Near extremal Khovanov homology of Turaev genus one links

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Motivating question

Links with Turaev genus one include almost-alternating links and non-alternating pretzel and Montesinos links. Such links are "close" to alternating.

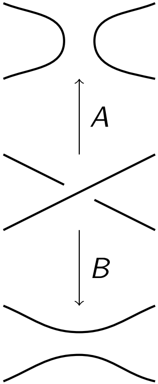
Question. What can we say about the Jones polynomial and Khovanov homology of a Turaev genus one link?

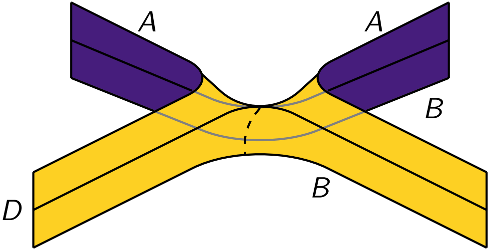
The Turaev surface

The *Turaev surface* of a link diagram D is obtained by

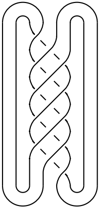
1. constructing a cobordism between the all-A and all-B Kauffman states of D that has saddles corresponding with crossings, and
2. capping off the boundary components of the above cobordism with disks.

The Turaev surface at a crossing



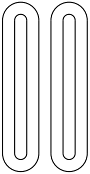


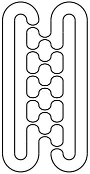
The Turaev surface



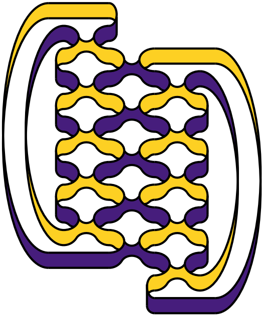
D

all-*A* state





all-*B* state



Turaev surface of *D*

The Turaev genus of a link

- For a connected link diagram D, the genus of the Turaev surface is

$$g_T(D) = \frac{1}{2}\left(2 + c(D) - s_A(D) - s_B(D)\right)$$

  where $c(D)$ is the number of crossings in D and $s_A(D)$ and $s_B(D)$ are the number of components in the all-A and all-B Kauffman states of D respectively.
- The *Turaev genus* $g_T(L)$ of a non-split link L is

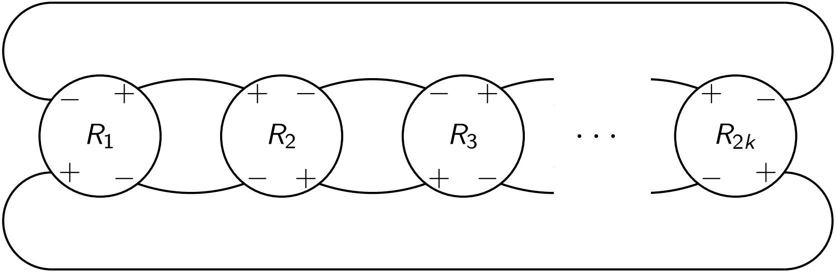
$$g_T(L) = \min\{g_T(D) \mid D \text{ is a diagram of } L\}.$$

- The Turaev genus of a non-split link is zero if and only if the link is alternating.

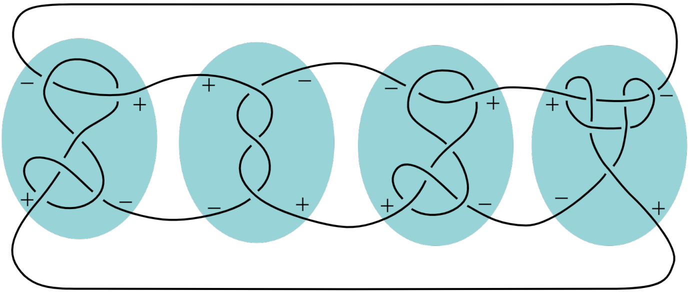
Turaev genus one link

Theorem (Armond, L.; Kim)

Every non-split Turaev genus one link has a diagram as depicted below where each R_i is an alternating tangle and $+$ or $-$ indicates that the first crossing that strand meets is an over or under crossing respectively.

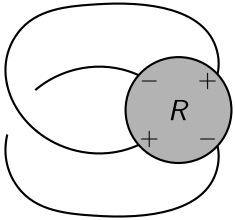


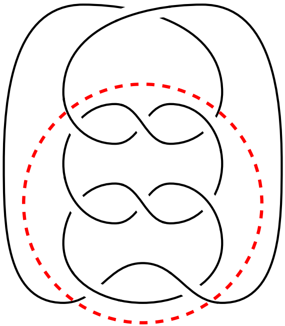
Example of a Turaev genus one knot



Almost-alternating link

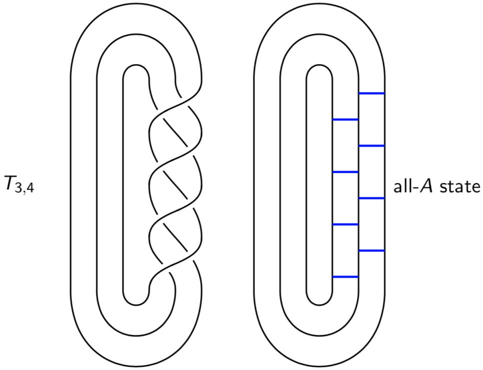
A non-alternating link is *almost-alternating* if it has a diagram that can be transformed into an alternating diagram via one crossing change (Adams et al.).





Adequacy

A link diagram is *A-adequate* if no two arcs in the resolution of the same crossing lie on the same component of the of the all-A Kauffman state. A link is *A-adequate* if it has an A-adequate diagram. Similarly define B-adequate.



Almost-alternating, semi-adequacy, mutation

The Turaev genus of an almost-alternating link is one.

Theorem (Armond, L.)

There is a sequence of mutations transforming every Turaev genus one link into an almost-alternating link.

Theorem (Kim)

Every Turaev genus one link is A-adequate, B-adequate, or almost-alternating.

The Jones polynomial of an alternating link

Theorem (Kauffman, Thistlethwaite)

Let L be a non-split alternating link with Jones polynomial

$$V_L(t) = a_m t^m + a_{m+1} t^{m+1} + \cdots + a_{n-1} t^{n-1} + a_n t^n$$

where a_m and a_n are nonzero. Then

- $|a_m| = |a_n| = 1$, *and*
- $a_i a_{i+1} \leq 0$ *for* $i = m, \ldots, n-1$.

The Jones polynomial of an adequate link

Theorem (Lickorish, Thistlethwaite, Stoimenow)

Let L be a non-split semi-adequate link with Jones polynomial

$$V_L(t) = a_m t^m + a_{m+1} t^{m+1} + \cdots + a_{n-1} t^{n-1} + a_n t^n$$

where a_m and a_n are nonzero. If L is A-adequate, then

$$|a_m| = 1 \text{ and } a_m a_{m+1} \leq 0.$$

If L is B-adequate, then

$$|a_n| = 1 \text{ and } a_n a_{n-1} \leq 0.$$

The Jones polynomial of a Turaev genus one link

Theorem (Dasbach, L., Spyropoulos)

Let L be a non-split link of Turaev genus one with Jones polynomial

$$V_L(t) = a_m t^m + a_{m+1} t^{m+1} + \cdots + a_{n-1} t^{n-1} + a_n t^n$$

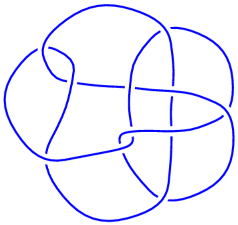
where a_m and a_n are nonzero. Then

$$|a_m| = 1 \text{ and } a_m a_{m+1} \leq 0,$$

or

$$|a_n| = 1 \text{ and } a_{n-1} a_n \leq 0.$$

Example $11n_{95}$

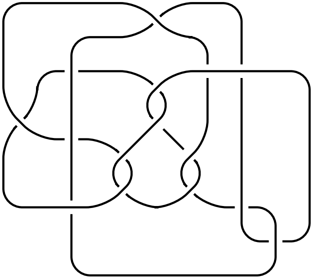


The Jones polynomial of $11n_{95}$ is

$$V_{11n_{95}}(t) = 2t^2 - 3t^3 + 5t^4 - 6t^5 + 6t^6 - 5t^7 + 4t^8 - 2t^9.$$

Since $11n_{95}$ has a diagram of Turaev genus two, it follows that $g_T(11n_{95}) = 2$.

Example $15n_{41133}$



The Jones polynomial of $15n_{41133}$ is

$$t^4 + t^5 - 3t^6 + 8t^7 - 12t^8 + 14t^9 - 15t^{10} + 13t^{11} - 10t^{12} + 6t^{13} - 2t^{14}.$$

Thus $g_T(15n_{41133}) \geq 2$.

Khovanov homology

Khovanov homology is a $\mathbb{Z}$-module equipped with two gradings i and j such that

$$\sum_{i,j} (-1)^i \operatorname{rank} Kh^{i,j}(L)\, t^j = (t + t^{-1}) V_L(t^2).$$

Example: the $(3,4)$-torus knot

$j \backslash i$	0	1	2	3	4	5
17						1
15						1
13				1	1	
11				1_2	1	
9			1			
7	1					
5	1					

$$\sum_{i,j} (-1)^i \operatorname{rank} Kh^{i,j}(T_{3,4})\, t^j = -\, t^{17} - t^{15} + t^{11} + t^9 + t^7 + t^5$$

$$V_{T_{3,4}}(t) = -\, t^8 + t^5 + t^3$$

Maximum and minimum gradings

Define

$$\begin{aligned} i_{\min}(L) &= \min\{i \mid Kh^{i,j}(L) \neq 0\}, \\ i_{\max}(L) &= \max\{i \mid Kh^{i,j}(L) \neq 0\}, \\ j_{\min}(L) &= \min\{j \mid Kh^{i,j}(L) \neq 0\}, \\ j_{\max}(L) &= \max\{j \mid Kh^{i,j}(L) \neq 0\}, \\ \delta_{\min}(L) &= \min\{2i-j \mid Kh^{i,j}(L) \neq 0\}, \text{ and} \\ \delta_{\max}(L) &= \max\{2i-j \mid Kh^{i,j}(L) \neq 0\}. \end{aligned}$$

Theorem (Champanerkar, Kofman, Stoltzfus)

If L is non-split and has Turaev genus one, then

$$2 \leq \delta_{\max}(L) - \delta_{\min}(L) \leq 4.$$

Extremal Khovanov homology

- The Khovanov homology of a non-split Turaev genus one link in either its maximal or minimal polynomial grading is isomorphic to $\mathbb{Z}$, and this $\mathbb{Z}$ summand must be on a specific diagonal.
- A certain summand of the Khovanov homology of a non-split Turaev genus one link in its near maximal or near minimal polynomial grading is trivial.

Extremal Khovanov homology

Theorem (Beldon, Dasbach, DeStefano, L., Milgrim, Villaseñor)

Let L be a non-split link with Turaev genus one. Either

1. $Kh^{*,j_{\min}(L)}(L) \cong Kh^{i_{\min}(L),j_{\min}(L)}(L) \cong \mathbb{Z}$,
2. $2i_{\min}(L) - j_{\min}(L) = \delta_{\min}(L) + 2$, *and*
3. $Kh^{i_{\min}(L)+2,j_{\min}(L)+2}(L)$ *is trivial.*

or

1. $Kh^{*,j_{\max}(L)}(L) \cong Kh^{i_{\max}(L),j_{\max}(L)}(L) \cong \mathbb{Z}$,
2. $2i_{\max}(L) - j_{\max}(L) = \delta_{\max}(L) - 2$, *and*
3. $Kh^{i_{\max}(L)-2,j_{\max}(L)-2}(L)$ *is trivial.*

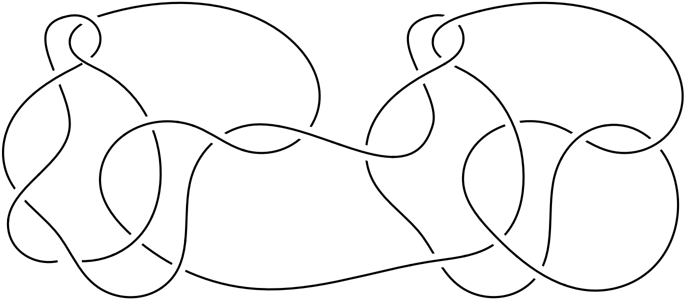
Example: the $(3, 4)$-torus knot

$j \backslash i$	0	1	2	3	4	5
17						1
15						1
13				1	1	
11				1_2	1	
9			1			
7	1					
5	1					

Example: the $(3, 4)$-torus knot

$j \backslash i$	0	1	2	3	4	5
17						1
15						1
13				1	1	
11				1_2	1	
9			1			
7	1					
5	1					

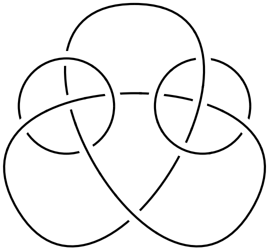
Example: $10_{132} \# \overline{10_{132}}$



Example: $Kh(10_{132}\#\overline{10_{132}})$

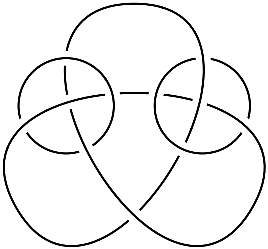
$j\backslash i$	-7	-6	-5	-4	-3	-2	-1	0	1	2	3	4	5	6	7
13														1	1
11													1	1_2	1_2
9												2	$3,1_2$	1	
7										1	5	$3,2_2$	2_2		
5									1	$3,1_2$	$3,5_2$	$2,2_2$			
3								1	$7,1_2$	$8,3_2$	$2,1_2$				
1							3	$8,1_2$	$4,6_2$	$1,3_2$					
-1						1	$4,3_2$	$8,6_2$	$3,1_2$						
-3					2	$8,1_2$	$7,3_2$	$1,1_2$							
-5				2	$3,2_2$	$3,5_2$	$1,1_2$								
-7				$3,2_2$	$5,2_2$	1									
-9		1	3	$2,1_2$											
-11		1_2	$1,1_2$												
-13	1	1													

Example: $11n_{376}$



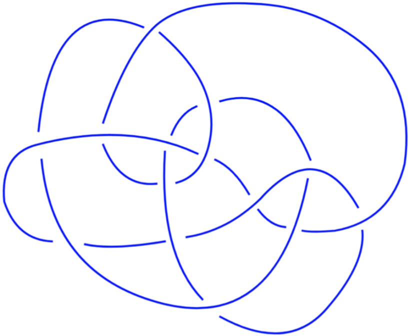
$j\backslash i$	-7	-6	-5	-4	-3	-2	-1	0
-3								2
-5							1	2
-7						2	1_2	
-9				1	1	$1,2_2$		
-11				4	$2,1_2$			
-13				$2,2_2$				
-15		1	2					
-17		1_2						
-19	1							

Example: $11n_{376}$



$j\backslash i$	-7	-6	-5	-4	-3	-2	-1	0
-3								2
-5							1	2
-7						2	1_2	
-9				1	1	$1, 2_2$		
-11				4	$2, 1_2$			
-13				$2, 2_2$				
-15		1	2					
-17		1_2						
-19	1							

Example: $14n_{21152}$



$j\backslash i$	-3	-2	-1	0	1	2	3	4	5	6	7
15											1
13									1		1_2
11								1	$1,1_2$	1	
9							3	$2,1_2$	1_2		
7						2	$1,3_2$	$1,1_2$			
5					3	$4,2_2$	1				
3				3	$2,3_2$	1_2					
1			1	$4,2_2$	1						
-1		2	$2,1_2$								
-3		$1,2_2$									
-5	2										

Proof ingredients

- Khovanov homology techniques: long exact sequence, direct computation, etc.
- Diagram specific results for Turaev genus one links, especially how they relate to almost-alternating and semi-adequate links.

Application: Rasmussen's invariant

Let $s(K)$ be the Rasmussen invariant of the knot K, and let $c_-(D)$ be the number of negative crossings in D.

Theorem (Beldon, DeStefano, L., Milgrim, Villaseñor)

Let D be a diagram of the knot K. If D is A-adequate, has Turaev genus one, and $c_-(D) = 2$, then

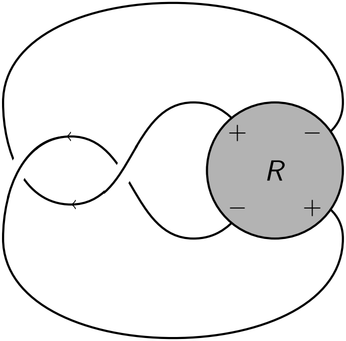
$$s(K) = c(D) - s_A(D) - 1.$$

Proof

Suppose D is A-adequate with two negative crossings and has Turaev genus one.

$j \backslash i$	-2	-1	0
$j_{\min}+6$			?
$j_{\min}+4$			?
$j_{\min}+2$			
$j_{\min}$	1		

Example



If R is a positive alternating tangle and D is A-adequate, then

$$s(K) = \frac{1}{2}(c(D) - s_A(D) - 1).$$

Thank you!