Invariants of 2-bridge knots

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Knots in Washington 50

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Question

What is the average of the genera of 2-bridge knots with a fixed crossing number?

More questions

What is the $\left\{ \begin{array}{c} \text{average} \\ \text{median} \\ \text{mode} \\ \text{variance} \end{array} \right\}$ of the $\left\{ \begin{array}{c} \text{genera} \\ \text{braid indices} \\ |\text{signatures}| \\ \text{4-genera} \\ \text{crosscap numbers} \end{array} \right\}$ of 2-bridge knots with a fixed crossing number?

Collaborators

- **Genus:** Moshe Cohen, Abigail Dinardo, Steven Raanes, Izabella Rivera, Andrew Steindl, Ella Wanebo
- **Braid index:** Tobias Clark, Jeremy Frank
- **Signature and 4-genus:** Moshe Cohen, Neal Madras, Steven Raanes
- **Crosscap number:** Moshe Cohen, Thomas Kindred, Patrick Shanahan, Cornelia Van Cott

Results

	average	variance	median	mode
genus	$\frac{c}{4}+\frac{1}{12}$ *	$\frac{c}{16}-\frac{17}{144}$	$\left\lfloor\frac{c+2}{4}\right\rfloor$	$\left\lfloor\frac{c+2}{4}\right\rfloor$
braid index	$\frac{c}{3}+\frac{11}{9}$ *	$\frac{2c}{27}-\frac{10}{81}$	$\left\lceil\frac{c+3}{3}\right\rceil$ **	$\left\lceil\frac{c+3}{3}\right\rceil$
$\lvert$signature$\rvert$	$\sqrt{\frac{2c}{\pi}}$	-	-	-
4-genus	$\sqrt{\frac{c}{2\pi}} \leq ? \leq \frac{9.75c}{\log c}$	-	-	-
crosscap number	$\frac{c}{3}+\frac{1}{9}$	-	-	-

* Suzuki and Tran independently proved these results.

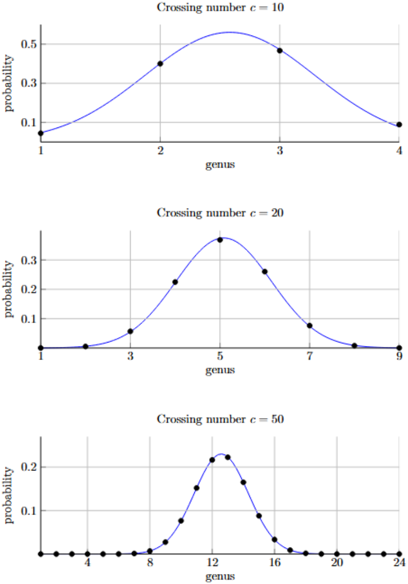
** Conjecture. True for $c \leq 10,000$.

Approaching normal

Theorem (Cohen, Dinardo, L., Raanes, Rivera, Steindl, Wanebo)

The probability distribution of the genera of 2-bridge knots with crossing number c approaches a normal distribution as $c \to \infty$.

Approaching normal



Counting 2-bridge knots with genus g

Theorem (Cohen, Dinardo, L., Raanes, Rivera, Steindl, Wanebo)

The number of 2-bridge knots with genus g and crossing number c is

$$\frac{1}{2}\left((-1)^{c'-g-1}\sum_{n=0}^{c'-g-1}(-1)^n\binom{n+g-1}{n} + (-1)^{c-1}\sum_{n=0}^{c-2g-1}(-1)^n\binom{n+2g-1}{n}\right)$$

where $c' = \left\lfloor \frac{c+1}{2} \right\rfloor$ and $1 \le g \le \left\lfloor \frac{c-1}{2} \right\rfloor$.

2-bridge knots with genus g and crossing number c

$c \setminus g$	1	2	3	4	5	6	7	8	9
3	1								
4	1								
5	1	1							
6	1	2							
7	2	4	1						
8	2	7	3						
9	2	12	9	1					
10	2	18	21	4					
11	3	26	45	16	1				
12	3	36	85	47	5				
13	3	49	151	123	25	1			
14	3	64	251	280	89	6			
15	4	82	400	588	276	36	1		
16	4	103	610	1141	736	151	7		
17	4	128	904	2094	1784	542	49	1	
18	4	156	1294	3648	3960	1658	237	8	
19	5	188	1814	6104	8230	4558	967	64	1
20	5	224	2486	9842	16126	11394	3339	351	9

Counting 2-bridge knots with braid index b

Theorem (Clark, Frank, L.)

Let $c \geq 3$. The number of 2-bridge knots with crossing number c and braid index b is

$$\begin{cases} 1 & \text{if } c \text{ is odd and } b = 2, \\ 2^{b-4}\binom{c-b}{b-2} & \text{if } 3 \leq b \leq \left\lceil \frac{c+1}{2} \right\rceil \text{ and } c+b \text{ is even}, \\ 2^{b-4}\binom{c-b}{b-2} + 2^{b'-2}\binom{c'-b'-1}{b'-1} & \text{if } 3 \leq b \leq \left\lceil \frac{c+1}{2} \right\rceil \text{ and } c+b \text{ is odd}, \\ 0 & \text{otherwise}, \end{cases}$$

where $c' = \lceil c/2 \rceil$ and $b' = \lfloor b/2 \rfloor$.

2-bridge knots with crossing number c and braid index b

$c \backslash b$	2	3	4	5	6	7	8	9	10	11
3	1									
4		1								
5	1	1								
6		2	1							
7	1	2	4							
8		3	6	3						
9	1	3	12	8						
10		4	15	22	4					
11	1	4	24	40	22					
12		5	28	73	60	10				
13	1	5	40	112	146	48				
14		6	45	172	280	174	16			
15	1	6	60	240	516	448	116			
16		7	66	335	840	1,020	448	36		
17	1	7	84	440	1,340	2,016	1,360	256		
18		8	91	578	1,980	3,716	3,360	1,168	64	
19	1	8	112	728	2,890	6,336	7,432	3,840	584	
20		9	120	917	4,004	10,326	14,784	10,600	2,880	136

Continued fractions

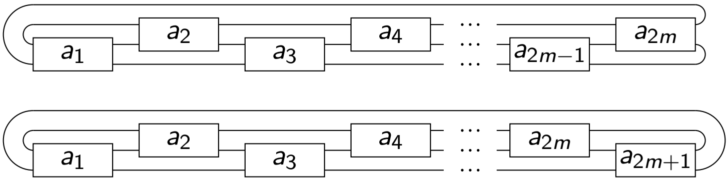
Every rational number $\frac{p}{q}$ has a continued fraction expansion

$$\frac{p}{q} = a_1 + \cfrac{1}{a_2 + \cfrac{}{\ddots + \cfrac{1}{a_{n-1} + \cfrac{1}{a_n}}}}$$

We write $\frac{p}{q} = [a_1, a_2, \ldots, a_n]$.

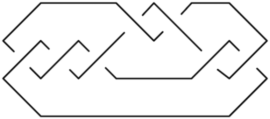
2-bridge knots

For each rational number $\frac{p}{q} = [a_1, \ldots, a_n]$, the rational knot $K\left(\frac{p}{q}\right)$ is the knot with diagram as below.

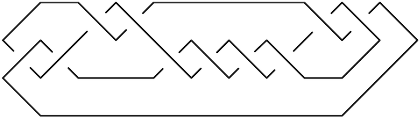


Top: $n = 2m$ is even. **Bottom:** $n = 2m+1$ is odd.

Two specific examples



[3, 2, 2]



[2, 2, -4, 2]

Classification of 2-bridge knots

Theorem (Schubert - 1956)

The 2-bridge knots $K(p/q)$ and $K(p'/q')$ are equivalent if and only if

1. $p = p'$ *and*
2. *either* $q \equiv q' \mod p$ *or* $qq' \equiv 1 \mod p$.

Why 2-bridge knots?

- Schubert's classification.
- Ernst and Sumners proved in 1987 that there are

$$\frac{1}{3}\left(2^{c-3}+2^{\lfloor \frac{c-3}{2} \rfloor}+\varepsilon(c)\right)$$

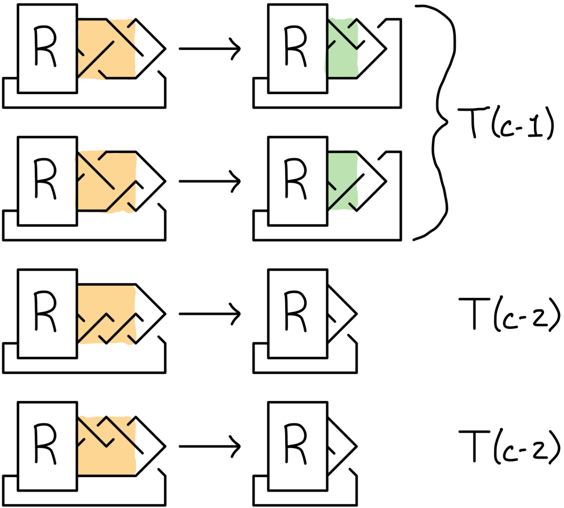
  2-bridge knots of crossing number c, where $\varepsilon(c) \in \{-1,0,1\}$.
- Invariants can often be easily computed for 2-bridge knots.

Our approach using $T(c)$

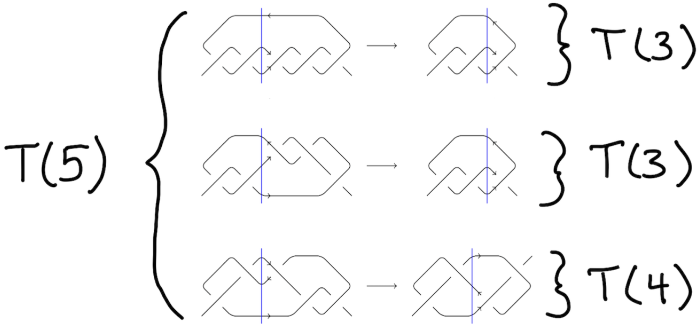
We define a set of alternating 2-bridge knot diagrams $T(c)$ with the following properties.

- Every 2-bridge knot has one or two diagrams in $T(c)$.
- Most 2-bridge knots have two diagrams in $T(c)$.
- If c is odd, then every diagram in $T(c)$ ends with a crossing in the bottom row.
- If c is even, then every diagram in $T(c)$ ends with a crossing in the top row.
- There is a bijection f_c between the sets $T(c)$ and $T(c-1) \sqcup T(c-2) \sqcup T(c-2)$.

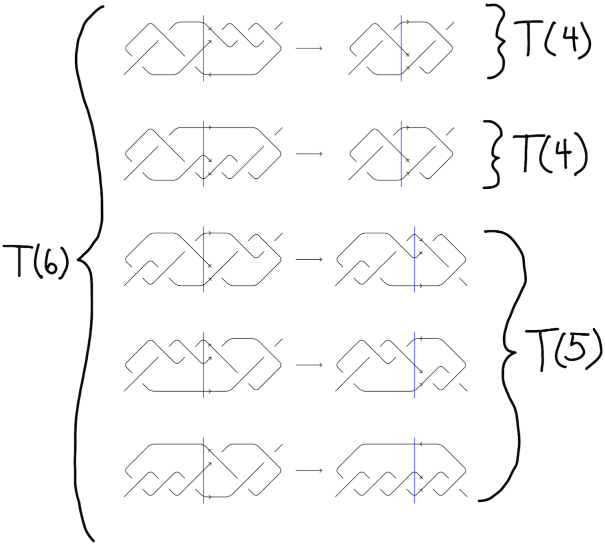
The bijection f_c



The bijection f_5



The bijection f_6



The size of $T(c)$

Let $t_c = |T(c)|$. Since f_c is a bijection,

$$t_c = t_{c-1} + 2t_{c-2}.$$

This recursion has characteristic polynomial

$$x^2 - x - 2 = (x-2)(x+1),$$

and hence the general form for a solution to this recursion is

$$t_c = \alpha 2^c + \beta(-1)^c.$$

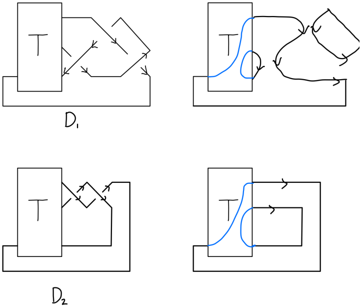
The size of $T(c)$

The initial values $t_3 = t_4 = 1$ determine α and β yielding

$$t_c = \frac{2^{c-2} - (-1)^c}{3}.$$

How f_c alters genus

If D is an alternating diagram of K with c crossings and s Seifert circles, then $g(K) = \frac{1}{2}(1 + c - s)$.



$$
\begin{aligned}
g(K_1) &= \frac{1}{2}(1 + c(D_1) - s(D_1)) \\
&= \frac{1}{2}(1 + (c(D_2) + 1) - (s(D_2) + 1)) = g(K_2)
\end{aligned}
$$

Counting diagrams in $T(c)$ of genus g.

Let $t_{c,g}$ be the number of diagrams in $T(c)$ whose genus is g. Examining how the bijection f_c affects genus leads to the following result.

The number $t_{c,g}$ satisfies the recursion

$$t_{c,g} = t_{c-1,g} + t_{c-2,g-1} + t_{c-2,g} + t_{c-3,g-1} - t_{c-3,g}.$$

$$t_{c,g} = t_{c-1,g} + t_{c-2,g-1} + t_{c-2,g} + t_{c-3,g-1} - t_{c-3,g}$$

$c \setminus g$	1	2	3	4	5	6	7	8	9
3	1								
4	1								
5	2	1							
6	2	3							
7	3	7	1						
8	3	13	5						
9	4	22	16	1					
10	4	34	40	7					
11	5	50	86	29	1				
12	5	70	166	91	9				
13	6	95	296	239	46	1			
14	6	125	496	553	174	11			
15	7	161	791	1163	541	67	1		
16	7	203	1211	2269	1461	297	13		
17	8	252	1792	4166	3544	1068	92	1	
18	8	308	2576	7274	7896	3300	468	15	
19	9	372	3612	12174	16414	9076	1912	121	1
20	9	444	4956	19650	32206	22748	6656	695	17

$$t_{c,g} = t_{c-1,g} + t_{c-2,g-1} + t_{c-2,g} + t_{c-3,g-1} - t_{c-3,g}$$

$c \setminus g$	1	2	3	4	5	6	7	8	9
3	1								
4	1								
5	2	1							
6	2	3							
7	3	7	1						
8	3	13	5						
9	4	22	16	1					
10	4	34	40	7					
11	5	50	86	29	1				
12	5	70	166	91	9				
13	6	95	296	239	46	1			
14	6	125	496	553	174	11			
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$$t_{c,g} = t_{c-1,g} + t_{c-2,g-1} + t_{c-2,g} + t_{c-3,g-1} - t_{c-3,g}$$

$c \setminus g$	1	2	3	4	5	6	7	8	9
3	1								
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19	9	372	3612	12174	16414	9076	1912	121	1
20	9	444	4956	19650	32206	22748	6656	695	17

Total genus

Define the total genus g_c by

$$g_c = \sum_{D \in T(c)} g(D).$$

Then

$$g_c = \sum_{i=1}^{\infty} g \cdot t_{c,g}.$$

Recursion for g_c

$$\begin{aligned}
g_c &= \sum_{g=1}^{\infty} g t_{c,g} \\
&= \sum_{g=1}^{\infty} g t_{c-1,g} + g t_{c-2,g-1} + g t_{c-2,g} + g t_{c-3,g-1} - g t_{c-3,g} \\
&= \sum_{g=1}^{\infty} g t_{c-1,g} + \left(\sum_{g=1}^{\infty} (g-1) t_{c-2,g-1} + \sum_{g=1}^{\infty} t_{c-2,g-1} \right) \\
&\quad + \sum_{g=1}^{\infty} g t_{c-2,g} + \left(\sum_{g=1}^{\infty} (g-1) t_{c-3,g-1} + \sum_{g=1}^{\infty} t_{c-3,g-1} \right) \\
&\quad - \sum_{g=1}^{\infty} g t_{c-3,g}
\end{aligned}$$

Recursion for g_c

$$
\begin{aligned}
&= \sum_{g=1}^{\infty} g t_{c-1,g} + \left(\sum_{g=1}^{\infty} (g-1) t_{c-2,g-1} + \sum_{g=1}^{\infty} t_{c-2,g-1} \right) + \sum_{g=1}^{\infty} g t_{c-2,g} \\
&\quad + \left(\sum_{g=1}^{\infty} (g-1) t_{c-3,g-1} + \sum_{g=1}^{\infty} t_{c-3,g-1} \right) - \sum_{g=1}^{\infty} g t_{c-3,g} \\
&= \sum_{g=1}^{\infty} g t_{c-1,g} + \left(\sum_{g=1}^{\infty} g t_{c-2,g} + \sum_{g=1}^{\infty} t_{c-2,g} \right) + \sum_{g=1}^{\infty} g t_{c-2,g} \\
&\quad + \left(\sum_{g=1}^{\infty} g t_{c-3,g} + \sum_{g=1}^{\infty} t_{c-3,g} \right) - \sum_{g=1}^{\infty} g t_{c-3,g}
\end{aligned}
$$

Recursion for g_c

$$
\begin{aligned}
&= \sum_{g=1}^{\infty} g t_{c-1,g} + \left(\sum_{g=1}^{\infty} g t_{c-2,g} + \sum_{g=1}^{\infty} t_{c-2,g} \right) + \sum_{g=1}^{\infty} g t_{c-2,g} \\
&\quad + \left(\sum_{g=1}^{\infty} g t_{c-3,g} + \sum_{g=1}^{\infty} t_{c-3,g} \right) - \sum_{g=1}^{\infty} g t_{c-3,g} \\
&= g_{c-1} + g_{c-2} + t_{c-2} + g_{c-2} + g_{c-3} + t_{c-3} - g_{c-3} \\
&= g_{c-1} + 2g_{c-2} + t_{c-2} + t_{c-3} \\
&= g_{c-1} + 2g_{c-2} + 2^{c-5}.
\end{aligned}
$$

Formula for g_c

The closed formula for

$$g_c = g_{c-1} + 2g_{c-2} + 2^{c-5}$$

is

$$g_c = \frac{(9c+3)2^{c-3} - 24(-1)^c}{54}.$$

Average genus

The average genus $\overline{g}(c)$ of 2-bridge knots with crossing number c is

$$\begin{aligned}\overline{g}(c) &\approx \frac{g_c}{t_c} \\ &= \frac{\frac{(9c+3)2^{c-3}-24(-1)^c}{54}}{\frac{2^{c-2}-(-1)^c}{3}} \\ &\approx \frac{(9c+3)2^{c-3}}{18\cdot 2^{c-2}} \\ &= \frac{c}{4}+\frac{1}{12}.\end{aligned}$$

Results again

	average	variance	median	mode
genus	$\frac{c}{4} + \frac{1}{12}$	$\frac{c}{16} - \frac{17}{144}$	$\lfloor \frac{c+2}{4} \rfloor$	$\lfloor \frac{c+2}{4} \rfloor$
braid index	$\frac{c}{3} + \frac{11}{9}$	$\frac{2c}{27} - \frac{10}{81}$	$\lceil \frac{c+3}{3} \rceil$	$\lceil \frac{c+3}{3} \rceil$
$\lvert$signature$\rvert$	$\sqrt{\frac{2c}{\pi}}$	-	-	-
4-genus	$\sqrt{\frac{c}{2\pi}} \leq\ ?\ \leq \frac{9.75c}{\log c}$	-	-	-
crosscap number	$\frac{c}{3} + \frac{1}{9}$	-	-	-

- Murasugi's formula for braid index.
- Traczyk's formula for signature.
- A new formula for crosscap number based on work of Adams and Kindred.

Average 4-genus

Theorem (Cohen, L., Madras, Raanes)

The average 4-genus $\overline{g_4}(c)$ *of a 2-bridge knot with crossing number* c *satisfies*

$$\overline{g_4}(c) \leq \frac{9.75c}{\log c}.$$

Corollary

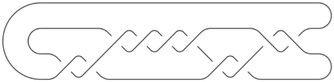
The quotient $\frac{\overline{g_4}(c)}{\overline{g}(c)}$ *approaches* 0 *as* $c \to \infty$.

Even continued fractions

Every 2-bridge knot has an even continued fraction representation

$$\mathbf{a} = [2a_1, 2a_2, \ldots, 2a_{2m}],$$

where $a_i \neq 0$.



$$K([2, -4, 2, 2])$$

Baader, Kjuchukova, Lewark, Misev, Ray

Let $\widehat{g_4}(n)$ be the average 4-genus of 2-bridge knots whose even continued fraction diagrams have n crossings.

Theorem (Baader, Kjuchukova, Lewark, Misev, Ray)

$$\lim_{n\to\infty} \frac{\widehat{g_4}(n)}{n} = 0.$$

In other words, the average 4-genus is sublinear with respect to n, which approximates the true crossing number c.

Proof strategy

Use saddle moves to decompose a 2-bridge knot into the connected sum of many smaller 2-bridge knots.

Proof strategy

For any knot K, the knot $K\# -\overline{K}$ is slice, and thus

$$g_4(K\# -\overline{K}) = 0.$$

Use a random walk argument to find the expected number of cancelling summands in our connected sum.

Bound the average 4-genus by the genus of the cobordism given by the saddle moves and the 3-genus of the non-cancelling summands.

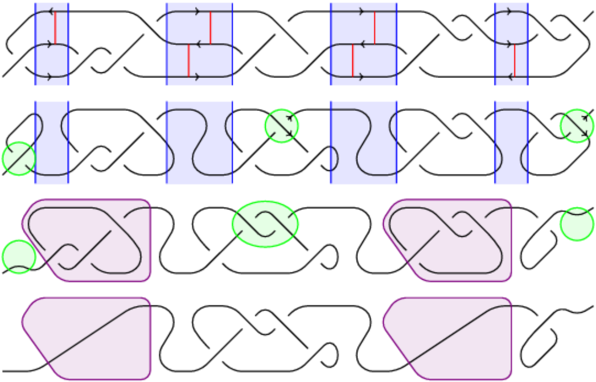
Our approach

We use alternating diagrams of 2-bridge knots instead of diagrams coming from even continued fraction. Hence our average 4-genus computations are with respect to the crossing number.

Our approach mirrors the strategy of Baadder et al., but we had to overcome technical hurdles not present in the even continued fraction case.

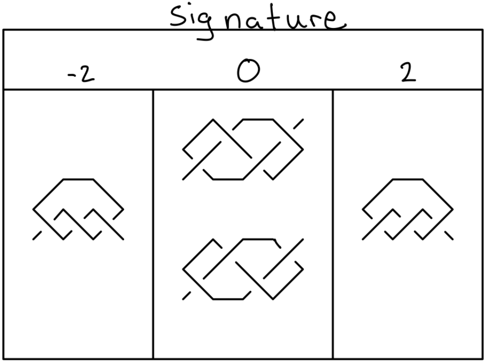
We give an explicit upper bound of $\frac{9.75c}{\log c}$ for $\overline{g_4}(c)$.

A glimpse of the technical hurdles

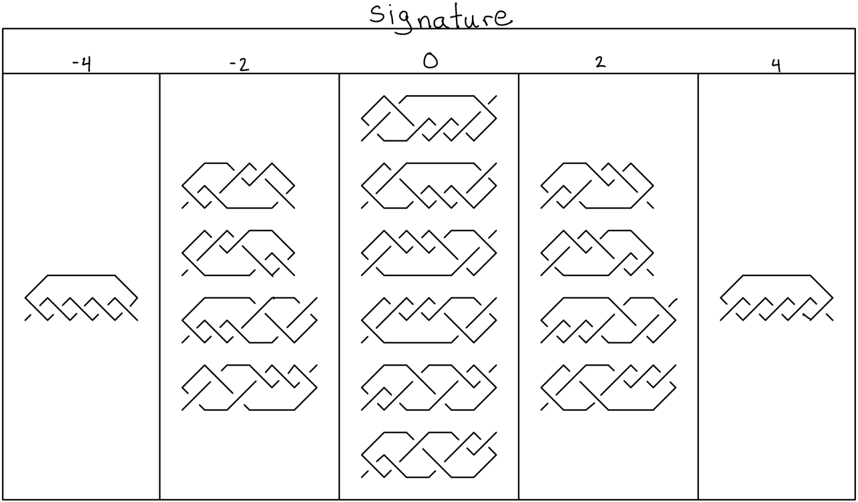


Signature and binomial coefficients

The number of 2-bridge diagrams (in some version of $T(c)$) of crossing number $c = 2m+1$ or $2m+2$ and of signature σ is $\binom{2m}{m+\sigma/2}$.



Signature and binomial coefficients



The signature distribution

Consider the probability that a 2-bridge knot with crossing number $c = 2m + 1$ or $2m + 2$ has a certain fixed signature.

This probability distribution approaches a binomial distribution, and hence is asymptotically normal.

What about the analogous question for a single crossing number?

Future directions

1. Other invariants?
2. Other statistical quantities?
3. Other families of knots?

Thank you!