The Jones polynomial and Khovanov homology of Turaev genus one links

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Outline

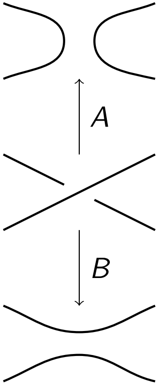
- What are Turaev genus one links?
- How are Turaev genus one links related to some familiar families of links?
- What can we say about the Jones polynomial and Khovanov homology of a Turaev genus one link?

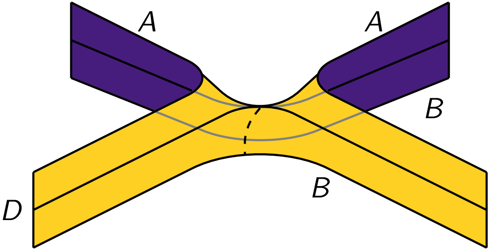
The Turaev surface

The *Turaev surface* of a link diagram D is obtained by

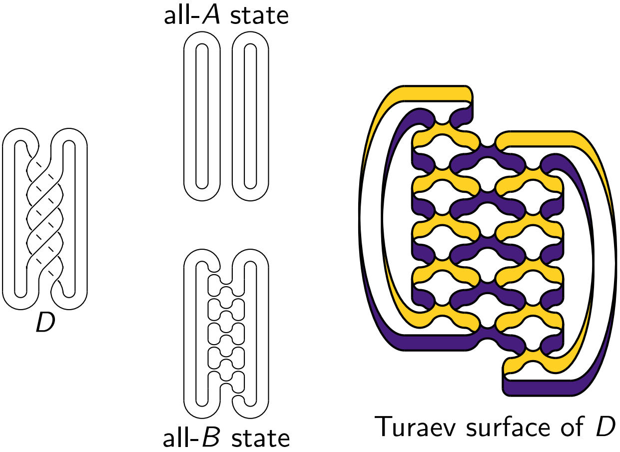
1. constructing a cobordism between the all-A and all-B Kauffman states of D that has saddles corresponding with crossings, and
2. capping off the boundary components of the above cobordism with disks.

The Turaev surface at a crossing





The Turaev surface



The Turaev genus of a link

- For a connected link diagram D, the genus of the Turaev surface is

$$g_T(D) = \frac{1}{2}\left(2 + c(D) - s_A(D) - s_B(D)\right)$$

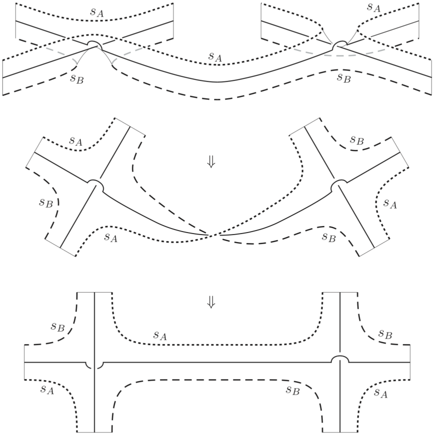
where $c(D)$ is the number of crossings in D and $s_A(D)$ and $s_B(D)$ are the number of components in the all-A and all-B Kauffman states of D respectively.

- The *Turaev genus* $g_T(L)$ of a link L is

$$g_T(L) = \min\{g_T(D) \mid D \text{ is a diagram of } L\}.$$

Alternating projection

A link L has an alternating projection on any of its Turaev surfaces.



The Turaev surface of an alternating diagram

Theorem (Turaev)

The Turaev genus of a link is zero if and only if the link is alternating.

Idea of proof: If the link has a genus zero Turaev surface, then it has an alternating projection to a sphere (i.e. it is alternating). If a link has an alternating diagram, then the components of the all-A and all-B states correspond to the complementary regions of the diagram. Thus the diagram has Turaev genus zero.

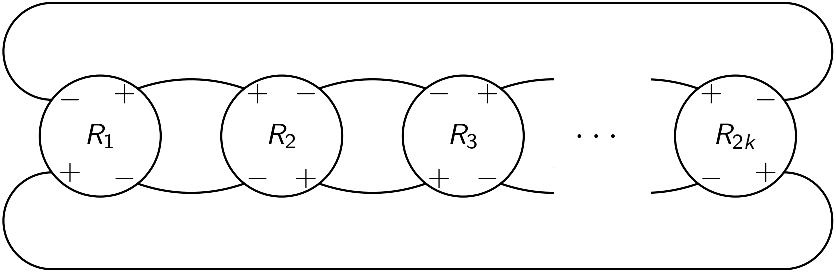
Turaev genus as an alternating distance

The Turaev genus of a link gives a filtration on all links starting with alternating links and becoming "more non-alternating" as the Turaev genus increases.

Turaev genus one links

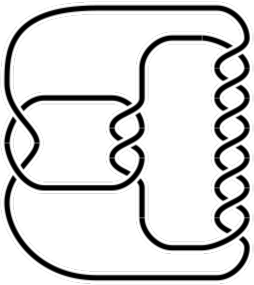
Theorem (Armond, L.; Kim)

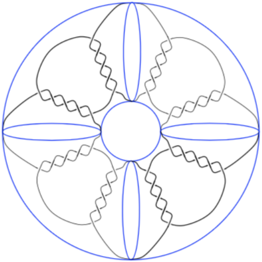
Every non-split Turaev genus one link has a diagram as depicted below where each R_i is an alternating tangle and $+$ or $-$ indicates that the first crossing that strand meets is an over or under crossing respectively.



Pretzel links

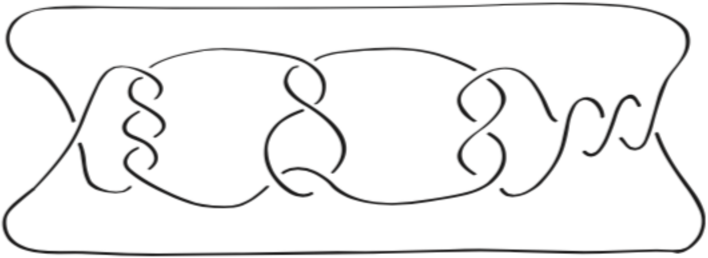
Non-alternating pretzel links are Turaev genus one.





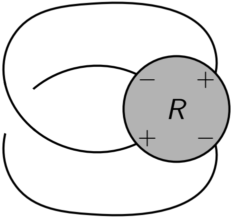
Montesinos links

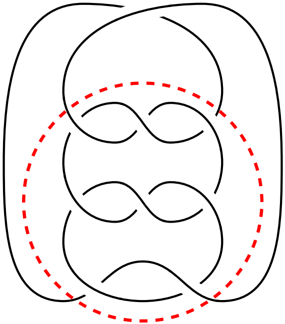
Non-alternating Montesinos links are Turaev genus one.



Almost-alternating links

A non-alternating link is *almost-alternating* if it has a diagram that can be transformed into an alternating diagram via one crossing change. Almost-alternating links were first defined by a group of undergraduates working with Colin Adams in 1991.





Almost-alternating links have Turaev genus one

Every almost-alternating link has Turaev genus one.

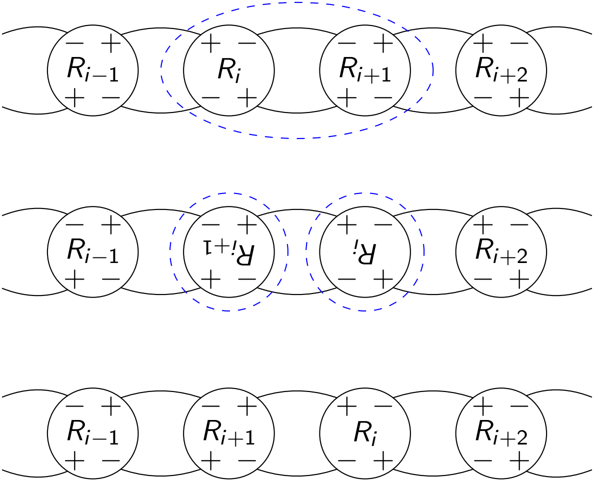
Open Question. Is there a Turaev genus one link that is not almost-alternating?

Almost-alternating links and mutation

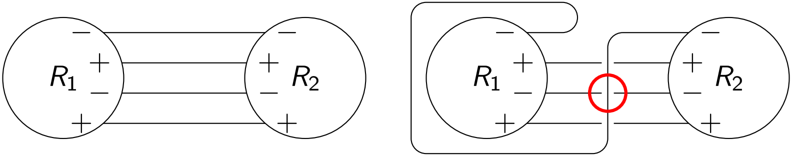
Theorem (Armond, L.)

There is a sequence of mutations transforming every Turaev genus one link into an almost-alternating link.

Mutation proof



Mutation proof continued

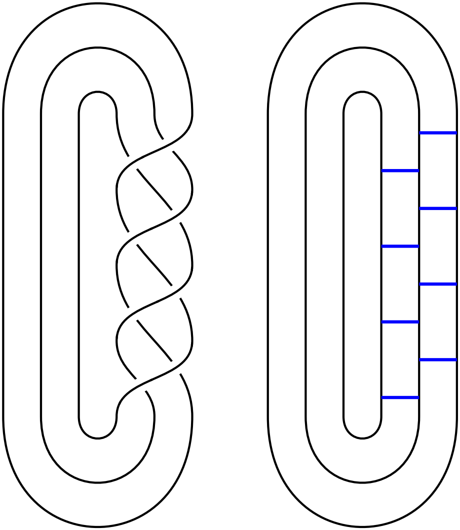


Semi-adequate links

A link diagram is *A-adequate* if no two arcs in the resolution of the same crossing lie on the same component of the of the all-A Kauffman state. Similarly, a link diagram is *B-adequate* if no two arcs in the resolution of the same crossing lie on the same component of the of the all-B Kauffman state. A link with either at least one A-adequate or B-adequate diagram is *semi-adequate*. A link with no A-adequate or B-adequate diagrams is *inadequate*.

A-adequate example

$T_{3,4}$



all-A state

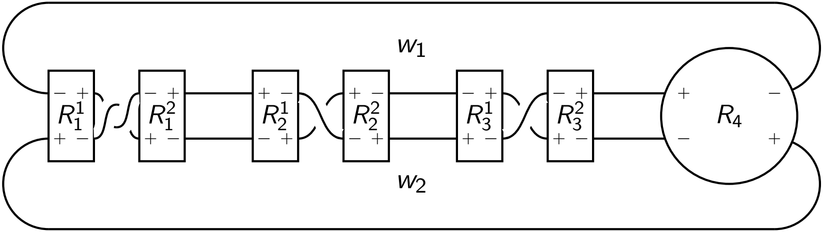
Turaev genus one, semi-adequate, and almost-alternating

Theorem (Kim)

Every Turaev genus one link is semi-adequate or almost-alternating.

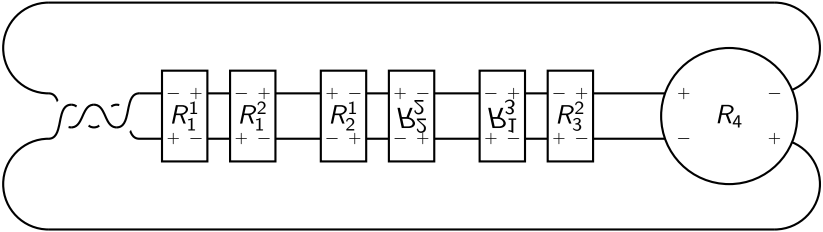
Idea of proof

Start with a diagram like below. Then flype repeatedly to collect all of the crossings in the boundary of w_1 and w_2 into one twist region.



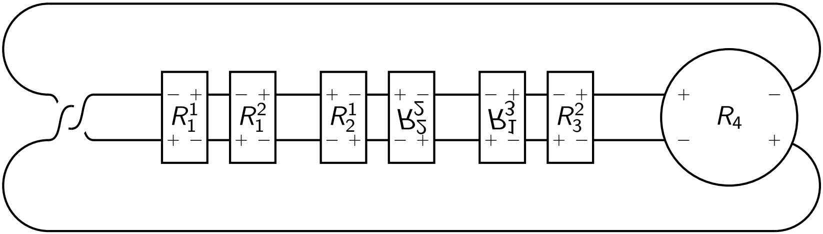
Idea of Proof

Cancel crossings in the twist region if possible.



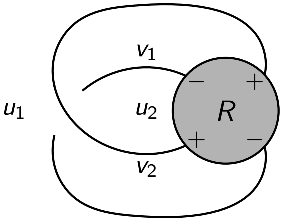
Idea of proof

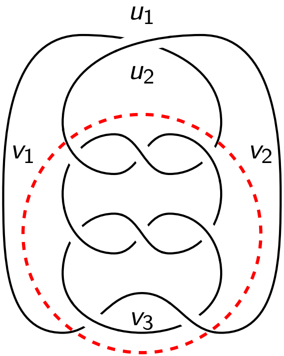
The resulting diagram is semi-adequate unless there is a single crossing in the twist region and exactly two alternating tangles, one of which is the single crossing in the twist region. In that case, the diagram is almost-alternating.



Adjacent faces

Label the regions surrounding the dealternator u_1, u_2, v_1, and v_2 as below. Define $\text{adj}(u_1, u_2)$ to be the number of regions in R that share crossings with both u_1 and u_2. Similarly define $\text{adj}(v_1, v_2)$.





$\text{adj}(u_1, u_2) = 0 \quad \text{adj}(v_1, v_2) = 1$

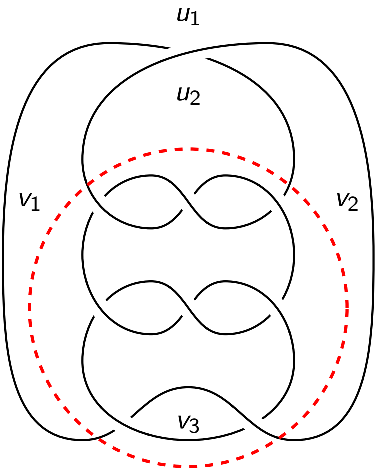
A- and B-almost-alternating

An almost-alternating link with a diagram where $\text{adj}(u_1, u_2) = 0$ is *A-almost alternating*, and an almost-alternating link with a diagram where $\text{adj}(v_1, v_2) = 0$ is *B-almost alternating*.

Theorem (Dasbach, L.)

Every almost-alternating link is either A- or B-almost alternating.

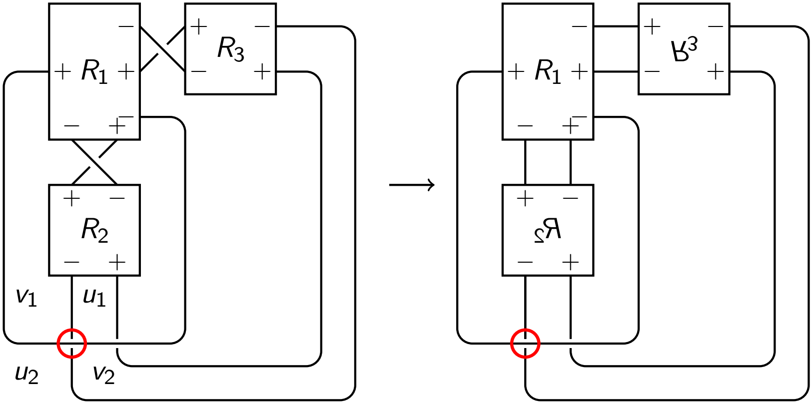
Example: A-almost alternating



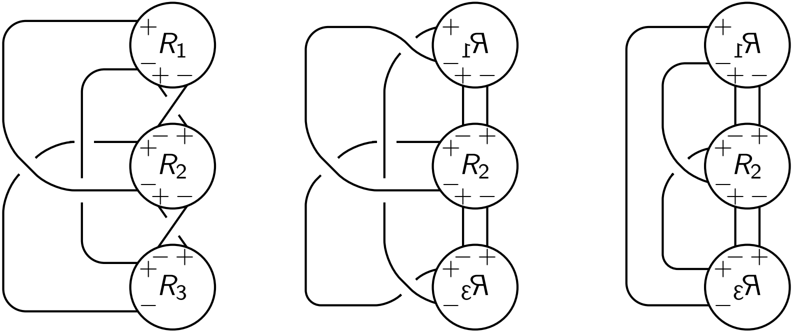
Sketch of Proof

1. If $\text{adj}(u_1, u_2) > 2$, then $\text{adj}(v_1, v_2) = 0$, and if $\text{adj}(v_1, v_2) > 2$, then $\text{adj}(u_1, u_2) = 0$.
2. If $\text{adj}(u_1, u_2) = \text{adj}(v_1, v_2) = 1$, then there is another almost-alternating diagram of the link with fewer crossings.
3. If $1 \leq \text{adj}(u_1, u_2) \leq 2$ and $\text{adj}(v_1, v_2) = 2$ (or vice versa), then the link is alternating.

$\mathrm{adj}(u_1, u_2) = \mathrm{adj}(v_1, v_2) = 1$



$\mathrm{adj}(u_1, u_2) = 2$ and $\mathrm{adj}(v_1, v_2) = 1$.



A- and B-Turaev genus one

A Turaev genus one link is *A-Turaev genus one* if it is A-almost-alternating or A-adequate. A Turaev genus one link is *B-Turaev genus one* if it is B-almost-alternating or B-adequate.

Every Turaev genus one link is either A-Turaev genus one or B-Turaev genus one.

The Kauffman bracket

Recursively define the Kauffman bracket $\langle D \rangle$ by

1. $\langle \diagup\!\!\!\!\diagdown \rangle = A \langle\ \rangle\langle\ \rangle + A^{-1} \langle \asymp \rangle$,
2. $\langle\, D \sqcup \bigcirc \,\rangle = (-A^2 - A^{-2}) \langle D \rangle$,
3. $\langle\, \bigcirc \,\rangle = 1$.

The Jones polynomial

If L is an oriented link with diagram D, then its Jones polynomial is

$$V_L(t) = (-A^3)^{-w(D)} \langle D \rangle \Big|_{A=t^{-1/4}}$$

where $w(D) = \#($  $) - \#($  $)$ is the writhe of D.

The Kauffman bracket of an alternating link

Theorem (Dasbach-Lin)

The Kauffman bracket of a reduced alternating diagram D with c crossings is given by

$$\langle D \rangle = \sum_{i=0}^{c} \gamma_i A^{c+2v-2-4i}$$

where

$$\gamma_0 = (-1)^{v-1},$$
$$\gamma_1 = (-1)^{v-2}(e - v + 1), \text{ and}$$
$$\gamma_2 = (-1)^{v-3}\left(\binom{v-1}{2} - e(v-2) + \mu + \binom{e}{2} - \tau\right).$$

The Kauffman bracket of an almost-alternating link

Theorem (Dasbach, L., Spyropoulos)

Let L be a link with an A-almost-alternating diagram D with c crossings. The Kauffman bracket of D can be expressed as

$$\langle D \rangle = \sum_{i=0}^{c-3} \alpha_i A^{c+2v-8-4i}$$

where

$$\alpha_0 = (-1)^v (P-1) \text{ and}$$

$$\alpha_1 = (-1)^{v-1} \left(\beta_1 (P-1) - \binom{P}{2} + P_2 - P_0 + Q - S \right).$$

The Jones polynomial of a Turaev genus one link

Theorem (Dasbach, L., Spyropoulos)

Let L be a Turaev genus one link with Jones polynomial

$$V_L(t) = a_m t^m + a_{m+1} t^{m+1} + \cdots + a_{n-1} t^{n-1} + a_n t^n$$

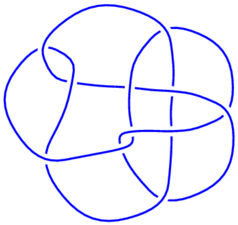
where a_m and a_n are nonzero. If L is A-Turaev genus one, then

$$|a_m| = 1 \text{ and } a_m a_{m+1} \leq 0.$$

If L is B-Turaev genus one, then

$$|a_n| = 1 \text{ and } a_{n-1} a_n \leq 0.$$

Example $11n_{95}$

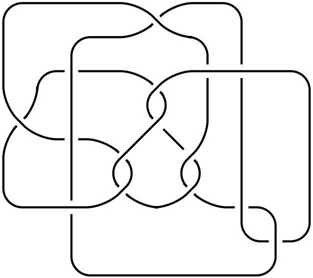


The Jones polynomial of $11n_{95}$ is

$$V_{11n_{95}}(t) = 2t^2 - 3t^3 + 5t^4 - 6t^5 + 6t^6 - 5t^7 + 4t^8 - 2t^9.$$

Since $11n_{95}$ has a diagram of Turaev genus two, it follows that $g_T(11n_{95}) = 2$.

Example $15n_{41133}$



The Jones polynomial of $15n_{41133}$ is

$$t^4 + t^5 - 3t^6 + 8t^7 - 12t^8 + 14t^9 - 15t^{10} + 13t^{11} - 10t^{12} + 6t^{13} - 2t^{14}.$$

Thus $g_T(15n_{41133}) \geq 2$.

Non-triviality of the Jones polynomial

Theorem (L., Spyropoulos)

Let L be an m-component link of Turaev genus one where $m \geq 1$, and let $V_L(t)$ be the Jones polynomial of L. Then

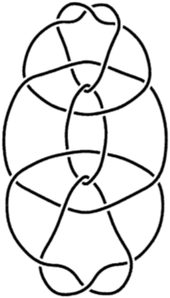
$$V_L(t) \neq t^k \left(-t^{\frac{1}{2}} - t^{-\frac{1}{2}}\right)^{m-1}$$

for any $k \in \mathbb{Z}$. In particular, the Jones polynomial of L is different from the Jones polynomial of the m-component unlink.

Notes about the previous theorem

- The $m = 1$ case says the Jones polynomial detects the unknot among all almost-alternating knots. (This is the hard part to prove).
- The $m \geq 2$ case follows from the previous theorem almost immediately.

Example with trivial Jones polynomial



Eliahou, Kauffman, and Thistlethwaite found non-trivial links whose Jones polynomials are trivial. Since the above link L has Jones polynomial $V_L(t) = (-t^{\frac{1}{2}} - t^{-\frac{1}{2}})^4$, it follows that $g_T(L) \geq 2$.

Khovanov homology

Khovanov homology is a $\mathbb{Z}$-module equipped with two gradings i and j such that

$$\sum_{i,j} (-1)^i \operatorname{rank} Kh^{i,j}(L)\; t^j = (t + t^{-1}) V_L(t^2).$$

Example: the $(3,4)$-torus knot

$j \backslash i$	0	1	2	3	4	5
17						1
15						1
13				1	1	
11				1_2	1	
9			1			
7	1					
5	1					

$$\sum_{i,j} (-1)^i \operatorname{rank} Kh^{i,j}(T_{3,4})\, t^j = -\,t^{17} - t^{15} + t^{11} + t^9 + t^7 + t^5$$

$$V_{T_{3,4}}(t) = -\,t^8 + t^5 + t^3$$

Maximum and minimum gradings

Define

$$
\begin{aligned}
j_{\min}(L) &= \min\{j \mid Kh^{i,j}(L) \neq 0\}, \\
j_{\max}(L) &= \max\{j \mid Kh^{i,j}(L) \neq 0\}, \\
\delta_{\min}(L) &= \min\{2i - j \mid Kh^{i,j}(L) \neq 0\}, \text{ and} \\
\delta_{\max}(L) &= \max\{2i - j \mid Kh^{i,j}(L) \neq 0\}.
\end{aligned}
$$

Extremal Khovanov homology

Theorem (Dasbach, L.)

Suppose that L is a non-split link.

1. *If L is A-Turaev genus one, A-almost alternating, or A-adequate, then there is an $i_0 \in \mathbb{Z}$ such that $Kh^{*,j_{\min}(L)}(L) = Kh^{i_0,j_{\min}(L)}(L) \cong \mathbb{Z}$ and $2i_0 - j_{\min}(L) = \delta_{\min}(L) + 2$.*
2. *If L is B-Turaev genus one, B-almost alternating, or B-adequate, then there is an $i_0 \in \mathbb{Z}$ such that $Kh^{*,j_{\max}(L)}(L) = Kh^{i_0,j_{\max}(L)}(L) \cong \mathbb{Z}$ and $2i_0 - j_{\max}(L) = \delta_{\max}(L) - 2$.*

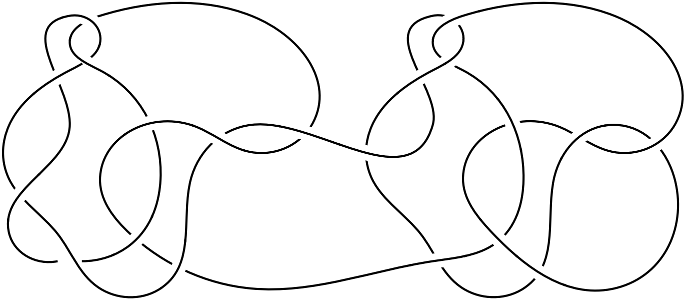
Example: the $(3,4)$-torus knot

$j \backslash i$	0	1	2	3	4	5
17						1
15						1
13				1	1	
11				1_2	1	
9			1			
7	1					
5	1					

Example: the $(3, 4)$-torus knot

$j \backslash i$	0	1	2	3	4	5
17						1
15						1
13				1	1	
11				1_2	1	
9			1			
7	1					
5	1					

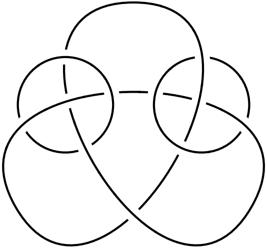
Example: $10_{132}\#\overline{10_{132}}$



Example: $Kh(10_{132}\#\overline{10_{132}})$

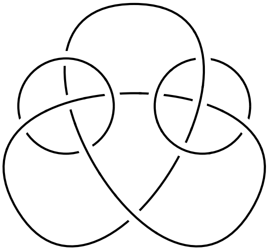
$j\backslash i$	−7	−6	−5	−4	−3	−2	−1	0	1	2	3	4	5	6	7
13														1	1
11													1	1_2	1_2
9												2	$3,1_2$	1	
7										1	5	$3,2_2$	2_2		
5									1	$3,1_2$	$3,5_2$	$2,2_2$			
3								1	$7,1_2$	$8,3_2$	$2,1_2$				
1							3	$8,1_2$	$4,6_2$	$1,3_2$					
−1						1	$4,3_2$	$8,6_2$	$3,1_2$						
−3					2	$8,1_2$	$7,3_2$	$1,1_2$							
−5				2	$3,2_2$	$3,5_2$	$1,1_2$								
−7				$3,2_2$	$5,2_2$	1									
−9		1	3	$2,1_2$											
−11		1_2	$1,1_2$												
−13	1	1													

Example: $11n_{376}$



$j \backslash i$	−7	−6	−5	−4	−3	−2	−1	0
−3								2
−5							1	2
−7						2	1_2	
−9				1	1	$1,2_2$		
−11				4	$2,1_2$			
−13				$2,2_2$				
−15		1	2					
−17		1_2						
−19	1							

Example: $11n_{376}$



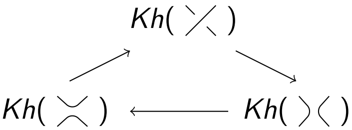
$j\backslash i$	−7	−6	−5	−4	−3	−2	−1	0
−3								2
−5							1	2
−7						2	1_2	
−9				1	1	$1,2_2$		
−11				4	$2,1_2$			
−13				$2,2_2$				
−15		1	2					
−17		1_2						
−19	1							

Sketch of proof: semi-adequate links

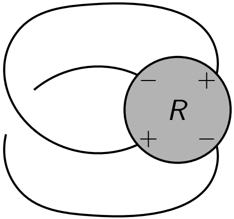
1. If D is an A-adequate diagram, then there is only one generator in the minimum j-grading of the Khovanov complex. This was observed by Khovanov.
2. The diagonal grading of the generator of $Kh^{*,j_{\min(L)}}(L)$ can be worked out using the Lee spectral sequence and work of Champanerkar, Kofman, and Stoltzfus.

Sketch of proof: almost-alternating links

1. Khovanov homology satisfies an exact triangle:



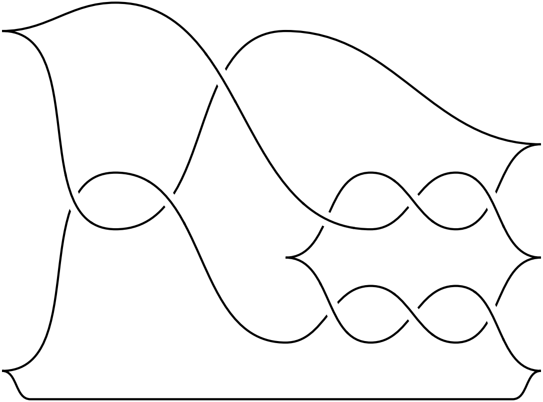
2. Resolve crossings inside the alternating tangle R of an almost-alternating diagram.



Sketch of proof: Turaev genus one links

Kim proved that a Turaev genus one link is either almost-alternating or semi-adequate.

Legendrian front diagrams: the $(3, 4)$ torus knot



Thurston Bennequin numbers

Let $\mathcal{L}$ be a Legendrian link with front diagram F. The *Thurston Bennequin number* $\mathrm{tb}(\mathcal{L})$ of $\mathcal{L}$ is the difference between the writhe of F and half the number of cusps of F. The maximal Thurston Bennequin number of a classical link L is

$$\overline{\mathrm{tb}}(L) = \max\{\mathrm{tb}(\mathcal{L}) \mid \mathcal{L} \text{ has topological type } L\}.$$

$\overline{\text{tb}}(L)$ and Khovanov homology

Theorem (Ng)

Let L be a non-split link. Then

$$\overline{\text{tb}}(L) \leq \min\{j - i \mid Kh^{i,j}(L) \neq 0\}.$$

Theorem (Ng)

Let L be a non-split link with reduced alternating diagram D. Then

$$\overline{\text{tb}}(L) = w(D) - s_A(D).$$

$\overline{\text{tb}}(L)$ for Turaev genus one links

Theorem (Dasbach, L.)

Let L be a link with Turaev genus one diagram D. At least one of the following inequalities hold:

$$\begin{aligned} w(D) - s_A(D) &\leq \overline{\text{tb}}(L) \leq w(D) - s_A(D) + 1, \text{ or} \\ -w(D) - s_B(D) &\leq \overline{\text{tb}}(\overline{L}) \leq -w(D) - s_B(D) + 1. \end{aligned}$$

Relationship to semi-adequate links

Every theorem above about the Jones polynomial or Khovanov homology of a Turaev genus one link also holds for semi-adequate links (Lickorish, Thistlethwaite; Stoimenov; Khovanov; Abe; Kálmán).

Open Question. Is every Turaev genus one link semi-adequate?

Thank you!