The genus distribution of 2-bridge knots

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AMS Spring Southeastern Sectional Meeting
Joint work with
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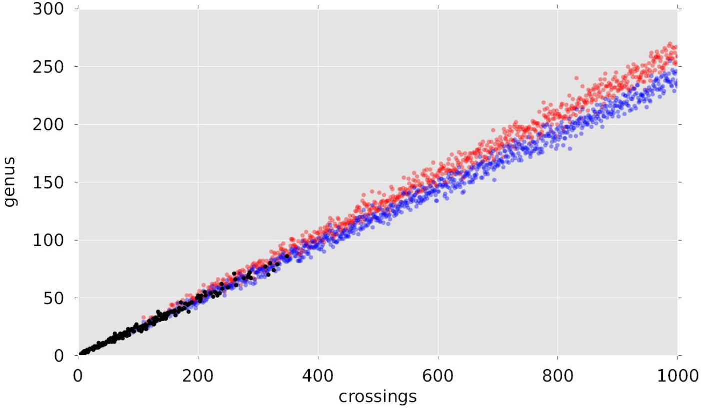
Answer: I don't know.

Some data

Dunfield et al. gave experimental data suggesting a linear relationship between the crossing number of a knot and its genus.



Make the question easier

Question: What is the expected value of the genus of a 2-bridge knot with crossing number c selected uniformly at random from the set of 2-bridge knots with crossing number c?

Theorem (Cohen, L. and Suzuki, Tran - 2022)

The expected value of the genus of a 2-bridge knot with crossing number c approaches $\frac{c}{4} + \frac{1}{12}$ *as* $c \to \infty$.

Vassar's Undergraduate Research Summer Institute (URSI)

URSI Question

URSI 2022 Question: What else can we say about the distribution of the genera of 2-bridge knots of crossing number c?

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Answer: Quite a lot!

Median, mode, and variance

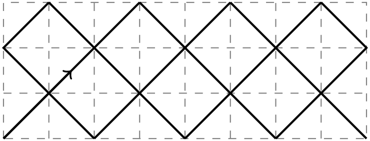
Theorem (Cohen, Dinardo, L., Raanes, Rivera, Steindl, Wanebo)

1. *The median and mode of the genera of 2-bridge knots with crossing number c is* $\left\lfloor \frac{c+2}{4} \right\rfloor$.
2. *The variance of the genera of 2-bridge knots with crossing number c approaches* $\frac{c}{16} - \frac{17}{144}$ *as* $c \to \infty$.

Theorem (L., Raanes)

The distribution of the genera of 2-bridge knots with crossing number c approaches a normal distribution as $c \to \infty$.

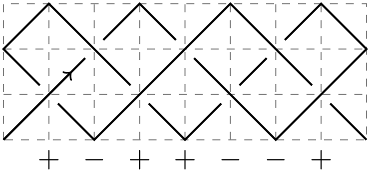
Billiard table knot diagram



Take a $3 \times b$ grid where $3 \nmid b$ and draw a piecewise-linear arc that starts by bisecting the bottom left corner and reflects when it hits an edge of the rectangle.

Each interior vertical grid line has exactly one double point.

Billiard table knot diagram



Choose crossing information at each double point. A crossing corresponding to a + looks like  , and a crossing corresponding to a − looks like  .

Connect the endpoints of the piecewise-linear segment outside of the rectangle.

A list of 2-bridge knots

Let $T(c)$ be the set of words in the symbols $\{+,-\}$ defined as follows. When c is odd, $T(c)$ consists of words of the form

$$+(-)^{\varepsilon_2}(+)^{\varepsilon_3}(-)^{\varepsilon_4}\ldots(-)^{\varepsilon_{c-1}}+,$$

where $\varepsilon_i \in \{1,2\}$ for $2 \leq i \leq c-1$, and the length of the word $\sum_{i=1}^{c} \varepsilon_i \equiv 1 \bmod 3$. When c is even, $T(c)$ consists of words of the form

$$+(-)^{\varepsilon_2}(+)^{\varepsilon_3}(-)^{\varepsilon_4}\ldots(+)^{\varepsilon_{c-1}}-,$$

where $\varepsilon_i \in \{1,2\}$ for $2 \leq i \leq c-1$ and the length of the word $\sum_{i=1}^{c} \varepsilon_i \equiv 1 \bmod 3$.

Examples of $T(c)$

$$
\begin{aligned}
T(3) &= \{+--+\} \\
T(4) &= \{+-+-\} \\
T(5) &= \{+--++-+, +-++--+, +--+--+\} \\
T(6) &= \{+-+-++-, +-+--+-, \\
&\qquad +--++--++-, +-++-+-, +--+-+-\}
\end{aligned}
$$

A list of 2-bridge knots

Using Schubert's 1956 classification of 2-bridge knots and work of Koseleff and Pecker (2011), Cohen, Even-Zohar, and Krishnan (2018) showed the following.

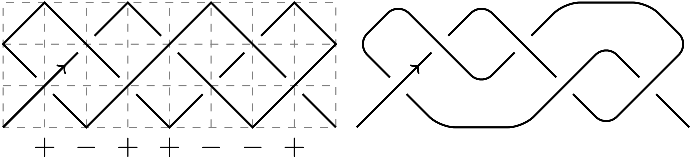
Theorem (Cohen, Even-Zohar, Kirshnan)

Every 2-bridge knot of crossing number c corresponds to one or two words in $T(c)$.

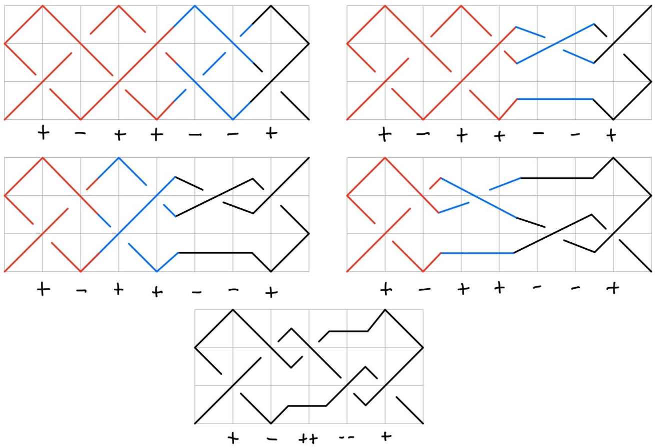
The 2-bridge knots that correspond to one word in $T(c)$ are rare compared to those represented by two words. For this talk, we pretend every 2-bridge knot of crossing number c is represented by two words in $T(c)$.

Alternating diagrams

Every billiard table diagram coming from a word in $T(c)$ has an equivalent alternating diagram obtained by replacing $+$ and $--$ with  on the first row and by replacing $-$ and $++$ with  on the second row.



Billiard table to alternating diagram



The genus of an alternating knot

Murasugi (1958) and Crowell (1959) proved that the genus of an alternating knot K is the genus of the surface obtained from Seifert's algorithm applied to an alternating diagram D of K. Thus

$$g(K) = \frac{1}{2}(1 + c - s)$$

where D has c crossings and s Seifert circles.

Average genus of a 2-bridge knot

Theorem (Cohen, L. and Suzuki, Tran - 2022)

The expected value of the genus of a 2-bridge knot with crossing number c approaches $\frac{c}{4} + \frac{1}{12}$ as $c \to \infty$.

Sketch of the proof

The set $T(c)$ lists 2-bridge knots with crossing number c twice.*

The set $T(c)$ can be recursively built from two copies of $T(c-2)$ and one copy of $T(c-1)$.

$T(5)$	$+-\ -++-()+$	$+-\ -++-(+--)+$	
	$+-++--()+$	$+-++--(+--)+$	
	$+-\ -+-\ -()+$	$+-\ -+-\ -(+--)+$	
$T(5)$	$+-\ -++-()+$	$+-\ -++-(++-)+$	
	$+-++--()+$	$+-++--(++-)+$	
	$+-\ -+-\ -()+$	$+-\ -+-\ -(++-)+$	$T(7)$
$T(6)$	$+-+-\ -(+)-$	$+-+-\ -(++--)+$	
	$+-++-(+)-$	$+-++-(++--)+$	
	$+-\ -+-(+)-$	$+-\ -+-(++--)+$	
	$+-+-(++)-$	$+-+-(+-)+$	
	$+-\ -++--(++)-$	$+-\ -++--(+-)+$	

Sketch of the proof, continued

Compare the alternating diagrams when constructing $T(c)$ from two copies of $T(c-2)$ and one copy of $T(c-1)$. The difference in the number of Seifert circles for each replacement can be determined. This yields a recursive formula for the sum of the genera of knots coming from $T(c)$. Call this sum $g_{\text{total}}(c)$.

Use the recursive formula for $g_{\text{total}}(c)$ to obtain a closed form.

Average genus is the quotient of $g_{\text{total}}(c)$ and the number of words in $T(c)$.

Median, mode, and variance

A similar approach yields an exact formula for the number of 2-bridge knots with crossing number c and genus g:

$$\frac{1}{2}\left((-1)^{c'-g-1}\sum_{n=0}^{c'-g-1}(-1)^n\binom{n+g-1}{n}+\right.$$
$$\left.(-1)^{c-1}\sum_{i=0}^{c-2g-1}(-1)^i\binom{i+2g-1}{i}\right),$$

where $c' = \left\lfloor \frac{c+1}{2} \right\rfloor$ and $1 \le g \le \left\lfloor \frac{c-1}{2} \right\rfloor$.

This formula and other recursive techniques lead to the computation of the median, mode, and variance.

Median, mode, and variance

Theorem (Cohen, Dinardo, L., Raanes, Rivera, Steindl, Wanebo)

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2. *The variance of the genera of 2-bridge knots with crossing number c approaches $\frac{c}{16} - \frac{17}{144}$ as $c \to \infty$.*

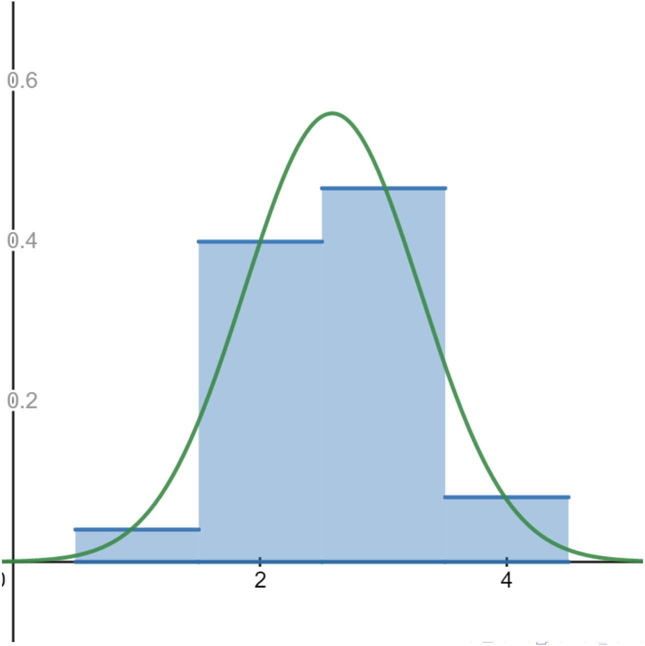
Theorem (L., Raanes)

The distribution of the genera of 2-bridge knots with crossing number c approaches a normal distribution as $c \to \infty$.

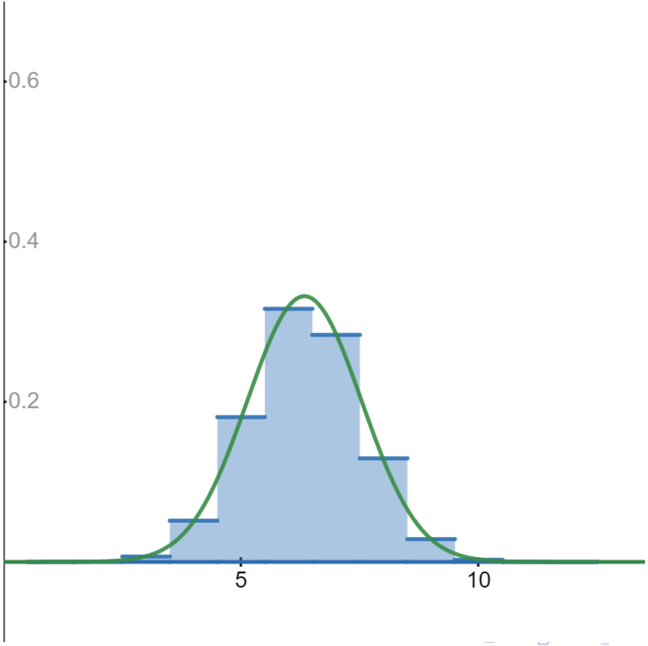
The number of 2-bridge knots of genus g with crossing number c

$c \backslash g$	1	2	3	4	5	6	7	8	9
3	1								
4	1								
5	1	1							
6	1	2							
7	2	4	1						
8	2	7	3						
9	2	12	9	1					
10	2	18	21	4					
11	3	26	45	16	1				
12	3	36	85	47	5				
13	3	49	151	123	25	1			
14	3	64	251	280	89	6			
15	4	82	400	588	276	36	1		
16	4	103	610	1141	736	151	7		
17	4	128	904	2094	1784	542	49	1	
18	4	156	1294	3648	3960	1658	237	8	
19	5	188	1814	6104	8230	4558	967	64	1
20	5	224	2486	9842	16126	11394	3339	351	9

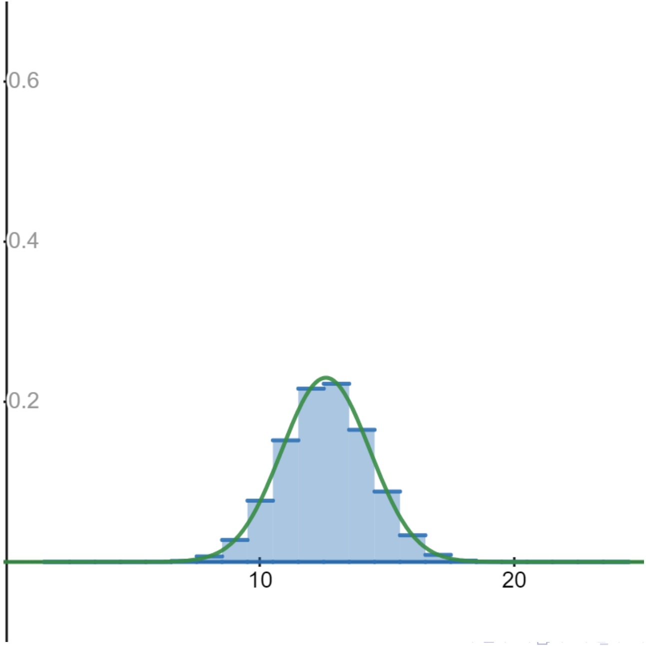
Approaching a normal distribution: $c = 10$



Approaching a normal distribution: $c = 25$



Approaching a normal distribution: $c = 50$



Future directions

- Study other invariants for 2-bridge knots using similar techniques.
  - (L., Raanes - in progress) A 2-bridge knot is p-colorable with probability $\frac{1}{p+1}$.
- What does the genus distribution look like for alternating knots?