Statistics of invariants of 2-bridge knots

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Vassar's Undergraduate Research Summer Institute (URSI)

Pick your favorite numerical knot invariant.

Question: What can we prove about the probability distribution of the knot invariant for **2-bridge** knots with a fixed crossing number?

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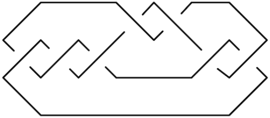
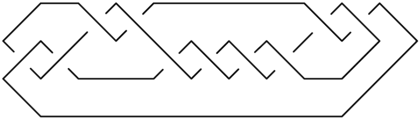
Sometimes, quite a lot!

2-bridge knots

Every 2-bridge knot can be represented by a tuple of nonzero integers $[a_1, a_2, \ldots, a_n]$ where $|a_i|$ is the number of crossings in the corresponding twist region and $\text{sign}(a_i)$ indicates the over/under information for the crossings in that twist region.



[3, 2, 2]



[2, 2, -4, 2]

Why 2-bridge knots?

- 2-bridge knots are completely classified.
- We can prove formulas for the average value, variance, median, mode etc. of some knot invariants on the set of 2-bridge knots with crossing number c.
- In some cases, the formulas above match with experimental data on arbitrary knots.

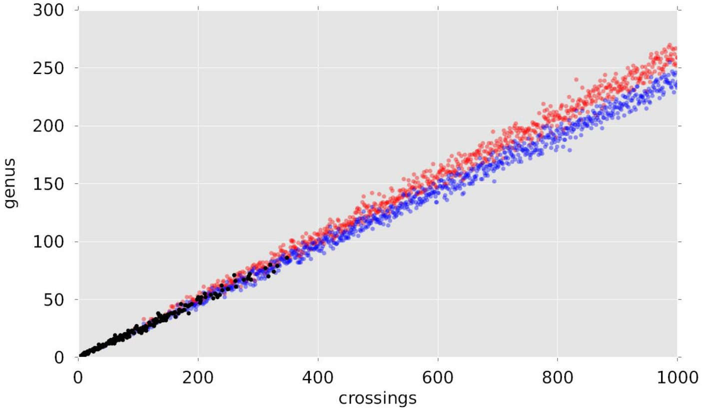
Genus

Theorem (Baader, Kjuchukova, Lewark, Misev, Ray). The average genus of a 2-bridge knot with crossing number c is at least $\frac{c}{4}$. Moreover, the average value of $\frac{g_4(K)}{g(K)}$ where K is a 2-bridge knot with crossing number c approaches zero as $c \to \infty$.

Theorem (Suzuki and Tran, Cohen and L). The average genus of a 2-bridge knot with crossing number c approaches $\frac{c}{4} + \frac{1}{12}$ as $c \to \infty$.

Experimental data

Dunfield et al. gave experimental data suggesting a linear relationship between the crossing number of a knot and its genus.



More results on the genus of 2-bridge knots

Theorem (Cohen, Dinardo, L, Raanes, Rivera, Steindl, Wanebo).

1. The variance of the genera of 2-bridge knots with crossing number c approaches $\frac{c}{16} - \frac{17}{144}$ as $c \to \infty$.
2. The median and mode of the genera of 2-bridge knots with crossing number c are both $\left\lfloor \frac{c+2}{4} \right\rfloor$.
3. The probability distribution of the genera of 2-bridge knots with crossing number c is asymptotically normal as $c \to \infty$.

Results about braid index

Theorem (Clark, Frank, L).

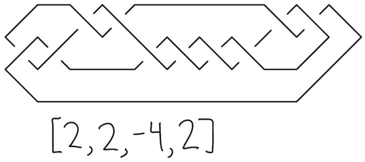
1. The average braid index of 2-bridge knots with crossing number c approaches $\frac{c}{3} + \frac{11}{9}$ as $c \to \infty$.
2. The variance of the braid indices of 2-bridge knots with crossing number c approaches $\frac{2c}{27} - \frac{10}{81}$ as $c \to \infty$.
3. The mode of the braid indices of 2-bridge knots with crossing number c is $b = \left\lceil \frac{c}{3} \right\rceil + 1$.

Representing 2-bridge knots by "even" tuples

Theorem (Murasugi). Every 2-bridge knot has a diagram of the form

$$[2a_1, 2a_2, \ldots, 2a_{2m}]$$

where $a_i \neq 0$ for $1 \leq i \leq 2m$.



The crossing number of a 2-bridge knot

Theorem (Suzuki). The crossing number of the 2-bridge knot with diagram $[2a_1, \ldots, 2a_{2m}]$ is

$$c = \left(\sum_{i=1}^{2m} 2|a_i| \right) - \ell$$

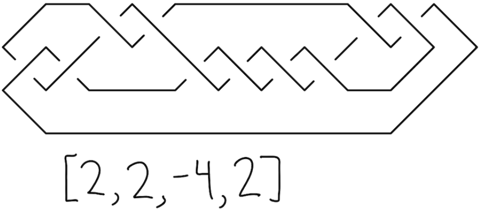
where ℓ is the number of sign changes in the sequence $[2a_1, \ldots, 2a_{2m}]$.

The braid index of a 2-bridge knot

Theorem (Murasugi). The braid index b of a two bridge knot represented by $[2a_1, \ldots, 2a_{2m}]$ is

$$b = \left(\sum_{i=1}^{2m} |a_i| \right) - \ell + 1.$$

Example



The crossing number is

$$c = \left(\sum_{i=1}^{2m} 2|a_i| \right) - \ell = (2+2+4+2) - 2 = 8,$$

and the braid index is

$$b = \left(\sum_{i=1}^{2m} |a_i| \right) - \ell + 1 = (1+1+2+1) - 2 + 1 = 4.$$

The set of even tuples

The 2-bridge knot $[-2a_1, -2a_2, \dots, -2a_{2m}]$ is the mirror image of the 2-bridge knot $[2a_1, 2a_2, \dots, 2a_{2m}]$. For this talk, we won't distinguish between a knot and its mirror image. Define

$$E(c) = \left\{ [2a_1, \dots, 2a_{2m}] \ : \ a_1 > 0 \text{ and } c = \left(\sum_{i=1}^{2m} 2|a_i| \right) - \ell \right\}.$$

Every 2-bridge knot of crossing number c appears in the set $E(c)$ exactly once (when $[2a_1, \dots, 2a_{2m}] = [2a_{2m}, \dots, 2a_1]$) or exactly twice (otherwise).

Towards computing average braid index

The average braid index of 2-bridge knots of crossing number c is

$$\text{average} = \frac{\text{sum of the braid indices of 2-bridge knots with } c \text{ crossings}}{\text{number of 2-bridge knots with } c \text{ crossings}}$$
$$\approx \frac{\text{sum of braid indices of } [2a_1, \ldots, 2a_{2m}] \in E(c)}{|E(c)|}.$$

Define the total braid index $tbi(c)$ to be the sum of the braid indices of diagrams in $E(c)$.

$E(c, b)$

Define

$$E(c,b) = \left\{ [2a_1, \ldots, 2a_{2m}] \in E(c) \ : \ b = \left(\sum_{i=1}^{2m} |a_i| \right) - \ell + 1 \right\}$$

to be the subset of $E(c)$ consisting of tuples corresponding to knots of braid index b.

Define $e(c, b) = |E(c, b)|$ to be the number of tuples in $E(c, b)$.

A recursive formula for $e(c, b)$

The number $e(c, b)$ of even tuples corresponding to 2-bridge knots with crossing number c and braid index b satisfies

$$e(c, b) = 2e(c - 3, b - 1) + 2e(c - 2, b - 1) + e(c - 2, b).$$

Idea of proof

- Take a tuple $[2a_1, \ldots, 2a_{2m}]$ in $E(c, b)$ and modify it to obtain a tuple in $E(c')$ where $c' < c$.
- For example, if $a_{2m} > 1$, replace $[2a_1, \ldots, 2a_{2m}]$ with $[2a_1, \ldots, 2a_{2m} - 2]$.
- This replacement (and the analogous one when $a_{2m} < -1$) shows that the subset

  $$\{[2a_1, \ldots, 2a_{2m}] \in E(c, b) \ : \ |a_{2m}| > 1\}$$

  of $E(c, b)$ is in bijection with $E(c - 2, b - 1)$.
- Several more cases yield the result.

$tbi(c)$ and the average

The total braid index is

$$tbi(c) = \sum_{b \geq 2} b\, e(c, b),$$

and satisfies the recursive formula

$$tbi(c) = 3tbi(c-2) + 2tbi(c-3) + 2^{c-4}.$$

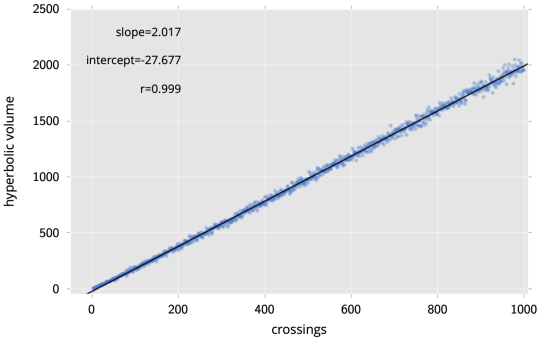
An exact formula for $tbi(c)$ follows from the recursive formula and yields the average braid index.

In progress results about determinant - joint with Raanes

1. The average determinant of 2-bridge knots with crossing number c is slightly more than 1.5^c.
2. For any odd prime p, a 2-bridge knot is p-colorable with probability $\frac{1}{p+1}$.

Related future work

1. Experimental data from Dunfield suggests the possibility that hyperbolic volume is asymptotically linear with respect to crossing number. Can we prove it for 2-bridge knots?



2. Statistics of other invariants for 2-bridge knots?

Thank you!