Arc index and Turaev genus

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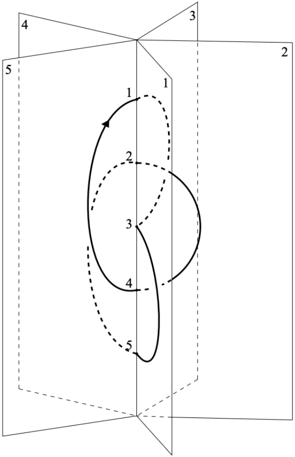
Plan for the talk

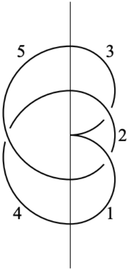
1. Define the arc index and Turaev genus of a link.
2. Describe a conjectural relationship between the crossing number, arc index, and Turaev genus of a link, and prove the conjecture for some infinite families.
3. Compute the arc index of some infinite families of links (e.g. adequate links).

Arc presentation

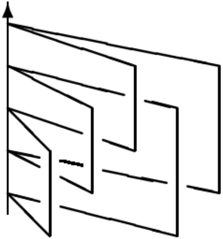
Consider the open-book decomposition of $\mathbb{R}^3$ where the z-axis is the binding and the pages are half planes. An *arc presentation* of a link L is an embedding of L in finitely many pages of the open-book decomposition so that each of these pages meets L in a single simple arc.

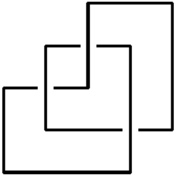
An arc presentation of the trefoil





Arc presentations and grid diagrams

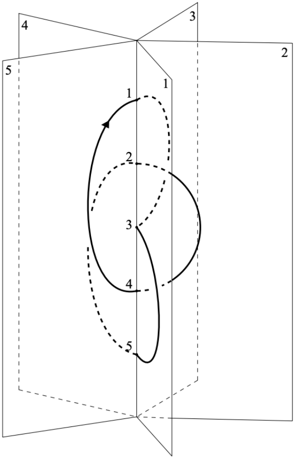


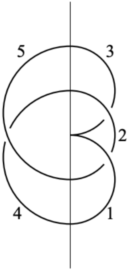


Arc index

Cromwell defined the *arc index* $\alpha(L)$ of a link L to be the minimum number of pages in any arc presentation of L.

Returning to the trefoil





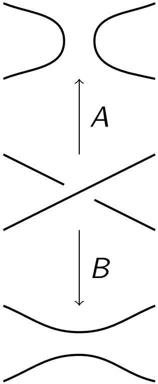
$$\alpha(3_1) = 5$$

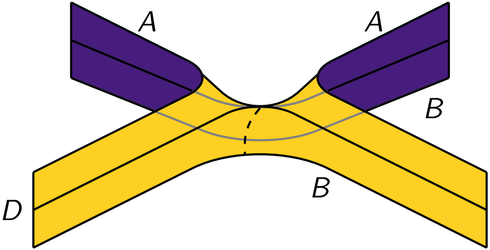
The Turaev surface

The *Turaev surface* of a link diagram D is obtained by

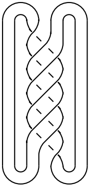
1. constructing a cobordism between the all-A and all-B Kauffman states of D that has saddles corresponding with crossings, and
2. capping off the boundary components of the above cobordism with disks.

The Turaev surface at a crossing



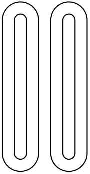


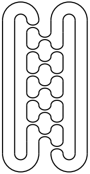
The Turaev surface



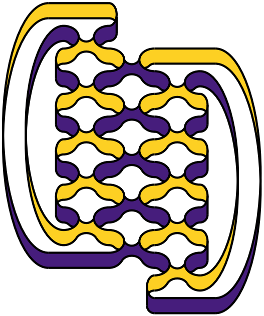
D

all-A state





all-B state



Turaev surface of D

The Turaev genus of a link

- For a connected link diagram D, the genus of the Turaev surface is

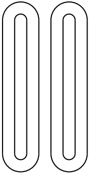
$$g_T(D) = \frac{1}{2}\left(2 + c(D) - |s_A D| - |s_B D|\right)$$

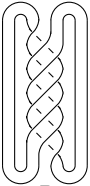
  where $c(D)$ is the number of crossings in D and $|s_A D|$ and $|s_B D|$ are the number of components in the all-A and all-B Kauffman states of D respectively.
- The *Turaev genus* $g_T(L)$ of a link L is

$$g_T(L) = \min\{g_T(D) \mid D \text{ is a diagram of } L\}.$$

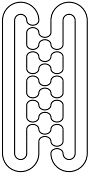
Computing the genus of the Turaev surface

all-A state

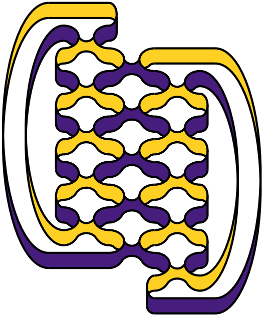




D



all-B state

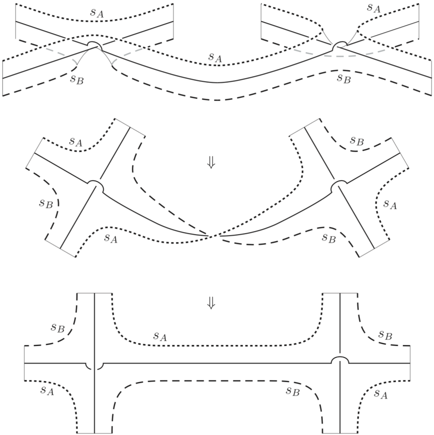


Turaev surface of D

$$c(D) = 15, |s_A D| = 4, |s_B D| = 5, g_T(D) = 4.$$

Alternating projection

A link L has an alternating projection on any of its Turaev surfaces.



The Turaev surface of an alternating diagram

Theorem (Turaev)

The Turaev genus of a link is zero if and only if the link is alternating.

Idea of proof: If the link has a genus zero Turaev surface, then it has an alternating projection to a sphere (i.e. it is alternating). If a link has an alternating diagram, then the components of the all-A and all-B states correspond to the complementary regions of the diagram. Thus the diagram has Turaev genus zero.

History of the Turaev surface

- (1987) - Turaev constructed the surface to give an alternate proof that span $V_L(t) \leq c(L)$. In fact,

$$\text{span } V_L(t) + g_T(L) \leq c(L).$$

- (2003, 2006) - Manturov extended the above inequality to virtual links and found a lower bound to $g_T(L)$ coming from Khovanov homology thickness. See also, Champanerkar, Kofman, and Stoltzfus (2007).
- (2008) - Dasbach, Futer, Kalfagianni, Lin, and Stoltzfus showed that the Jones polynomial of a link is an evaluation of the Bollobas-Riordan-Tutte polynomial of the all-A state graph embedded on the Turaev surface.

History, continued

- (2008) - L. showed that knot Floer homology thickness gives a lower bound to $g_T(L)$.
- (2015) S. Kim and Armond/L. showed that the genus of the Turaev surface only depends on the alternating tangle decomposition of the link diagram.
- (Since 2010) Many other results connecting the Turaev genus to Khovanov homology, knot Floer homology, and to coefficients of the Jones polynomial.

The main conjecture

Conjecture (Del Valle Vílchez, L.)

The following inequality holds for all prime links L:

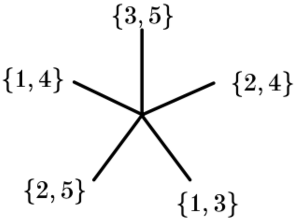
$$c(L) + 2 - \alpha(L) \geq 2g_T(L).$$

Families of knots/links where the conjecture is true

The conjecture holds for

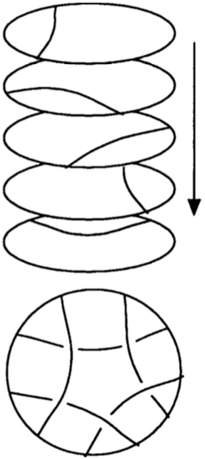
- alternating links,
- Turaev genus one links,
- adequate links,
- torus knots,
- closures of positive 3-braids, and
- all knots with at most 12 crossings.

A top down view of an arc presentation



Binding as a stacked tangle

Perturbing the binding of an arc presentation results in a stacked tangle, and vice versa.

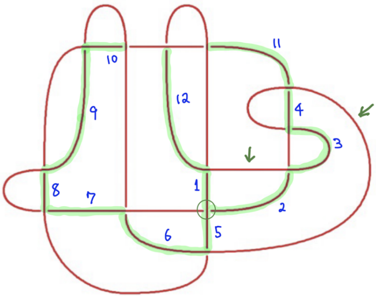


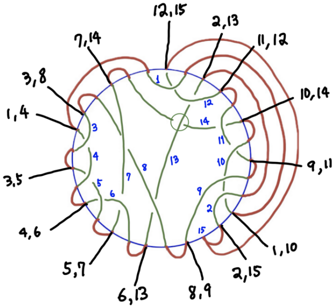
From an arbitrary diagram to a stacked tangle

Jin and Lee (2012) gave a procedure that transforms an arbitrary prime diagram D into a stacked tangle with $c(D) + 2$ arcs on the exterior. Their procedure uses a filtered spanning tree of D.

This is a combinatorial rephrasing of a procedure given by Bae and Park (2000).

From an arbitrary diagram to a stacked tangle





The edges of D not in the spanning tree become the arcs on the exterior of the stacked tangle. There are $c + 1$ such edges, one of which gets split into two arcs, resulting in $c + 2$ arcs on the exterior of the stacked tangle.

Splitting one arc into two

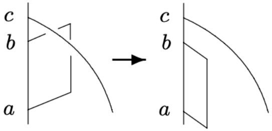
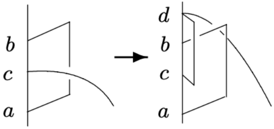


Fig. 5.



Top. A portion of one arc pushed past another arc.
Bottom. An arc split in two, so that half of it can be pushed past an existing arc.

An upper bound on $\alpha(L)$

Theorem (Bae, Park)

For any prime link L,

$$\alpha(L) \leq c(L) + 2.$$

Alternating links

Theorem (Beltrami, Morton)

Let $F_L(v,z)$ be the Kauffman polynomial of L. Then

$$\operatorname{span}_v F_L(v,z) + 2 \leq \alpha(L).$$

When L is alternating, $\operatorname{span}_v F_L(v,z) = c(L)$.

Corollary

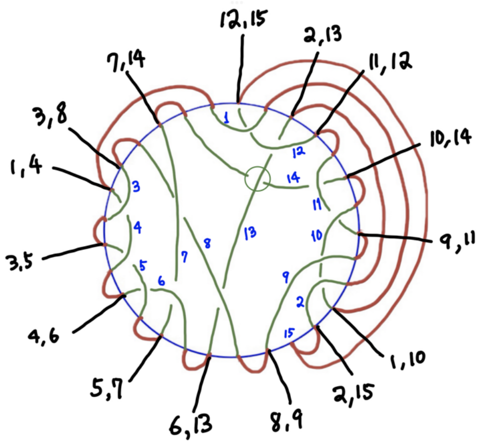
If L is a prime alternating link, then

$$\alpha(L) = c(L) + 2.$$

Our conjecture. If L is alternating, then

$$c(L) + 2 - \alpha(L) = 0 = 2g_T(L).$$

Removing arcs from a stacked tangle



The two arcs labeled $\{8, 9\}$ and $\{11, 12\}$ can be pushed inside the stacked tangle.

Arc index of non-alternating links

Theorem (Jin, Park)

If L is a prime, non-alternating link, then

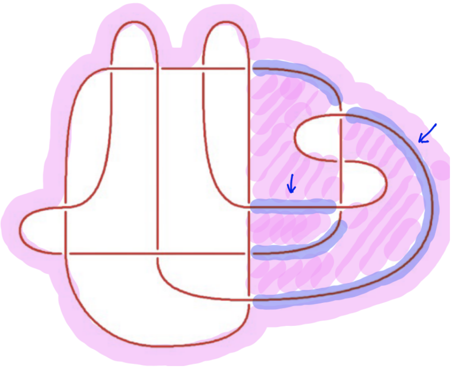
$$\alpha(L) \leq c(L).$$

Our conjecture. If $g_T(L) = 1$, then L is non-alternating, and

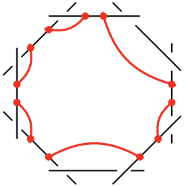
$$c(L) + 2 - \alpha(L) \geq 2 = 2g_T(L).$$

Non-alternating edges

In a series of papers, Gyo Taek Jin, Hwa Jeong Lee, Alexander Stoimenow and other collaborators give a technique for removing arcs that correspond to non-alternating edges from the stacked tangle.



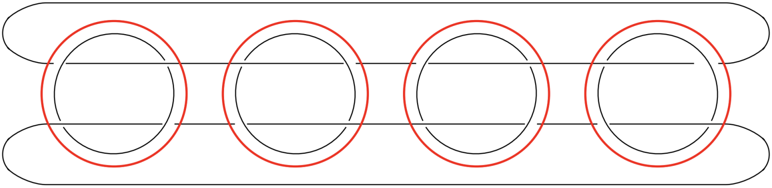
Thistlethwaite's alternating decomposition



Mark each non-alternating edge with two points. Inside each face with a non-alternating edge, connect each marked point to the marked point that is nearest to it on the boundary and not on the same non-alternating edge.

The resulting collection of curves along with the non-alternating edges of D is the *alternating decomposition* of D.

Alternating decomposition example



Alternating decompositions and Turaev genus

Theorem (Armond, L. and S. Kim)

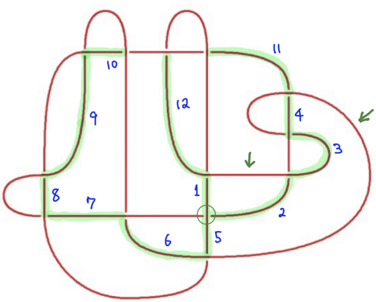
If diagrams D_1 and D_2 have isomorphic alternating decompositions, then $g_T(D_1) = g_T(D_2)$, that is, the genera of the Turaev surfaces of those diagrams are equal.

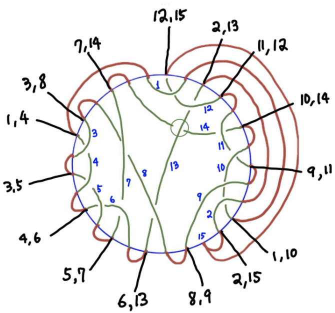
Alternating decompositions and the conjecture

Strategy. Start with the alternating decomposition of a crossing number minimizing diagram D. The alternating decomposition allows us to compute $g_T(D)$. Use the techniques of Jin, Lee, Stoimenow, and others to find an upper bound for $\alpha(L)$.

Results. The strategy works for some families, like the closure of positive 3-braids, but fails for other families.

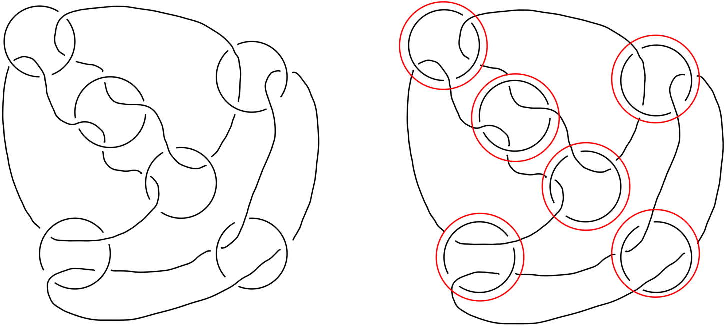
An example where this strategy works





$$c(13_{n3003}) = 13, \alpha(13_{n3003}) = 13, g_T(13_{n3003}) = 1$$

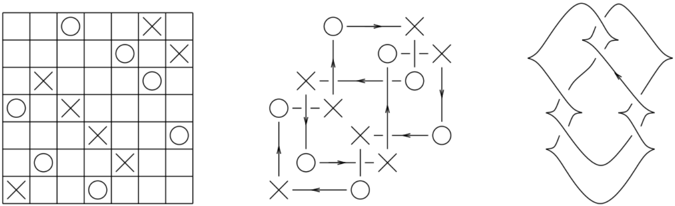
An example where this strategy fails



$$c(L) = 24, g_T(L) = 2, \alpha(L) = 22.$$

Cannot remove 4 edges from the stacked tangle.

Grid diagrams and Legendrian fronts

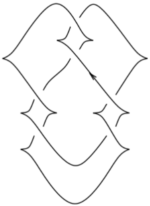


Rotating a grid diagram $\pi/4$ counterclockwise and smoothing north/south corners results in a Legendrian front diagram.

Similarly, rotating $\pi/4$ clockwise results in a Legendrian front diagram for the mirror.

Thurston-Bennequin number

The *Thurston-Bennequin number* $tb(\mathcal{L})$ of a Legendrian link $\mathcal{L}$ with front F is the writhe of F minus the number of right cusps of F.



$w(F) = 6$
$c(F) = 5$
$tb(\mathcal{L}) = 6 - 5 = 1$

The *maximum Thurston-Bennequin number* $\overline{tb}(L)$ of a link L is

$$\overline{tb}(L) = \max\{tb(\mathcal{L}) \mid \mathcal{L} \text{ is a Legendrian representative of } L\}.$$

Arc index and maximum Thurston-Bennequin number

Theorem (Matsuda - 2006)

Let L be a link with mirror $\overline{L}$. Then

$$\alpha(L) \geq -\overline{tb}(L) - \overline{tb}(\overline{L}).$$

Proof. Suppose that L has a grid diagram of size $\alpha(L)$ yielding Legendrian fronts F and $\overline{F}$. Then

$$\begin{aligned} -\overline{tb}(L) - \overline{tb}(\overline{L}) &\leq -tb(F) - tb(\overline{F}) \\ &= -w(F) + c(F) - w(\overline{F}) + c(\overline{F}) \\ &= c(F) + c(\overline{F}) \\ &= \alpha(L). \end{aligned}$$

Arc index and maximum Thurston-Bennequin number

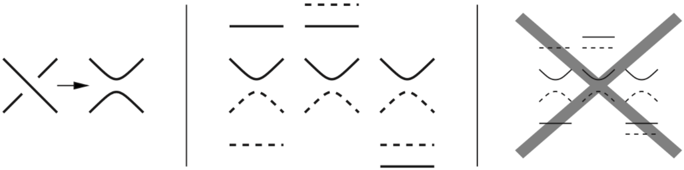
Theorem (Dynnikov, Prasolov - 2013)

Let L be a link with mirror $\overline{L}$. Then

$$\alpha(L) = -\overline{tb}(L) - \overline{tb}(\overline{L}).$$

Ungraded rulings

An *ungraded ruling* is a partial A-resolution of a front F such that each component of the resolution has one right cusp, every (unresolved) crossing is between distinct components of the resolution, and the components involved in any resolution look like the middle of the picture below.



Ungraded rulings and Thurston-Bennequin number

Theorem (Rutherford - 2006)

Let F be a front of a Legendrian link $\mathcal{L}$ with classical link type L. If F has an ungraded ruling, then $\overline{tb}(L) = tb(\mathcal{L})$.

Adequate links

A link diagram D is *A-adequate* if no two arcs in the A-resolution of any crossing lie on the same component of the all-A state $s_A D$ of D. Similarly define B-adequate. A link is *adequate* if it has a diagram that is both A- and B-adequate.

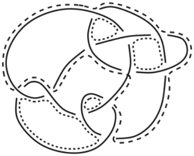
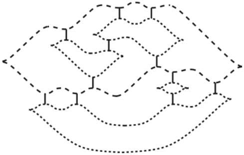
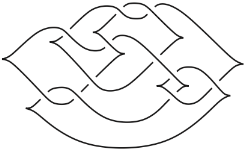
Theorem (Kálmán - 2008)

If D is an A-adequate diagram of the link L, then

$$\overline{tb}(L) = w(D) - |s_A D|.$$

Sketch of Kálmán's proof

There is a front diagram F that is planar isotopic to D such that $c(F) = |s_A D|$. Choosing to resolve every crossing in F results in an ungraded ruling.

The arc index of adequate links

Conjecture (Park, Seo - 2000)

Let D be an adequate diagram of a link L. The arc index of L is

$$\alpha(L) = c(L) + 2\rho(D),$$

where $\rho(D)$ is a quantity computed from the checkerboard graphs of D.

Theorem (Del Valle Vílchez, L.)

Let D be an adequate diagram of a link L. The arc index of L is

$$\alpha(L) = |s_A D| + |s_B D| = c(L) + 2\rho(D).$$

Proof

Kálmán implies that

$$\begin{aligned}\overline{tb}(L) &= w(D) - |s_A D| \text{ and}\\ \overline{tb}(\overline{L}) &= w(\overline{D}) - |s_A \overline{D}|\\ &= -w(D) - |s_B D|.\end{aligned}$$

Dynnikov and Prasolov imply that

$$\alpha(L) = -\overline{tb}(L) - \overline{tb}(\overline{L}) = |s_A D| + |s_B D|.$$

Our conjecture for adequate links

Lickorish and Thistlethwaite (1988) proved that adequate diagrams minimize crossing number. Abe (2009) proved that adequate diagrams minimize Turaev genus.

Our conjecture. If D is an adequate diagram of a link L, then

$$c(L) + 2 - \alpha(L) = c(D) + 2 - |s_A D| - |s_B D| = 2g_T(D) = 2g_T(L).$$

Our conjecture for torus knots

Matsuda (2006) proved that the arc index of the (p, q)-torus knot $T_{p,q}$ is $\alpha(T_{p,q}) = |p| + |q|$.
Assume $0 < p < q$. We find a diagram of D of $T_{p,q}$ such that

$$c(T_{p,q}) + 2 - \alpha(T_{p,q}) = pq - 2q - p + 2 \geq 2g_T(D) \geq 2g_T(T_{p,q}).$$

Our conjectured inequality is strict for $T_{3,4}$:

$$c(T_{3,4}) + 2 - \alpha(T_{3,4}) = 8 + 2 - 7 = 3 > 2 = 2g_T(T_{3,4}).$$

Work in progress

1. Prove that $c(L) + 2 - \alpha(L) \geq 2g_T(L)$ for other families.
2. Prove that $c(L) + 2 - \alpha(L) \geq \eta g_T(L)$ for some η with $0 < \eta < 2$.
3. Use rulings, Kálmán, and Dynnikov and Prasolov to find new infinite families where we can compute arc index.

Happy Birthday Lou!