Unknot detection and the Jones polynomial

Adam Lowrance - Vassar College

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Important problems in knot theory

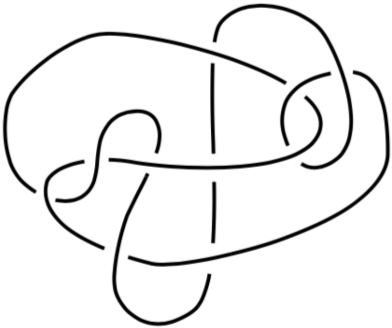
1. **Knot recognition.** Determine whether two given knots K_1 and K_2 are equivalent.

2. **Unknot detection.** Determine whether a given knot K is equivalent to the unknot.

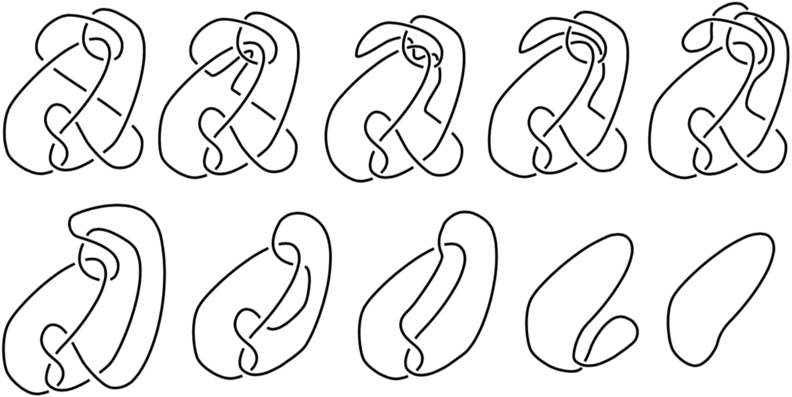
Unknot detection methods

1. Haken (1968) via normal surface theory.
2. Birman and Hirsch (1998) via braid foliations.
3. Haas and Lagarias (1998) proved that any diagram D of the unknot with n crossings can be unknotted by a sequence of Reidemeister moves where each intermediate diagram has at most $2^{10^{11}n}$ crossings.
4. Dynnikov (2002) proved that any grid diagram of the unknot can be transformed into the 2×2 grid diagram of the unknot by a sequence of grid moves that are non-increasing in grid number.

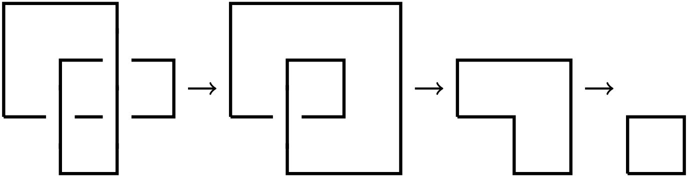
The culprit



Unknotting the culprit



Monotonic simplification of grid diagrams



More unknot detection methods

5. Ozsváth and Szabó (2003) proved that knot Floer homology detects the unknot.
6. Kronheimer and Mrowka (2010) proved that Khovanov homology detects the unknot.
7. Many more I've excluded.

Unknot detection and the Jones polynomial

The Jones polynomial of a knot K is a Laurent polynomial $V_K(t) \in \mathbb{Z}[t, t^{-1}]$.

Conjecture (Jones unknot conjecture)

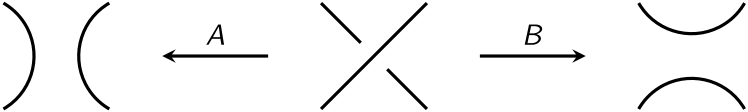
If $V_K(t) = 1$, then K is the unknot.

Plan for the rest of the talk

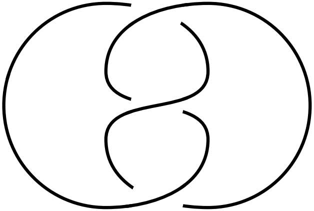
- Define the Jones polynomial.
- Evidence for/against the conjecture.
- Some classes where the conjecture is true.
- Strategies to prove or disprove the conjecture.

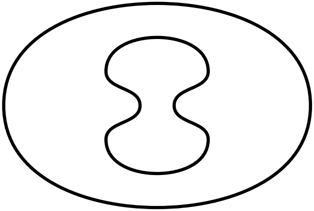
Kauffman states

A Kauffman state is the set of curves resulting from choosing an A-resolution or a B-resolution at each crossing of a knot diagram.

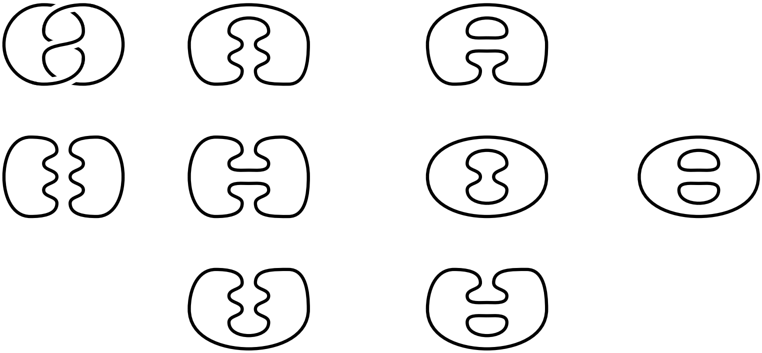


Kauffman state example





Kauffman state examples



The Kauffman bracket via state sums

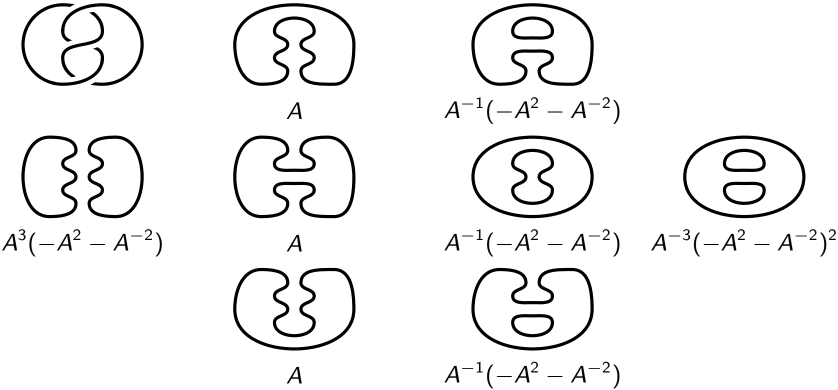
- For each Kauffman state S, define $|S|$ to be the number of components of S.
- Define

$$\langle D|S\rangle = A^{\#A\text{-resolutions}-\#B\text{-resolutions}}(-A^2 - A^{-2})^{|S|-1}.$$

- Define the Kauffman bracket of D by

$$\langle D\rangle = \sum_S \langle D|S\rangle.$$

Kauffman bracket example



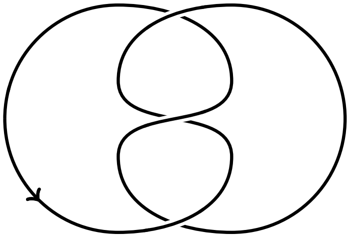
$$\langle D \rangle = -A^5 - A^{-3} + A^{-7}.$$

Writhe

$+$ $\qquad$ $-$

The writhe $w(D)$ is the difference between number of positive and negative crossings in the knot diagram D.

Writhe example



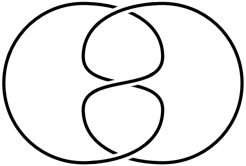
$$w(D) = 3$$

The Jones polynomial

If a knot K has diagram D, then its Jones polynomial $V_K(t)$ is defined by

$$V_K(t) = (-A)^{-3w(D)} \langle D \rangle |_{A=t^{-1/4}}.$$

The Jones polynomial of our example



$$V_K(t) = (-A)^{-9}(-A^5 - A^{-3} + A^{-7})|_{A=t^{-1/4}}$$
$$= t + t^3 - t^4.$$

The Jones polynomial of the unknot

- $\langle \bigcirc \rangle = 1.$
- $w(\bigcirc) = 0.$
- $V_{\bigcirc}(t) = 1.$

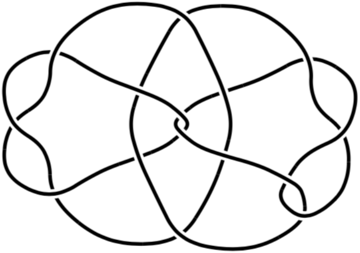
Evidence for the conjecture

1. If K has at most 23 crossings and $V_K(t) = 1$, then K is the unknot.
2. Khovanov homology is a generalization of the Jones polynomial, and it detects the unknot.
3. We haven't found a counterexample yet.

Evidence against the conjecture

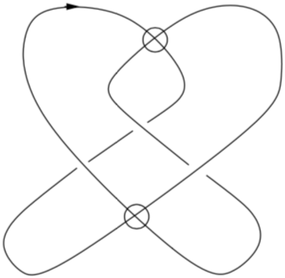
1. A generalization of the conjecture for links is false.
2. A generalization of the conjecture for virtual knots is false.
3. We haven't proven it yet.

Link with trivial Jones polynomial



$$V_L(t) = -t^{-1/2} - t^{1/2} = V_{\bigcirc\bigcirc}(t)$$

Virtual knot with trivial Jones polynomial

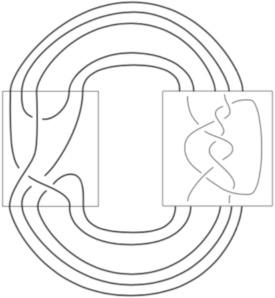


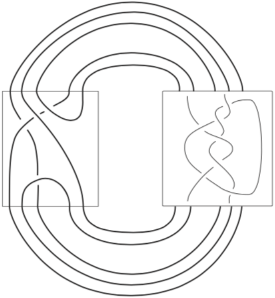
$$V_K(t) = 1$$

Some strategies to disprove the conjecture

- Some virtual knot diagrams with trivial Jones polynomial may secretly be classical.
- (Bigelow, Ito) If the Burau representation of B_4 is not faithful, then the conjecture is false.
- Generalized forms of mutation may change knot type but not the Jones polynomial.

Genus two mutation

$S_{55} =$ 

$S'_{55} =$ 

Genus two mutation preserves the Jones polynomial but can change the knot type.

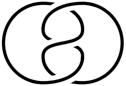
Families where the Jones polynomial detects the unknot

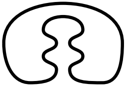
1. Alternating knots (Kauffman, Murasugi, Thistlethwaite).
2. Adequate knots (Lickorish, Thistlethwaite).
3. Semi-adequate knots (Stoimenow)
4. Positive knots (Stoimenow)
5. Almost-alternating (L, Spyropolous)

Alternating/adequate proof

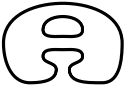
- Let D be an alternating (or adequate) diagram of a knot K. Let S_A and S_B be the all-A and all-B Kauffman states of D.
- If $S \neq S_A$, then $\max\deg\langle D|S_A\rangle > \max\deg\langle D|S\rangle$.
- If $S \neq S_B$, then $\min\deg\langle D|S_B\rangle < \min\deg\langle D|S\rangle$.
- The Kauffman bracket $\langle D\rangle$ has two different powers of A. So $V_K(t) \neq 1$.

Kauffman bracket example

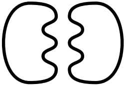




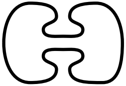
A



$A^{-1}(-A^2 - A^{-2})$



$A^3(-A^2 - A^{-2})$



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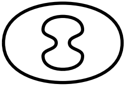
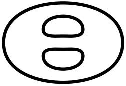
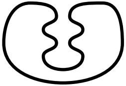


$A^{-1}(-A^2 - A^{-2})$



$A^{-3}(-A^2 - A^{-2})^2$



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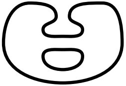


$A^{-1}(-A^2 - A^{-2})$

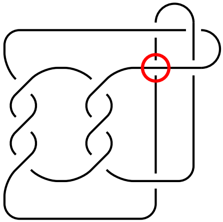
$$\langle D \rangle = -A^5 - A^{-3} + A^{-7}.$$

Alternating/semi-adequate proof

- If K is alternating or semi-adequate, then there are formulas for the extreme coefficients of $V_K(t)$.
- Analyze those formulas to ensure that if K is a nontrivial alternating/semi-adequate knot, then $V_K(t)$ has at least two nonzero coefficients.

Almost-alternating knots

A knot is almost-alternating if it is non-alternating and has a diagram that can be transformed into an alternating diagram via one crossing change.



$T_{3,4}$ is almost alternating.

Almost alternating knots and the Jones polynomial

Theorem (L, Spyropolous)

If K is almost alternating, then $V_K(t) \neq 1$.

Proof sketch. Find formulas for the first or last few coefficients of the Jones polynomial. Show that if K is almost-alternating, at least two coefficients are non-zero.

An optimistic approach

- A knot K is n-almost-alternating if it has a diagram that can be transformed into an alternating diagram via n crossing changes and no diagram of K can be transformed into an alternating diagram with fewer than n crossing changes.
- *Optimistic goal:* Show that if the Jones unknotting conjecture is true for n-almost-alternating knots, then it must be true for $(n+1)$-almost-alternating knots.
- *More realistic goal:* Show the Jones unknotting conjecture is true for 2-almost-alternating knots.

Thank you!