Unknotting and Region Crossing Change

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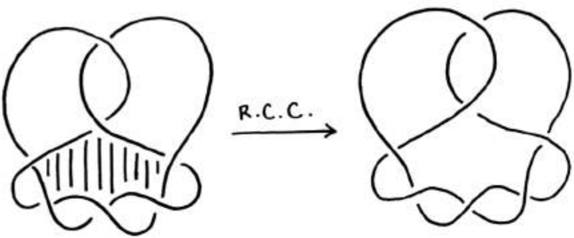
July 20, 2019

Overview

- Define a region crossing change, an unknotting operation, and the region index of a knot.
- Introduce our knot invariant, the multi-region index.
- Explain the Goeritz matrix of a knot diagram and show how it can be used to find a lower bound on the multi-region index.
- Demonstrate an upper bound for the multi-region index.
- Present a list of unanswered questions regarding our research.

Region Crossing Change

If a knot diagram D has a region R, which contains crossings $c_1, \ldots, c_m$, then a *region crossing change* on R is the operation that yields a diagram D' obtained from D by changing crossings $c_1, \ldots, c_m$.



A region crossing change on 7_2

Unknotting Operation

- Shimizu proved that a region crossing change is an unknotting operation.
- Shimizu defined the *region unknotting number* $u_R(D)$ to be the fewest number of region crossing changes necessary to obtain the unknot from a minimal crossing diagram D.

$u_R(8_2) = 2$

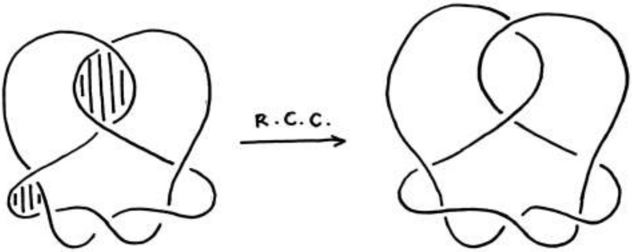
Unknotting Regions and Region Index

- Aida proved that for any knot K, there is a diagram D of K with a single region crossing change necessary to produce the unknot. We call such a region an *unknotting region*.
- Tanaka defined the *region index* $\mathrm{Reg}(K)$ of a knot K to be the fewest number of crossings in any unknotting region of any diagram of K.

$\mathrm{Reg}(8_5) = 2$

Multi-Region Index

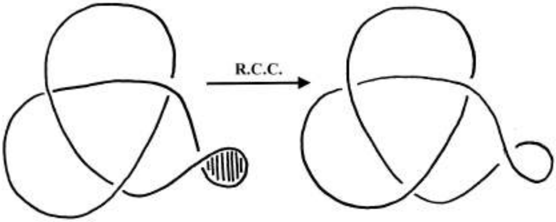
- A *set of unknotting regions* is a set of regions such that performing a region crossing change on all of the regions in the set produces the unknot.
- The *multi-region index* $\mathrm{MRI}(K)$ of a knot K is the fewest number of crossings changed in any set of unknotting regions of any diagram of K.



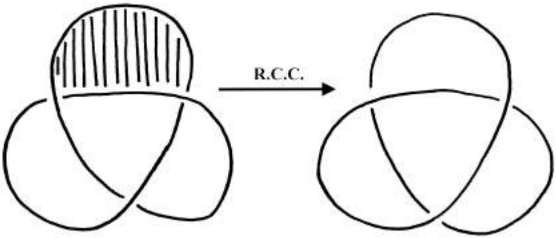
$\mathrm{MRI}(7_2) \leq 4$

Multi-Region Index

Since changing a region with only one crossing does not change the knot, $1 < \mathrm{MRI}(K)$ for any nontrivial knot K.



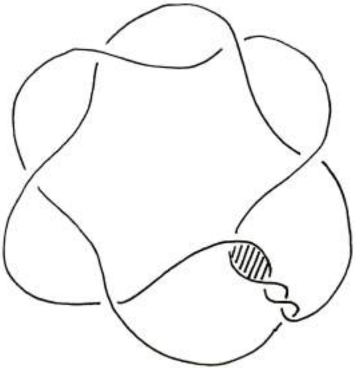
Example



$$\mathrm{MRI}(3_1) = 2$$

Multi-Region Index

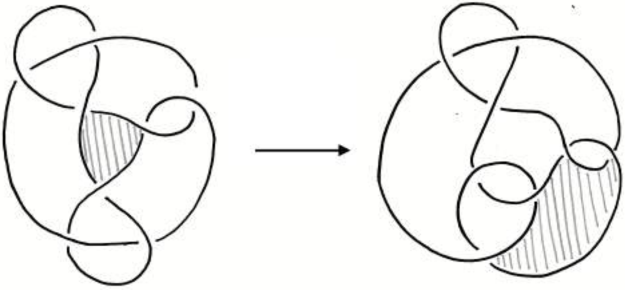
- Let $u(K)$ be the unknotting number of K.
- For any knot K, $u(K) \leq \mathrm{MRI}(K) \leq \mathrm{Reg}(K)$.



$$u(9_4) = \mathrm{MRI}(9_4) = \mathrm{Reg}(9_4) = 2$$

Multi-Region Index

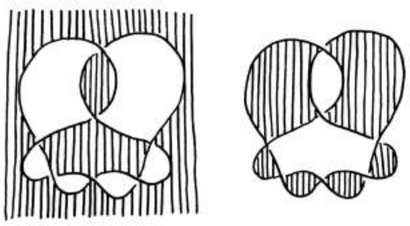
- The multi-region index of a diagram can change for different minimal crossing diagrams of the same knot.



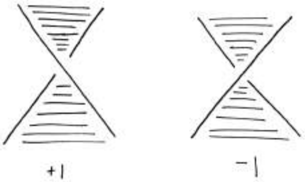
D_1 $\qquad\qquad$ D_2

$\mathrm{MRI}(D_1) = 3$ $\qquad\qquad$ $\mathrm{MRI}(D_2) = 4$

Checkerboard Shading and Crossings



Checkerboard shading



$\zeta(c) = \pm 1$

Goeritz Matrix

- Checkerboard shade the knot diagram D and label the shaded regions $R_0, \ldots, R_m$.
- Let $\overline{G}_D$ be the symmetric matrix whose (i, j)th entry is $\sum \zeta(c)$. The entries on the diagonal are the negative of the sum of the column.

$$\overline{G}_D = \begin{array}{c} \\ R_0 \\ R_1 \\ \vdots \\ R_m \end{array} \begin{array}{c} \begin{array}{cccc} R_0 & R_1 & \ldots & R_m \end{array} \\ \left[\begin{array}{cccc} & & & \\ & & & \\ & & * & \\ & & & \\ & & & \end{array} \right] \end{array}$$

Goeritz Matrix

- The *Goeritz matrix* of D is obtained by deleting the first row and column.

$$
G_D = \begin{array}{c c} & \begin{array}{ccc} R_1 & \dots & R_m \end{array} \\ \begin{array}{c} R_1 \\ \vdots \\ R_m \end{array} & \left[\begin{array}{ccc} & & \\ & * & \\ & & \end{array} \right] \end{array}
$$

Smith Normal Form

- There are invertible $m \times m$ matrices S and T with integer entries such that

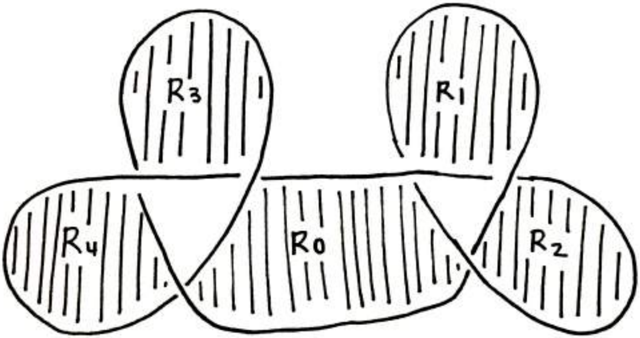
$$SG_DT = \begin{bmatrix} \alpha_1 & 0 & 0 & & \cdots & & 0 \\ 0 & \alpha_2 & 0 & & \cdots & & 0 \\ 0 & 0 & \ddots & & & & 0 \\ \vdots & & & \alpha_r & & & \vdots \\ & & & & 0 & & \\ & & & & & \ddots & \\ 0 & & & \cdots & & & 0 \end{bmatrix},$$

  where each α_j is a positive integer such that $\alpha_j | \alpha_{j+1}$ for $1 \leq j < r$.
- The matrix SG_DT is called the *Smith normal form* of G_D.

Smith Normal Form

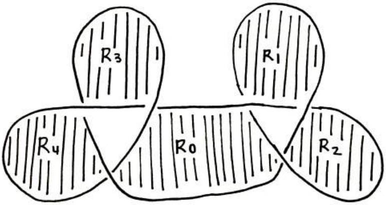
- A matrix with integer entries can be transformed into its Smith normal form by a sequence of the following row and column operations:
  - Replacing row or column i with $i + m * j$, where j is another row or column and m is an integer.
  - Switching rows and columns.
  - Scaling rows or columns by ± 1.

Example



Knot $3_1\#3_1$

Example



$$\overline{\mathbf{G}}_{\mathbf{D}} \qquad\qquad \mathbf{G}_{\mathbf{D}}$$

$$\begin{array}{c|c|cccc} & R_0 & R_1 & R_2 & R_3 & R_4 \\ R_0 & 4 & -1 & -1 & -1 & -1 \\ \hline R_1 & -1 & 2 & -1 & 0 & 0 \\ R_2 & -1 & -1 & 2 & 0 & 0 \\ R_3 & -1 & 0 & 0 & 2 & -1 \\ R_4 & -1 & 0 & 0 & -1 & 2 \end{array} \rightarrow \begin{array}{c|cccc} & R_1 & R_2 & R_3 & R_4 \\ \hline R_1 & 2 & -1 & 0 & 0 \\ R_2 & -1 & 2 & 0 & 0 \\ R_3 & 0 & 0 & 2 & -1 \\ R_4 & 0 & 0 & -1 & 2 \end{array}$$

Example

$$G_D = \begin{bmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & 0 & 0 \\ 0 & 0 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -1 & 0 & 0 \\ 1 & 2 & 0 & 0 \\ 0 & 0 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{bmatrix} \rightarrow$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 3 & 0 & 0 \\ 0 & 0 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{bmatrix} \rightarrow \cdots$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix}$$

$d(K)$

- Let K be a knot with diagram D. The set of diagonal entries of the Smith normal form of G_D that are greater than one is an invariant of K.
- Define $d(K)$ to be the number of entries α_j on the diagonal of the Smith normal form of G_D where $\alpha_j > 1$.

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix}$$

$$d(3_1\#3_1) = 2.$$

Lower Bounds

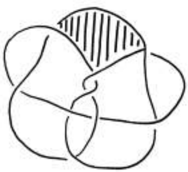
Theorem (Nakanishi)
If K is a knot, then $d(K) \leq u(K)$.

Theorem (Tanaka)
If K is a knot, then $d(K) < \mathrm{Reg}(K)$.

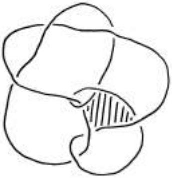
Theorem (G, L, MG, R, Z)
If K is a knot, then $d(K) < \mathrm{MRI}(K)$.

Examples

$\mathrm{MRI}(5_1) = 2,\ d(5_1) = 1$



$\mathrm{MRI}(9_{41}) = 3,\ d(9_{41}) = 2$



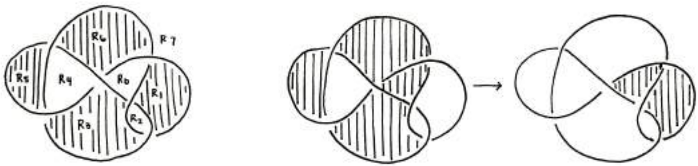
$\mathrm{MRI}(9_{48}) \leq 4,\ d(9_{48}) = 2$

Upper Bound on MRI(K)

Theorem

For any knot K, $\mathrm{MRI}(K) \leq 2c(K)$.

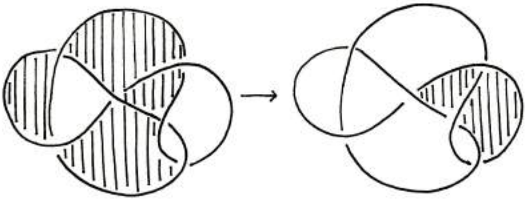
- A set of unknotting regions is $\{R_0, R_3, R_5, R_6\}$.
- All regions in a single checkerboard coloring has $2c(K)$ crossings.
- The set of shaded unknotting regions is $\{R_3, R_5, R_6\}$ with 9 crossing changes, its complement is $\{R_1\}$ with 3 crossing changes.



Knot 6_3

Upper Bound on MRI(K)

- The set of unshaded unknotting regions is $\{R_0\}$ with 3 crossing changes, and $3 < c = 6$.
- Crossings in new set of unknotting regions $= 6 \leq 2c(K) = 12$.



Knot 6_3

Questions

- Is there a knot K where $\mathrm{MRI}(K) < \mathrm{Reg}(K)$?
- Can $\mathrm{MRI}(K) - u(K)$ be arbitrarily large?
- What are other lower bounds on $\mathrm{MRI}(K)$?

Thank You!