Transfer of the Ramsey Property between Classes

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BLAST 2015 @ UNT
We consider classes of **finite** structures such as

\[ \mathcal{K}_< = \{(V, <) \mid < \text{linearly orders } V\} \]
\[ \mathcal{K}_g = \{(V, E, <) \mid < \text{linearly orders } V, \ E \text{ is a symmetric edge relation on } V\} \]

Say that \( L \) is the language of the finite structures in \( \mathcal{K} \).

For \( A, C \in \mathcal{K} \) define

\[ \left( \begin{array}{c} C \\ A \end{array} \right) := \{ \text{the } L\text{-substructures of } C \text{ isomorphic to } A \} \]
Ramsey classes

The previous classes $\mathcal{K}_<, \mathcal{K}_g$ have the **Ramsey property**: for all $A, B \in \mathcal{K}$ there is a $C \in \mathcal{K}$ so that

$$C \rightarrow (B)_2^A$$

i.e. for all 2-colorings

$$c : \binom{C}{A} \rightarrow \{\text{red, blue}\}$$

there is some homogeneous copy $B' \cong B, B' \subseteq C$

so that $c$ is constant on $\binom{B'}{A}$. 

Example

We look at some members of $\mathcal{K}_g$.

Nodes are numbered to indicate the linear ordering. Let

$A : \bullet^2 \rightarrow \bullet^3 \quad B : \bullet^2 \rightarrow \bullet^3$

$\bullet^1 \quad \bullet^4$

$B$ contains two subcopies of $A$ so in general we need $C$ much bigger and more complicated to ensure

$C \rightarrow (B)^A_2$
Indiscernibles

Suppose we want to study some structure $\mathbb{U}$ in a model theoretic context. (Let $\mathbb{U}$ be very saturated.)

Let $I$ be a countable structure with age $\mathcal{K}$.

It is useful to consider $I$-indexed indiscernible sets $\{b_i \mid i \in I\}$.

These are sets of same-length tuples $b_i \in \mathbb{U}^m$ indexed by $I$ so that

$$(i_1, \ldots, i_n) \sim_I (j_1, \ldots, j_n) \Rightarrow \text{tp}^\mathbb{U}(b_{i_1}, \ldots, b_{i_n}) = \text{tp}^\mathbb{U}(b_{j_1}, \ldots, b_{j_n})$$

i.e.

$$\bar{i} \sim_I \bar{j} \Rightarrow \bar{b_i} \equiv^\mathbb{U} \bar{b_j}$$

where $\sim_I$ means “same quantifier-free type in $I$”.
$I$-indexed indiscernibles have the **modeling property** if:

given parameters $\mathcal{I} = \{a_i \mid i \in I\}$ there exist $I$-indexed indiscernible parameters $\mathcal{J} = \{b_i \mid i \in I\}$ that are **locally based on** $\mathcal{I}$, i.e.

for every $j$ from $I$ there exists $i \sim_I j$ such that

$$a_i \equiv \bigcup b_j.$$  

Let $\theta$ be the partial “color-preserving” $L$-embedding that sends $j \mapsto \bar{i}$.  

Translation

Fix $|A| = n$, $|B| = m$ and $\text{age}(I) = \mathcal{K}$.
Assume $I$-indexed indiscernibles have the modeling property. Show $I \rightarrow (B)_2^A$.
We can require $\mathbb{U}$ to code the following 2-coloring on $n$-tuples from $I$

$$c : \binom{I}{A} \rightarrow \{0, 1\}$$

by way of an injection

$$i \in I \mapsto a_i \in \mathbb{U}$$

and a choice of language $L(\mathbb{U}) = \{P_0(x_0, \ldots, x_{n-1}), P_1(x_0, \ldots, x_{n-1})\}$

$$\models_{\mathbb{U}} P_0(\bar{a}_i) \quad \text{if} \quad c(\bar{i}) = 0 \quad (1)$$
$$\models_{\mathbb{U}} P_1(\bar{a}_i) \quad \text{if} \quad c(\bar{i}) = 1 \quad (2)$$

Suppose we find $I$-indexed indiscernible $\mathcal{J} = \{b_i \mid i \in I\}$ locally based on $\mathcal{I} = \{a_i \mid i \in I\}$. $\mathcal{J}$ gives a different coloring $c'$ on $I$:

$$c'(\bar{i}) = 0 \quad \text{if} \quad \models_{\mathbb{U}} P_0(\bar{b}_i) \quad (3)$$
$$c'(\bar{i}) = 1 \quad \text{if} \quad \models_{\mathbb{U}} P_1(\bar{b}_i) \quad (4)$$
Modeling prop: $c'$ copies $c$ locally

Let’s find a copy of $B$ in $I$ homogeneous for the coloring $c$.

1. Fix any subset $B' \subseteq I$ isomorphic to $B$. Enumerate $B'$ as $\bar{j}'$.
2. All copies of $A$ in $B'$ get the same $c'$-color: there is some $* \in \{0, 1\}$ so that for all $(i_0, \ldots, i_{n-1}) \cong A$ in $I$,
   \[
   \models \bigwedge_i P_* (\bar{b}_i)
   \]
   (in particular, for all $\bar{i}$ subsequences of $\bar{j}'$.)
3. By the modeling property there is $\bar{j} \cong \bar{j}'$ in $I$ so that
   \[
   \bar{a}_\bar{j} \equiv \bigwedge \bar{b}_{\bar{j}'} \leftarrow \text{(detecting the value of } s\text{)}
   \]

4. $\bar{j} = (j_0, \ldots, j_{m-1}) \in {}^m I$ is the homogeneous copy of $B$ for $c$:
   for all $\bar{i} = (i_0, \ldots, i_{n-1}) \cong A$ subseq. of $\bar{j}$,
   \[
   \models \bigwedge_i P_s (\bar{a}_i)
   \]
   i.e. $c(\bar{i}) = s$. [[Sco15] is a source for further details.]
Definitions

The theory of $\mathbb{U}$ is stable if it does not admit a set of parameters $\{a_i \mid i \in \mathbb{N}\} \subseteq \mathbb{U}^m$ linearly ordered by some definable relation $\varphi(x; y)$.

The theory of $\mathbb{U}$ is NIP if it does not admit a set of parameters $\{a_i \mid i \in \mathbb{N}\} \subseteq \mathbb{U}^m$ such that some definable relation $\varphi(x; y)$ forms a random graph edge relation between the points.

<table>
<thead>
<tr>
<th>structure</th>
<th>signature</th>
<th>age</th>
<th>$I$-indexed indiscernible</th>
</tr>
</thead>
<tbody>
<tr>
<td>$I_= $</td>
<td>$\emptyset$</td>
<td>all finite sets</td>
<td>total indiscernible</td>
</tr>
<tr>
<td>$I_&lt;$</td>
<td>${&lt;}$</td>
<td>$\mathcal{K}_&lt;$</td>
<td>order-indiscernible</td>
</tr>
<tr>
<td>$I_g$</td>
<td>${&lt;, E}$</td>
<td>$\mathcal{K}_g$</td>
<td>ordered graph-indiscernible</td>
</tr>
</tbody>
</table>
**Theorem.** ([She90]) Theory $T$ is stable if and only if any order indiscernible in a model of $T$ is totally indiscernible.

**Theorem.** ([Sco12]) Theory $T$ is NIP if and only if any ordered graph-indiscernible in a model of $T$ is order indiscernible.

Thanks to M. Malliaris who pointed out the following observation.

**Corollary.** If $U$ has stable theory, any ordered graph-indiscernible in $U$ is totally indiscernible, thus, graph-indiscernible.
Old example

For unordered graphs

\[ A : \bullet -- \bullet \quad B : \bullet -- \bullet \]

the partition property

\[ C \rightarrow (B)_k^A \]

can be defeated by imposing an external ordering and coloring according to
this ordering.

In contrast: let \( U \) model a stable theory. Consider an initial set of
parameters \( \{a_i \mid i \in I\} \subset U \) where \( I = (V, E) \) embeds all finite graphs. Let
\( \{\varphi_1, \ldots, \varphi_k\} \) be formulas in \( L(U) \), exclusive and exhaustive on 3-tuples and define

\[ c(\bar{i}) = s \iff \models_U \varphi_s(a_{\bar{i}}) \]

We are always able to find a copy of \( B \) homogeneous for this coloring.
Sometimes we have two signatures that are somehow interchangeable with respect to the Ramsey property. See [KKS14] for a discussion of the origin of $L_1$ and thanks to C. Rosendal for letting us present example $L_2$ which he worked out in some notes.

**Example.** $I = (\omega > Q, \leq, \wedge, <_{\text{lex}}, (P_n)_n)$ \quad ($L_1 = \{\leq, \wedge, <_{\text{lex}}, (P_n)_n\}$)  
$I = (\omega > Q, f, <_{\text{lex}})$ \quad ($L_2 = \{f, <_{\text{lex}}\}$)

$P_n(\omega > Q)$ = all sequences of length $n$  
$\leq$ = partial tree order given by extension of sequences  
$\wedge$ = meet in the partial order  
$<_{\text{lex}}$ = lexicographical ordering on sequences

$f(\eta) = \nu$ if $\nu$ is the immediate $\leq$-predecessor of $\eta$ if it exists  
i.e. $\nu \upharpoonright \langle i \rangle = \eta$ for some $i \in Q$,  
otherwise $f(\eta) = \eta$ (if $\eta$ is the root.)

These structures are $L_{\omega_1, \omega}$-interdefinable. Every $L_1$-aut. of $\omega > Q$ is an $L_2$-aut. and vice versa.
Theorem. (Kechris, Pestov, Todorcevic)
Let $G \leq S_\infty$ be a closed subgroup. Then $G$ is extremely amenable if and only if $G$ is the automorphism group of the Fraïssé limit of a Fraïssé order class with the Ramsey property.

Assuming local finiteness we can easily adapt languages with function symbols to this context.

The theorem can be seen as commenting on the previous slide. There is the canonical (Hodges) infinite relational language and then there are the more natural finite functional languages. All map to the same automorphism group.
We cannot allow $A$ to embed in both $B_1$ and $B_2$ or risk losing the amalgamation property (thus losing the Ramsey property.)

So we know the canonical language would include relations forbidding such a thing to happen (a relation for the isomorphism type of $A = (a_0, a_1, a_2, a_3)$, not satisfied by either $(b_0, b_1, b_2, b_3)$ or $(c_0, c_1, c_2, c_3)$.)
Let’s look at the case where \( \tau_s \) is a countable relational signature and \( \tau_f \) extends \( \tau_s \) by countably many function symbols.

Suppose underlying set \( X \) can be given a \( \tau_s \)-structure \((X_s)\) so as to be ultrahomogeneous.

Further assume that \( \tau_s, \tau_f \) are interdefinable on \( X \) by quantifier-free \( L_{\omega_1,\omega} \)-formulas.

These assumptions immediately give us that \( \mathcal{K}_s := \text{age}(X_s) \) is a Fraïssé class.

By the interdefinability, \( \tau_f \)-embeddings are \( \tau_s \)-embeddings which extend to automorphisms on \( X \). So in fact the induced structure \( X_f \) is also ultrahomogeneous and \( \mathcal{K}_f := \text{age}(X_f) \) is a Fraïssé class.
At this point we could use the KPT technology to analyze both classes together, but what is the category theoretic perspective?

Let’s think of $\mathcal{K}_s, \mathcal{K}_f$ as categories $\mathcal{C}, \mathcal{D}$ with the finite structures as objects (up to isomorphism) and embeddings as maps.

Then $\text{Ob}(\mathcal{D}) \subset \text{Ob}(\mathcal{C})$ and $\mathcal{D}$ is a full subcategory of $\mathcal{C}$ whose objects are cofinal (under embeddings) in the objects of $\mathcal{C}$.

Thus it is clear that if $\mathcal{C}$ has the Ramsey property, so does $\mathcal{D}$.

What about from $\mathcal{D}$ to $\mathcal{C}$?
We know that if $\mathbf{D}$ has the Ramsey property, so must $\mathbf{C}$ (imm. by KPT.)

Let’s look at how the property is moved on the level of categories.

**Definition.** Define $\mathcal{F} : \text{Ob}(\mathbf{C}) \to \text{Ob}(\mathbf{D})$ by

$$A \mapsto \mathcal{F}(A)$$

$\mathcal{F}(A)$ = take a copy of $A$ in $X_s$ and close under the function symbols in $\tau_f$

$$f \in \text{Mor}_\mathbf{C}(A, B) \mapsto \mathcal{F}(f) \in \text{Mor}_\mathbf{D}(\mathcal{F}(A), \mathcal{F}(B))$$

$\mathcal{F}(f)$ is the natural “extension” of $f$

**Observation.** $\mathcal{F}$ cannot have an inverse, because two objects in $\text{Ob}(\mathbf{C})$ can be mapped to the same object in $\text{Ob}(\mathbf{D})$.

**Fact.** $\mathcal{F}$ is the left adjoint to the “forgetful functor” $\mathcal{G} : \text{Ob}(\mathbf{D}) \to \text{Ob}(\mathbf{C})$. 

Ramsey classes and indiscernibles
applications
transfer between classes

Diagrams

\[
\begin{array}{ccc}
\text{C:} & B \xrightarrow{\sigma_{B,C}(h)} \mathcal{G}(C) \\
& \uparrow f \\
\mathcal{A} & \xrightarrow{c(\sigma_{B,C}(h) \circ f) = r} \\
\end{array}
\]

\[
\text{D:} & \mathcal{F}(B) \xrightarrow{h} \mathcal{C} \\
& \uparrow F = \mathcal{F}(f) \\
\mathcal{F}(\mathcal{A}) & \xrightarrow{c'(\tau) := c(\sigma_{A,C}(\tau))} \\
\end{array}
\]

\[
\begin{array}{ccc}
\text{Mor}_D(\mathcal{F}(B), \mathcal{C}) & \xrightarrow{\sigma_{B,C}} & \text{Mor}_C(B, \mathcal{G}(C)) \\
& \downarrow \_ \circ \mathcal{F}(f) \\
\text{Mor}_D(\mathcal{F}(A), \mathcal{C}) & \xrightarrow{\sigma_{A,C}} & \text{Mor}_C(A, \mathcal{G}(C)) \\
& \exists \tau \\
\end{array}
\]

\[
\sigma : \text{Mor}_D(\mathcal{F}(\_), \_) \cong \text{Mor}_C(\_, \mathcal{G}(\_))
\]

\[
c'(\tau) := c(\sigma_{A,C}(\tau))
\]
Some questions

- Is there a meaningful calculus on categories that transfers the Ramsey property.

- Do these operations shadow the group constructions that preserve extreme amenability in the automorphism group.

- What is the best way to assign a language to a Ramsey class.
Thanks!

- B. Kim, J. Kim, and L. Scow. 
  Tree indiscernibilities, revisited. 

- L. Scow. 
  Characterization of NIP theories by ordered graph-indiscernibles. 

- L. Scow. 
  Indiscernibles, EM-types, and Ramsey classes of trees, 2015. 
  to appear in NDJFL.

- S. Shelah. 
  *Classification Theory and the number of non-isomorphic models (revised edition).* 