An introduction to Khovanov homology

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Hudson River Undergraduate Mathematics Conference

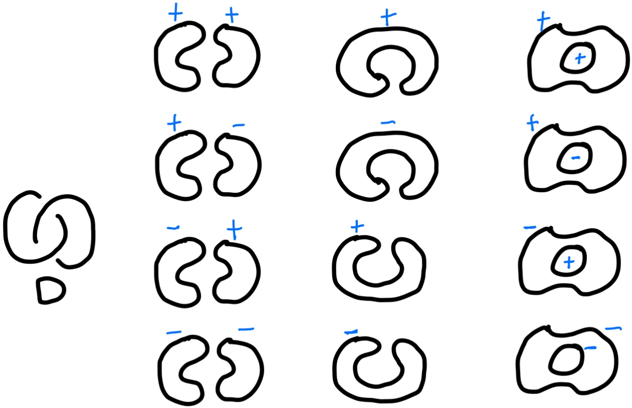
April 1, 2023

Kauffman states

A *Kauffman state* of a link diagram D is the collection of curves obtained by performing an A-resolution or a B-resolution at each crossing of D.

A *signed Kauffman state* S is a Kauffman state where each component has been labeled by "$+$" or "$-$".

Kauffman state example

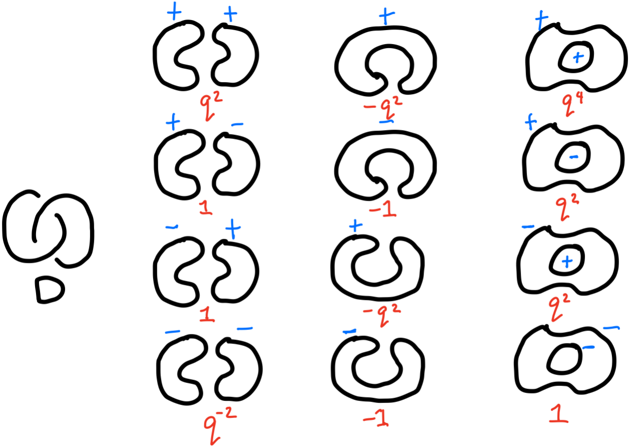


Monomials associated to Kauffman states

If S is a signed Kauffman state of D, define

$$\begin{aligned} i(S) &= \#(B\text{-resolutions}), \\ j(S) &= \#(B\text{-resolutions}) + \#(+\text{ signs}) - \#(-\text{ signs}), \text{ and} \\ \langle D|S\rangle &= (-1)^{i(S)} q^{j(S)}. \end{aligned}$$

Kauffman state example

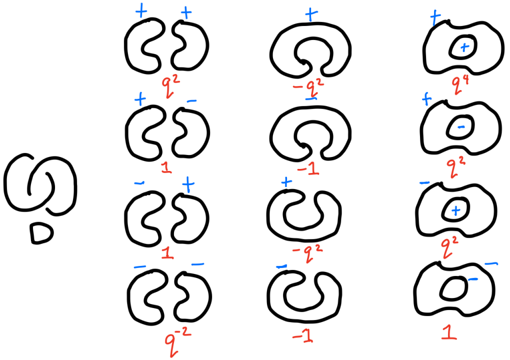


Jones polynomial

Let L be a link with diagram D. The Jones polynomial $V_L(q)$ is defined as

$$\begin{aligned} V_L(q) &= (-1)^{c_-} q^{c_+ - 2c_-} \sum_S \langle D|S\rangle \\ &= (-1)^{c_-} q^{c_+ - 2c_-} \sum_S (-1)^{i(S)} q^{j(S)}. \end{aligned}$$

Jones polynomial example



$$V_L(q) = q^2(q^{-2} + 1 + q^2 + q^4) = 1 + q^2 + q^4 + q^6$$

The Jones polynomial and crossing number

Theorem (Murasugi, Kauffman, Thistlethwaite).

1. The span of the Jones polynomial is at most $2c(L) + 2$ where $c(L)$ is the minimum crossing number of the link L.

2. The span of the Jones polynomial is exactly $2c(L) + 2$ if and only if the link L is non-split and alternating.

3. Any "reduced" alternating diagram of a link realizes the minimum crossing number of the link.

The Jones polynomial and unknot detection

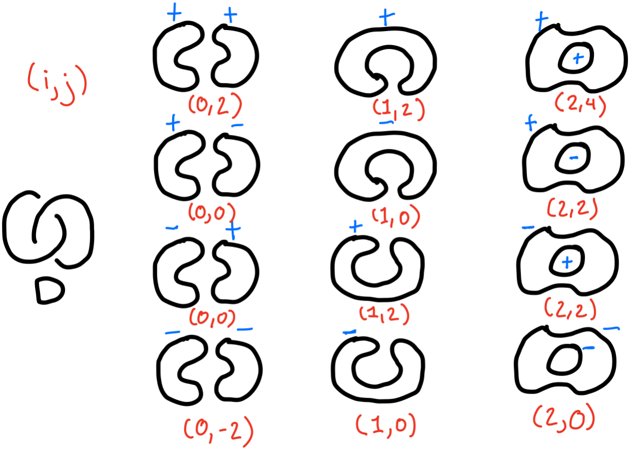
Conjecture. Let U be the unknot. If K is a knot such that $V_K(q) = V_U(q) = q + q^{-1}$, then K is the unknot.

Note. The conjecture holds for alternating knots by the previous slide.

Khovanov homology

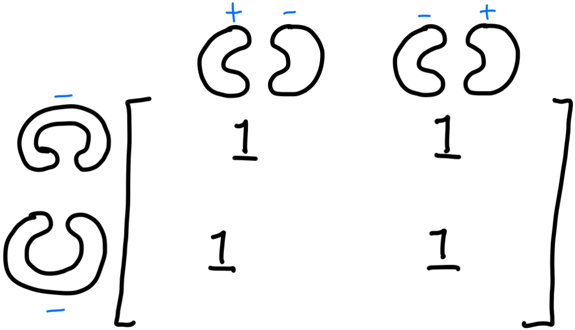
Big idea. Start with signed Kauffman states. Instead of associating a monomial, define a vector space $C^{a,b}$ with basis the signed Kauffman states S such that $i(S) = a$ and $j(S) = b$. Instead of adding together monomials, define linear maps between the vector spaces and take homology.

Khovanov example



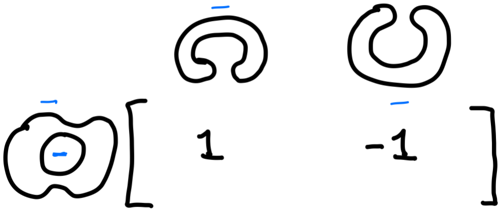
Maps $d^{a,b} : C^{a,b} \to C^{a+1,b}$ via example

$$d^{0,0} : C^{0,0} \to C^{1,0}$$



Maps $d^{a,b} : C^{a,b} \to C^{a+1,b}$ via example

$$d^{1,0} : C^{1,0} \to C^{2,0}$$



Khovanov homology

The Khovanov homology $Kh_L(h, q)$ is the two variable polynomial

$$Kh_L(h,q) = h^{-c_-} q^{c_+ - 2c_-} \sum_{a,b \in \mathbb{Z}} n_{a,b} h^a q^b$$

where $n_{a,b} = \operatorname{nullity} d^{a,b} - \operatorname{rank} d^{a-1,b}$.

Example: The Hopf link

If L is the Hopf link, then

$$Kh_L(h, q) = 1 + q^2 + h^2 q^4 + h^2 q^6.$$

Observe

$$\begin{aligned} Kh_L(-1, q) &= 1 + q^2 + (-1)^2 q^4 + (-1)^2 q^6 \\ &= 1 + q^2 + q^4 + q^6 \\ &= V_L(q) \end{aligned}$$

Advantages of Khovanov homology

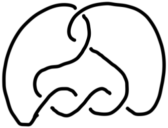
Let L be a link.

1. $Kh_L(-1, q) = V_L(q)$.
2. $2c(L) + 2 \geq \operatorname{span}_q Kh_L(h, q) \geq \operatorname{span} V_L(q)$.
3. (Kronheimer and Mrowka) Let U be the unknot. $Kh_U(h, q) = Kh_L(h, q)$ if and only if $U = L$.

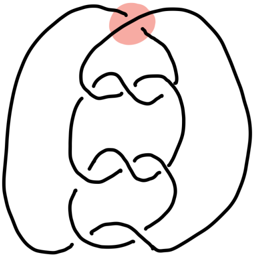
Alternating and almost-alternating links

A link diagram is alternating if the crossings alternate between over and under. A knot diagram is almost-alternating if one crossing change makes the diagram alternating.

A link is alternating if it has an alternating diagram and is almost-alternating if it is not alternating and has an almost alternating diagram.



Alternating



Almost-alternating

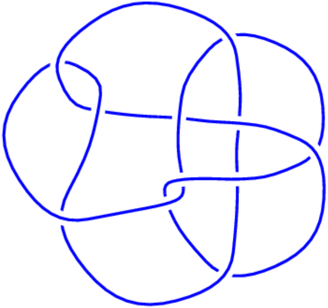
$V_L(q)$ and $Kh_L(h,q)$ for almost-alternating

Theorem (Dasbach, L). If L is almost-alternating, then the highest or lowest coefficient of $V_L(q)$ has absolute value 1.

Theorem (L, Spyropoulos). If L is almost alternating, then $V_L(q) \neq V_U(q)$.

Theorem (Dasbach, L). If L is almost-alternating, then the highest or lowest q-degree in $Kh_L(h,q)$ looks like $h^a q^b$.

$11n_{95}$



$$V_{11n_{95}}(q) = 2q^3 - q^5 + 2q^7 - q^9 + q^{13} - q^{15} + 2q^{17} - 2q^{19}$$