is a simplicial approximation to a continuous mapping $f\colon |K| \to |L|$ if and only if, for any vertex v of K, we have*

$$f(O_K(v)) \subset O_L(\phi(v)),$$

that is, the image of the open star of v under f is contained in the open star of $\phi(v)$, a vertex of L.

Proof: Suppose $\mathbf{p} \in O_K(v)$ for some vertex $v \in K$. Then $\mathbf{p} \in \text{int } \Delta^q[S]$ for some unique $S \in K$ with $v \in S$. Because ϕ is a simplicial mapping, $\phi(S) = T$ for some simplex in L, and $|\phi|(p) \in \text{int } \Delta^{q'}[T'] \subset O_L(\phi(v))$ for some $T' \prec T$. Since ϕ is a simplicial approximation to f, if $\mathbf{p} \in \Delta^r[S']$ for $S \prec S'$ and $f(\mathbf{p}) \in \text{int } \Delta^s[T'']$ for some $T'' \in L$, then $|\phi|(\mathbf{p}) \in \Delta^s[T'']$. Since points lie in unique interiors of simpices, $|\phi|(\mathbf{p}) \in \text{int } \Delta^{q'}[T']$ implies that $T' \prec T''$ and so $\phi(v) \in T''$. Therefore, $f(\mathbf{p}) \in O_L(\phi(v))$.

We introduce a weaker notion than a simplicial mapping. Let $K_0 = \{v \in K \mid \{v\}$, a 0-simplex in $K\}$. A *vertex map* $\phi\colon K_0 \to L_0$ satisfies if $v \in K$ is a vertex, then $\phi(v) \in L$ is also a vertex. Suppose also, for every vertex $v \in K_0$, that $f(O_K(v)) \subset O_L(\phi(v))$. Suppose that $S \in K$ is a simplex and $S = \{v_0, \dots, v_q\}$. Then

$$f\left(\bigcap\nolimits_i O_K(v_i)\right) \subset \bigcap\nolimits_i f(O_K(v_i)) \subset \bigcap\nolimits_i O_L(\phi(v_i)).$$

Since int $\Delta^q[S] \subset \bigcap_i O_K(v_i)$, this intersection is nonempty, and $\phi(S) = \{\phi(v_0), \dots, \phi(v_q)\}$ is a simplex in L. This establishes that a vertex mapping ϕ with $f(O_K(v)) \subset O_L(\phi(v))$, for all v, is a simplicial mapping. Furthermore, if $\mathbf{p} \in \text{int } \Delta^q[S]$ and $f(\mathbf{p}) \in \text{int } \Delta^r[T]$ for some $T \in L$, then for each vertex v_i of S, $f(\mathbf{p}) \in f(O_K(v_i)) \subset O_L(\phi(v_i))$, and so $\phi(v_i) \in T$. It follows that $\phi(S) \prec T$ and so $|\phi|(\mathbf{p}) \in \Delta^r[T]$. Therefore, ϕ is a simplicial approximation to f. $\diamondsuit$

Example: In Theorem 10.12 we proved that $|\text{sd } K| = |K|$. Is there a simplicial approximation to the identity mapping? Consider the vertex mapping $\lambda\colon \text{sd } K \to K$, defined by

$$\lambda\colon \beta(S) = \beta(\{v_0, \dots, v_q\}) \mapsto v_q.$$

To see that we have a simplicial approximation, we check that $O_{\text{sd } K}(\beta(S)) \subset O_K(v_q)$. A simplex with $\beta(S)$ as a vertex takes the form $T = \{\beta(S_0), \beta(S_1), \dots, \beta(S_n)\}$ with $S_1 \prec S_2 \prec \cdots \prec S_n$ in K and $S = S_j$ for some j. If $\mathbf{p} \in \text{int } \Delta^q[T]$, then $\mathbf{p} = \sum_i t_i \beta(S_i)$ with $t_i > 0$. We can rewrite the barycenters as the averages of the vertices in S_i for $i = 0$ to q, and we get $\mathbf{p} = \sum_k u_k w_k$ with $u_k > 0$ and $w_k \in K$ for all k. Since v_q is among the vertices and its barycentric coordinate is positive, $\mathbf{p} \in O_K(v_q)$. Thus λ is a simplicial approximation to id: $|\text{sd } K| \to |K|$. In fact, we did not need to choose the last vertex v_q to define λ. As the argument shows, any choice of vertex from S for each $S \in K$ will do. This added flexibility will come in handy later.

The topology of a triangulable space may be used to show that simplicial approximations are plentiful.

Simplicial Approximation Theorem. *Given two simplicial complexes K and L and a continuous mapping $f\colon |K| \to |L|$, then there is a nonnegative integer r and a simplicial mapping $\phi\colon \mathrm{sd}^r K \to L$ with ϕ a simplicial approximation to f.*

Proof: We use the fact that $|K|$ and $|L|$ are compact metric spaces. Suppose $\dim K = n$. The collection $\{f^{-1}(O_L(w)) \mid w \text{ a vertex in } L\}$ is an open cover of $|K|$. By Lebesgue's Lemma (Chapter 6) the cover has a Lebesgue number $\delta_K > 0$. Iterating barycentric subdivision, we can subdivide K until

$$\mathrm{mesh}\,(\mathrm{sd}^r K) \leq \left(\frac{n}{n+1}\right)^r \mathrm{mesh}\,(K) < \delta_K/2.$$

This is possible because $(\frac{n}{n+1})^r$ goes to zero as r goes to infinity. It follows that $\mathrm{sd}^r K$ has all simplices of diameter less than $\delta_K/2$ and so, for each $v \in \mathrm{sd}^r K$, the diameter of $O_K(v)$ is less than δ_K. Thus each $O_K(v)$ is contained in some $f^{-1}(O_L(w))$. This determines a vertex map $\phi\colon v \mapsto w$, which satisfies $f(O_K(v)) \subset O_L(\phi(v))$, a simplicial approximation.$\diamondsuit$

Simplicial approximations exist in abundance. How are these combinatorial mappings related to their approximated topological mappings? What relation is there between two simplicial approximations of the same continuous mapping? We can answer these questions with the homotopy relation between continuous mappings. This relationship formed the basis for the combinatorial nature of some of the earliest developments in topology (see, for example, [Brouwer1]).

Proposition 10.17. *If a simplicial mapping $\phi\colon K \to L$ is a simplicial approximation to a continuous mapping $f\colon |K| \to |L|$, then $|\phi|$ is homotopic to f.*

Proof: Suppose that $\mathbf{p} \in \mathrm{int}\ \Delta^q[S]$ for $S \in K$ and $S = \{v_0, \ldots, v_q\}$. By Lemma 10.14, $\mathbf{p} \in \bigcap_{v_i \in S} O_K(v_i)$. It follows that

$$f(\mathbf{p}) \in \bigcap\nolimits_{v_i \in S} f(O_K(v_i)) \subset \bigcap\nolimits_{v_i \in S} O_L(\phi(v_i)).$$

Therefore, $\{\phi(v_0), \ldots, \phi(v_q)\}$ is a simplex in L and the convex set $\Delta^q[\phi(S)]$ contains both $|\phi|(\mathbf{p})$ and $f(\mathbf{p})$. We define a homotopy on $\mathrm{int}\ \Delta^q[S]$ by

$$H(\mathbf{p}, t) = tf(\mathbf{p}) + (1-t)|\phi|(\mathbf{p}).$$

The homotopy extends to all of $|K|$ by Theorem 4.4 and so $f \simeq |\phi|$. $\diamondsuit$

It follows from the proposition that two, possibly different, simplicial approximations to a given continuous function have homotopic realizations. The simplicial mappings also enjoy a further combinatorial property.

Definition 10.18. *Two simplicial mappings ϕ and $\psi\colon K \to L$ are said to be* **contiguous** *if, for all simplices $S \in K$, the set $\phi(S) \cup \psi(S)$ is a simplex in L.*

Lemma 10.19. *Suppose $f\colon |K| \to |L|$ is a continuous function for which ϕ and $\psi\colon K \to L$ are simplicial approximations to f. Then ϕ and ψ are contiguous.*

Proof: Suppose S is a simplex in K with $S = \{v_0, \ldots, v_q\}$. Then for $\mathbf{p} \in \text{int } \Delta^q[S]$, we have

$$f(\mathbf{p}) \in f\left(\bigcap\nolimits_i O_K(v_i)\right) \subset \bigcap\nolimits_i f(O_K(v_i)) \subset \bigcap\nolimits_i O_L(\phi(v_i)) \cap O_L(\psi(v_i)).$$

Since this intersection is not empty, the collection $\phi(S) \cup \psi(S)$ is a simplex in L. $\diamondsuit$

The condition of being contiguous is combinatorial—we are only checking that unions of images of sets of vertices in K appear among the sets of vertices of L. The following results show that contiguity encodes the relation of homotopy very well.

Proposition 10.20. *Contiguous simplicial mappings have homotopic realizations.*

Proof: If $\mathbf{p} \in \text{int } \Delta^q[S] \subset |K|$, then the points $|\phi|(\mathbf{p})$ and $|\psi|(\mathbf{p})$ lie in the simplex of L given by $\phi(S) \cup \psi(S)$. The homotopy $H(\mathbf{p}, t) = (1-t)|\phi|(\mathbf{p}) + t|\psi|(\mathbf{p})$ is well-defined, continuous, and establishes $|\phi| \simeq |\psi|$. $\diamondsuit$

A partial converse to Proposition 10.20 is the following theorem.

Theorem 10.21. *Suppose that f and g are continuous mappings $|K| \to |L|$ and f is homotopic to g. Then there exists simplicial mappings ϕ and $\psi\colon \text{sd}^N K \to L$ with ϕ a simplicial approximation to f, ψ a simplicial approximation to g, and there is a sequence of simplicial mappings $\phi = \phi_0$, ϕ_1, $\ldots$, $\phi_{n-1}, \phi_n = \psi$ with ϕ_i contiguous to ϕ_{i+1} for $0 \leq i \leq n-1$.*

Proof: Let $H\colon |K| \times [0,1] \to |L|$ be a homotopy with $H(\mathbf{p}, 0) = f(\mathbf{p})$ and $H(\mathbf{p}, 1) = g(\mathbf{p})$. Cover $|K| \times [0,1]$ with the open cover $\{H^{-1}(O_L(w)) \mid w \text{ is a vertex of } L\}$. Since $|K| \times [0,1]$ is compact, by a careful use of Lebegue's Lemma, we can find a partition of $[0,1]$, $0 = t_0 < t_1 < \cdots < t_{n-1} < t_n = 1$ such that, for any $\mathbf{p} \in |K|$, $H(\mathbf{p}, t_{i-1})$ and $H(\mathbf{p}, t_i)$ lie in $O_L(w)$ for some vertex $w \in L$. Define the functions $h_i\colon |K| \to |L|$ by $h_i(\mathbf{p}) = H(\mathbf{p}, t_i)$. Construct another open cover of $|K|$ defined as $\mathcal{U} = \mathcal{U}_1 \cup \cdots \cup \mathcal{U}_n$ where

$$\mathcal{U}_i = \{h_i^{-1}(O_L(w)) \cup h_{i-1}^{-1}(O_L(w)) \mid w \text{ a vertex in } L\}.$$

Subdivide K enough times so that the simplices in $\text{sd}^N K$ are finer than the cover $\mathcal{U}$. Let $\phi_i\colon \text{sd}^N K \to L$ be the vertex mapping which satisfies $h_i(O_K(v)) \cup h_{i-1}(O_K(v)) \subset O_L(\phi_i(v))$ for each vertex $v \in \text{sd}^N K$. By construction, ϕ_i is a simplicial approximation to h_i and h_{i-1}. Regrouping these data, we find that ϕ_i and ϕ_{i+1} are both simplicial approximations to h_i and hence ϕ_i and ϕ_{i+1} are contiguous by Proposition 10.19. Since $h_0 = f$ and $h_n = g$, $\phi = \phi_0$ is a simplicial approximation of f, and $\psi = \phi_n$ is a simplicial approximation to g. This proves the theorem. $\diamondsuit$

We close with a consequence of these ideas. Suppose X and Y are triangulable spaces. Then the set of homotopy classes of mappings from X to Y, is denoted by $[X, Y]$, as introduced in Chapter 7. We can replace this set by $[|K|, |L|]$ where $|K|$ is homeomorphic to X and $|L|$ homeomorphic to Y. By the Simplicial Approximation Theorem, for each homotopy class $[f] \in [|K|, |L|]$, there is a simplicial mapping $\phi\colon \text{sd}^r K \to L$ with $[|\phi|] = [f]$. Furthermore, by Proposition 10.20 and Theorem 10.21, different choices of representative for $[f]$ always stay in the same homotopy class of the realization of the simplicial approximation.

Let $\mathcal{S}(K, L)$ denote the set of simplicial mappings from K to L. Because K and L involve only finitely many simplices, $\mathcal{S}(K, L)$ is a finite set. With this notation, the Simplicial Approximation Theorem implies that the mapping

$$\Theta: \bigcup_{N \geq 0} \mathcal{S}(\mathrm{sd}^N K, L) \longrightarrow [|K|, |L|], \qquad \Theta(\phi) = [|\phi|],$$

is onto. The union of countably many finite sets is countable and so we have proved that $[X, Y]$ is countable whenever X and Y are triangulable. This implies, for example, since $\pi_1(X, x_0) \subset [S^1, X]$, the fundamental group of a triangulable space is countable.

Exercises

1. Suppose that K is an abstract simplicial complex of dimension n. To find a geometric realization of K, we want to identify vertices of K with points in some $\mathbb{R}^N$ in such a way that, whenever $\{v_0, \ldots, v_q\}$ is a simplex in K, then the associated points $\{\mathbf{v}_0, \ldots, \mathbf{v}_q\}$ are in general position in $\mathbb{R}^N$. In $\mathbb{R}^{2n+1}$ consider the curve

   $$\mathcal{C} = \{(r, r^2, \ldots, r^{2n+1}) \mid r \in \mathbb{R}\}.$$

   Using the Vandermonde determinant, any $2n+2$ distinct points on $\mathcal{C}$ are in general position ([35]). Assign to each vertex in K, a distinct point on $\mathcal{C}$. Since $\dim K = n$, a simplex in K determines at most n points on $\mathcal{C}$ and hence a set in general position. We next worry about intersections of these geometric simplices. Suppose $\{\mathbf{v}_0, \ldots, \mathbf{v}_i, \ldots, \mathbf{v}_{i+k}\}$ and $\{\mathbf{v}_i, \ldots, \mathbf{v}_{i+k}, \ldots, \mathbf{v}_m\}$ are simplices with a shared face. Then $m < 2n+2$ because $\dim K = n$ and so the union of these sets is in general position. Show that this guarantees that the intersection between these simplices is along a common face alone. Thus we can take an abstract simplicial complex as a geometric simplicial complex without hesitation.

2. Draw a picture (or better yet, make a model) of the first and second barycentric subdivisions of Δ^3.

3. If K and L are simplicial complexes, their **join**, $K * L$ is the set consisting of the simplices of K, the simplices of L, and the set of 1-simplices $\{\{a, b\} \mid a$ a vertex in K, b a vertex in $L\}$. Show that $K * L$ is a simplicial complex. When $L = \{v_0\}$ and $v_0 \notin K$, show that $K * \{v_0\}$ has CK, the cone on K, as realization.

4. Suppose that $\phi: K \to L$ is a simplicial mapping. Suppose that $\psi: K \to L$ is a simplicial approximation to $|\phi|: |K| \to |L|$. Show that $\psi = \phi$. Thus a simplicial mapping is its own simplicial approximation.

5. Suppose that $f: |K| \to |L|$ has a simplicial approximation $\phi: K \to L$. Show that $\mathrm{sd}\,\phi: \mathrm{sd}\,K \to \mathrm{sd}\,L$ is also a simplicial approximation of f.

6. Prove that composites of contiguous simplicial mappings are contiguous.

7. Suppose K has dimension m and $\phi\colon K \to \operatorname{bdy} \Delta^n$ is a simplicial mapping. If $m < n$, show that $|\phi|$ is null homotopic by showing that the image of $|\phi|$ is not all of $|\operatorname{bdy} \Delta^n|$. This implies that $[S^m, S^n]$ has cardinality one for $m < n$.