I. Quantum-mechanical Systems and Photon Polarization

Systems of states in quantum mechanics are described by linear combinations of eigenvectors of that state. The most commonly encountered "state," a generic term for a mathematic encapsulation of a system's properties, is that of a particle's wavefunction, $|\psi\rangle$. This state, which describes the probability amplitude of a particle, is a linear combination (the *superposition*) of infinitely many orthogonal eigenstates, $|\varphi_n\rangle$, which represent the particle in its various available energy levels. In essence, a wavefunction is a vector in Hilbert space with an infinite number of components, and can be expressed as a sum of these:

$$|\psi\rangle = \sum c_n |\varphi_n\rangle$$

Just as a vector in three-dimensional Cartesian space is represented by the sum of three components, $\mathbf{r} = a\hat{\mathbf{x}} + b\hat{\mathbf{y}} + c\hat{\mathbf{z}}$, each eigenvector acts as a sort of unit vector in Hilbert space that defines theoretical directionality.

Another fundamental property of quantum-mechanical systems is the *generalized statistical interpretation* of operators, which tells us that if any operator corresponding to an observable (e.g. position, momentum, angular momentum, energy, etc.) is measured on a state, the measurement will only return one eigenvalue of that operator:

$$\hat{Q}|\psi\rangle = q_n|\psi\rangle$$

The probability, furthermore, for measuring any given eigenvalue q_n, is given by the square value of the coefficient that corresponds with that value's eigenstate, $|c_n|^2$. c_n can likewise be found through the inner product of a state and the coefficient's respective eigenstate:

$$c_n = \langle \varphi_n | \psi \rangle$$

This expression in Dirac notation can also be interpreted as the *projection* of the state $|\psi\rangle$ in the theoretical directionality of the eigenstate $|\varphi_n\rangle$.

The sum of these conceptual tenets is this: In quantum mechanics, the act of measurement changes a state by reducing it to the eigenstate whose value is measured. This is called the *Reduction*

Postulate. In regards to the wavefunction of a particle, this means that the act of measuring a discrete quantized value such as energy will reduce the state to a substate of a single energy level. This is called *wavefunction collapse*. One can also conceptualize the Reduction Postulate in terms of the Uncertainty Principle. Because position and momentum representations are both eigenstates of the wavefunction in Hilbert space, measuring one precisely will result in a higher uncertainty for the value of the other. States which can only be measured one at a time, such as those related by the Uncertainty Principle, or various energy levels of a particle's wavefunction, are said to be *conjugate* to each other.

The polarization of a photon is a two-state quantum-mechanical system. Just as the classical equivalent of a beam of light can be polarized into a coherent pattern of electromagnetic waves, on the quantum level, each individual photon, the smallest possible energy packet in that wave, can have a polarization state. Photons, additionally, can have a polarization in several different *bases*, each of which allows for a conjugate pair of two orthonormal eigenstates. For example, photons can have a rectilinear polarization basis, with either a horizontal or vertical polarization state. Alternatively, they can have diagonal polarization, which involves rotating a rectilinear polarizing filter by 45 degrees, so that the two available states are 45° and 135°, or even a circular polarization, which allows for either right- or left-circular polarization states.

Formally, a photon's polarization is given by the system:

$$|P\rangle = a|1\rangle + b|2\rangle$$

where $|1\rangle$ and $|2\rangle$ are the eigenvectors representing the two allowed states for a given basis. The probability of measuring one of these polarization states is the square value of the coefficient: there is a probability $|a|^2$ of measuring the photon in state $|1\rangle$, and a probability $|b|^2$ of measuring it in state $|2\rangle$. For a photon with a linear (either rectilinear or diagonal) polarization basis, this becomes:

$$|P\rangle = \cos\theta\, e^{i\alpha}\, |1_L\rangle + \sin\theta\, e^{i\alpha}\, |2_L\rangle$$

where α is the phase angle and θ is the angel of polarization. For the simplest case of $\alpha = 2\pi$, we can see that for a rectilinear polarization basis, with $\theta = \{0°, 90°\}$, $|P\rangle$ either takes the value of $|1\rangle$ or $|2\rangle$

with a probability of 1. For a diagonal basis, with $\theta = \{45°, 135°\}$, and still $\alpha = 2\pi$,

$$|P\rangle = \frac{1}{\sqrt{2}}|1_D\rangle + \frac{1}{\sqrt{2}}|2_D\rangle$$

Therefore there is an equal probability of 1/2 that either state is measured.

For a circular polarization basis:

$$|P\rangle = \frac{1}{\sqrt{2}}(\cos\theta\, e^{i\alpha_x} - i\sin\theta\, e^{i\alpha_y})\,|1_C\rangle + \frac{1}{\sqrt{2}}(\cos\theta\, e^{i\alpha_x} + i\sin\theta\, e^{i\alpha_y})\,|2_C\rangle$$

In this case, $|1_C\rangle$ corresponds to a right-circular polarization, and $|2_C\rangle$ to left-circular.

Polarization is an observable, therefore the Reduction Postulate applies to $|P\rangle$: if any measurement of polarization occurs, the system collapses to the eigenstate that is measured. Additionally, because they are also orthonormal, only the polarization in one basis can be measured at a time. This phenomenon is similar to that of electron spin, another two-state system, in which only one component of the spin, S_x, S_y, or S_z, can be known at a time. If a photon with a rectilinear polarization basis is measured in state $|1_R\rangle$, all following measurements of $\hat{P}_R$ (the polarization operator in that basis) will measure the same state. However, if a later observer measures, say, $\hat{P}_D$, the polarization in a diagonal basis, the photon will realign to either state $|1_D\rangle$ or $|2_D\rangle$. Because each of these states has probability $|\frac{1}{\sqrt{2}}|^2 = 1/2$, the realignment will be effectively random, with an equal chance of returning either state. Now that this photon is in an eigenstate of the diagonal basis, the Uncertainty Principle says that the previous measurement of $\hat{P}_R$ no longer is valid – that information is lost, because a precise measurement of $\hat{P}_D$ has resulted in a maximum uncertainty for the former observable.

This is very similar to the results of the Stern-Gellach experiment: if one measures the spin of an electron along one of its components, say S_x, and finds a certain value, then measures a different component, that component will be randomly measured according to the relative probabilities of each spin eigenstate in that direction. When S_x is measured again, the results will again be random, and will not necessarily match the previous measurement.

In summary, once a photon is measured in a certain polarization basis, all future measurements of the same basis will return the same value, but all measurements in different bases will be random.