The expected value of invariants of rational knots

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AMS Eastern Sectional - Spring 2024

April 7, 2024

Collaborators

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Chronology

- (2014-2021) Cohen and collaborators study random models of 2- and 3-bridge knots coming from Chebyshev billiard table diagrams. Cohen finds a lower bound for the average genus of 2-bridge knots.
- (2022) Suzuki-Tran, Diao-Ray, and Cohen-L. compute the average genus of 2-bridge knots.
- (2022-2023) Cohen-L. and Vassar undergrads Dinardo, Raanes, Rivera, Steindl, and Wanebo compute median, mode, and variance of the genus of 2-bridge knots and show that the distribution of genera is asymptotically normal.

Chronology continued

- (2023-2024) Suzuki-Tran and Clark-Frank-L. compute the average braid index of 2-bridge knots.
- (In progress) Cohen-L.-Raanes compute the average signature of 2-bridge knots.
- (In progress) Cohen, Kindred, L., Shanahan, and Van Cott compute the average crosscap number of 2-bridge knots.

Results

Invariant	Average value as $c \to \infty$
Genus	$\frac{3c+1}{12}$
Braid index	$\frac{3c+11}{9}$
Absolute value of signature	$\sqrt{\frac{2c}{\pi}}$
Crosscap number	$\frac{11c+2}{32}$

The signature of a knot

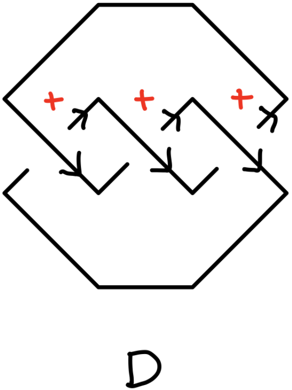
- (Trotter, 1968) The signature of a knot K is the difference in the number of positive and negative eigenvalues of $V + V^T$ for any Seifert matrix V of K.
- (Traczyk, 2004) The signature of an alternating knot K with reduced alternating diagram D is

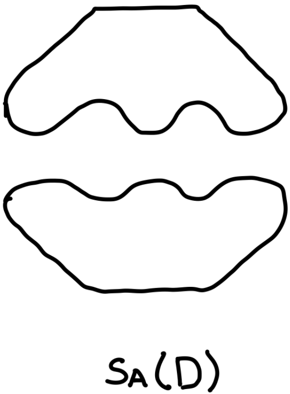
$$\sigma(K) = s_A(D) - c_+(D) - 1$$

where $s_A(D)$ is the number of components in the all-A Kauffman state and $c_+(D)$ is the number of positive crossings in D.

Signature example

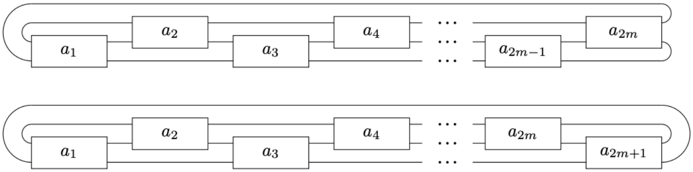
$$\sigma(K) = s_A(D) - c_+(D) - 1 = -2$$





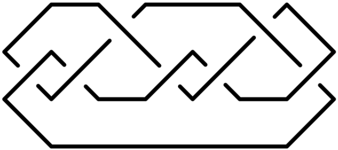
Rational or 2-bridge knots

Every rational knot has an alternating diagram of the form $D[a_1, \ldots, a_n]$ where the ith twist region has $|a_i|$ crossings.

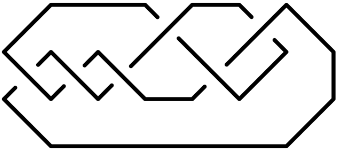


$D[a_1, \ldots, a_n]$ for $n = 2m$ and $n = 2m + 1$

Alternating diagrams of rational knots

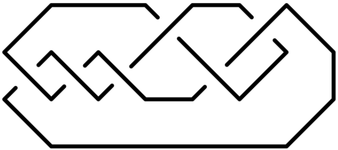


D[2, 1, 2, 1, 1]



D[-3, -1, -1, -1]

Equivalent diagrams of rational knots



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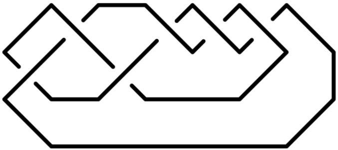
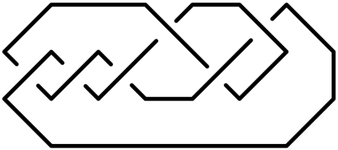


D[-3,-1,-1,-1]

D[1,1,1,3]



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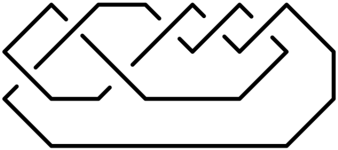
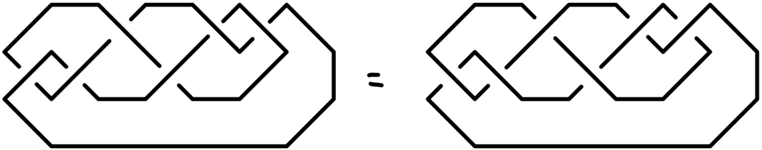


D[3,1,1,1]

D[-1,-1,-1,3]

Palindromic diagrams



D[2,1,1,2] D[-2,-1,-1,-2]

The set R_m

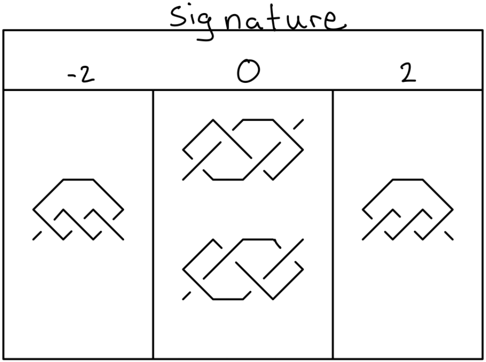
- The set R_m contains alternating diagrams $D[a_1, \dots, a_n]$ of crossing numbers $c = 2m + 1$ and $c = 2m + 2$.
- The set R_m can be combinatorially related to the set R_{m-1}. Let $r_{m,\sigma}$ be the number of diagrams in R_m whose signature is σ. Then

$$r_{m,\sigma} = r_{m-1,\sigma-2} + 2r_{m-1,\sigma} + r_{m-1,\sigma+2}.$$

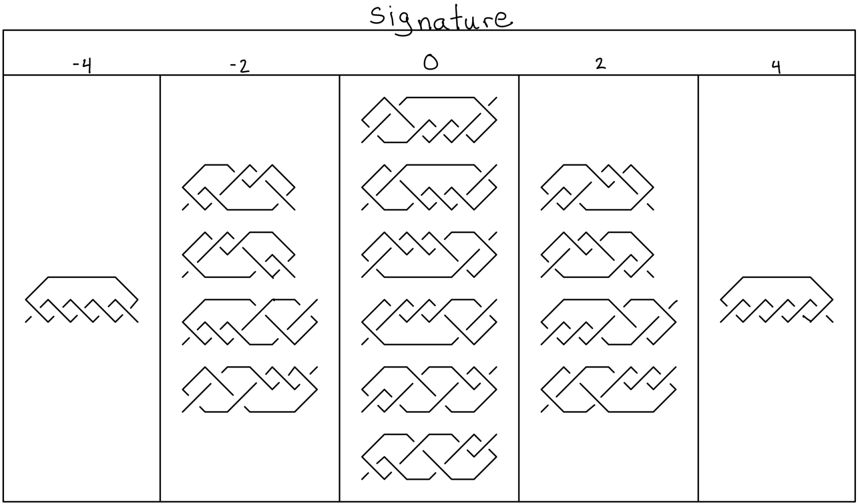
- Consequently,

$$r_{m,\sigma} = \binom{2m}{m + \sigma/2}.$$

The set R_1



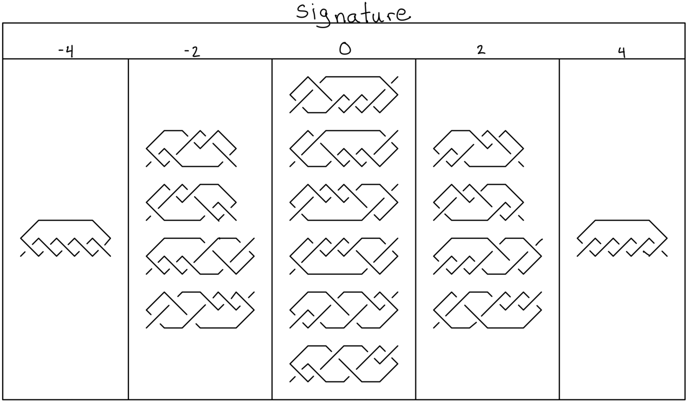
The set R_2



More about R_m

- Most rational knots appear twice in R_m. A small number only appear once.
- The average of the absolute value of signature of a rational knot approaches the average over the set R_m.

Total signature example: R_2



$$\mathrm{Tot}_\sigma(2) = 4 \cdot 1 + 2 \cdot 4 + 6 \cdot 0 + 2 \cdot 4 + 4 \cdot 1 = 24.$$

Total signature

$$
\begin{aligned}
\mathrm{Tot}_\sigma(m) &= \sum_{D \in R(m)} |\sigma(D)| \\
&= \sum_{\sigma} |\sigma| \; r_{m,\sigma} \\
&= \sum_{k=-m}^{m} 2|k| \binom{2m}{m+k} \\
&= 2m\binom{2m}{m} \approx 2^{2m+1}\sqrt{\frac{m}{\pi}}
\end{aligned}
$$

Average signature

$$
\begin{aligned}
\text{Average} &= \frac{\mathrm{Tot}_\sigma(c)}{|R_m|} \\
&\approx \frac{2^{2m+1}\sqrt{\frac{m}{\pi}}}{2^{2m}} \\
&= 2\sqrt{\frac{m}{\pi}}
\end{aligned}
$$

Average signature theorem

Theorem (Cohen, L., Raanes)

The average $\sigma_{avg}(c)$ of the absolute value of the signature of rational knots with c crossings satisfies

$$\lim_{c\to\infty}\left(\sigma_{avg}(c) - \sqrt{\frac{2c}{\pi}}\right) = 0.$$

Average 4-genus

The upper bound is implied by Baader, Kjuchukova, Lewark, Misev, and Ray (2019).

Corollary

The average 4-genus of a 2-bridge knot is sublinear and bounded from below by $\sqrt{\frac{c}{2\pi}}$

Thank you!