Invariants of Turaev genus one links

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Outline

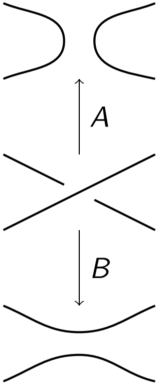
- What are Turaev genus one links?
- How are Turaev genus one links related to some familiar families of links?
- What can we say about the Jones polynomial and Khovanov homology of a Turaev genus one link?

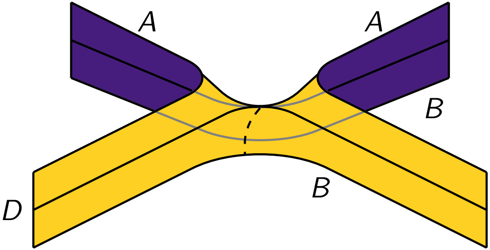
The Turaev surface

The *Turaev surface* of a link diagram D is obtained by

1. constructing a cobordism between the all-A and all-B Kauffman states of D that has saddles corresponding with crossings, and
2. capping off the boundary components of the above cobordism with disks.

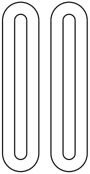
The Turaev surface at a crossing

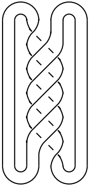




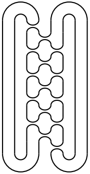
The Turaev surface

all-A state

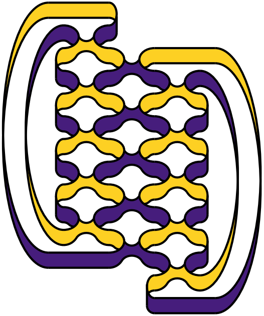




D



all-B state



Turaev surface of D

The Turaev genus of a link

- For a connected link diagram D, the genus of the Turaev surface is

$$g_T(D) = \frac{1}{2}\left(2 + c(D) - s_A(D) - s_B(D)\right)$$

  where $c(D)$ is the number of crossings in D and $s_A(D)$ and $s_B(D)$ are the number of components in the all-A and all-B Kauffman states of D respectively.
- The *Turaev genus* $g_T(L)$ of a link L is

$$g_T(L) = \min\{g_T(D) \mid D \text{ is a diagram of } L\}.$$

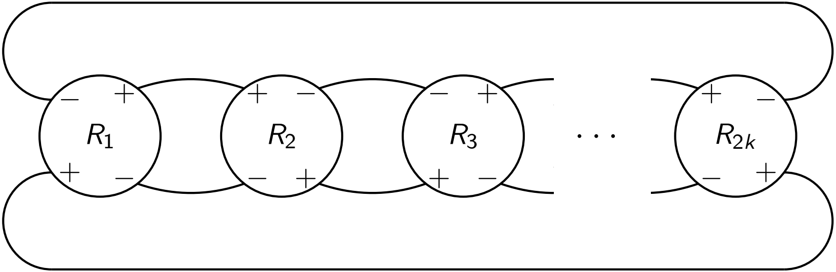
Facts about the Turaev surface and Turaev genus

- The link L has an alternating projection on any of its Turaev surfaces.
- The Turaev genus of a link is zero if and only if the link is alternating.

Turaev genus one links

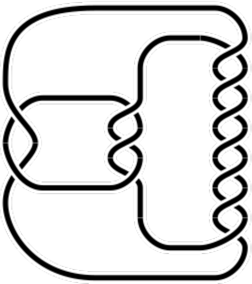
Theorem (Armond, L.; Kim)

Every non-split Turaev genus one link has a diagram as depicted below where each R_i is an alternating tangle and $+$ or $-$ indicates that the first crossing that strand meets is an over or under crossing respectively.



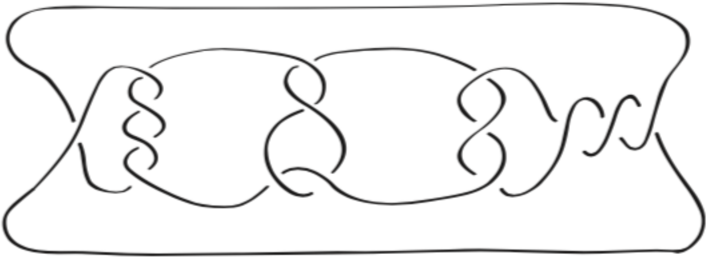
Pretzel links

Non-alternating pretzel links are Turaev genus one.



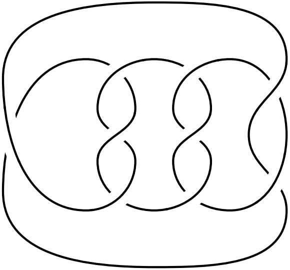
Montesinos links

Non-alternating Montesinos links are Turaev genus one.



Almost-alternating links

- A non-alternating link is *almost-alternating* if it has a diagram that can be transformed into an alternating diagram via one crossing change.
- Every almost-alternating link is Turaev genus one.
- **Open Question.** Is every Turaev genus one link almost-alternating?

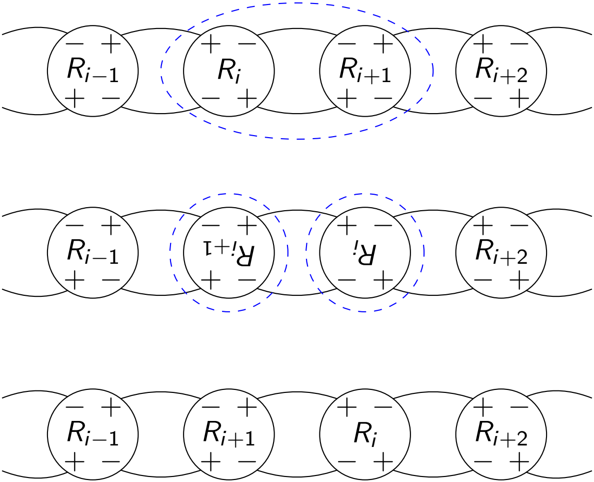


Almost-alternating links and mutation

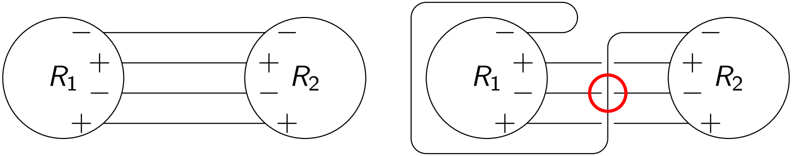
Theorem (Armond, L.)

Every Turaev genus one links is mutant to an almost-alternating link.

Mutation proof



Mutation proof continued



The Jones polynomial of a Turaev genus one link

Theorem (Dasbach, L., Spyropoulos)

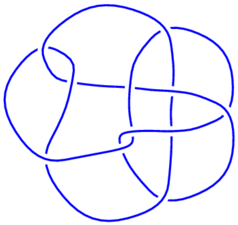
Let L be a Turaev genus one link with Jones polynomial

$$V_L(t) = a_m t^m + a_{m+1} t^{m+1} + \cdots + a_{n-1} t^{n-1} + a_n t^n$$

where a_m and a_n are nonzero. Either

- $|a_m| = 1$ *and* $a_m a_{m+1} \leq 0$, *or*
- $|a_n| = 1$ *and* $a_{n-1} a_n \leq 0$.

Example $11n_{95}$

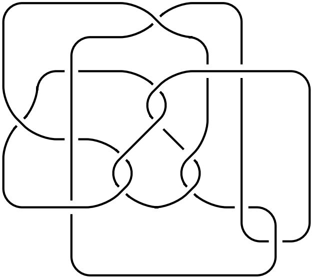


The Jones polynomial of $11n_{95}$ is

$$V_{11n_{95}}(t) = 2t^2 - 3t^3 + 5t^4 - 6t^5 + 6t^6 - 5t^7 + 4t^8 - 2t^9.$$

Since $11n_{95}$ has a diagram of Turaev genus two, it follows that $g_T(11n_{95}) = 2$.

Example $15n_{41133}$



The Jones polynomial of $15n_{41133}$ is

$$t^4+t^5-3t^6+8t^7-12t^8+14t^9-15t^{10}+13t^{11}-10t^{12}+6t^{13}-2t^{14}.$$

Thus $g_T(15n_{41133}) \geq 2$.

Non-triviality of the Jones polynomial

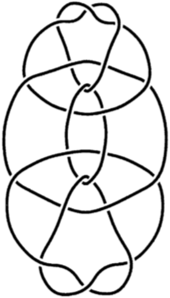
Theorem (L., Spyropoulos)

Let L be an m-component link of Turaev genus one where $m \geq 1$, and let $V_L(t)$ be the Jones polynomial of L. Then

$$V_L(t) \neq t^k \left(-t^{\frac{1}{2}} - t^{-\frac{1}{2}}\right)^{m-1}$$

for any $k \in \mathbb{Z}$. In particular, the Jones polynomial of L is different from the Jones polynomial of the m-component unlink.

Example with trivial Jones polynomial



Eliahou, Kauffman, and Thistlethwaite found non-trivial links whose Jones polynomials are trivial. Since the above link L has Jones polynomial $V_L(t) = (-t^{\frac{1}{2}} - t^{-\frac{1}{2}})^4$, it follows that $g_T(L) \geq 2$.

Khovanov homology

Khovanov homology is a $\mathbb{Z}$-module equipped with two gradings i and j such that

$$\sum_{i,j}(-1)^i \operatorname{rank} Kh^{i,j}(L)\ t^j = (t+t^{-1})V_L(t^2).$$

Let $j_{\min}(L)$ and $j_{\max}(L)$ denote the minimum and maximum j-gradings where $Kh(L)$ is supported respectively.

Extremal Khovanov homology

Theorem (Champanerkar, Kofman, Stotlzfus)

If L is a Turaev genus one link, then $Kh(L)$ is supported on at most three adjacent diagonals, i.e. there is an integer k such that $Kh^{i,j}(L) = 0$ unless $2i - j = k, k+2$, or $k+4$.

Theorem (Dasbach, L.)

Let L be a non-split link of Turaev genus one. Either

$$\bigoplus_{i\in\mathbb{Z}} Kh^{i,j_{\min}}(L) \cong \mathbb{Z} \text{ or } \bigoplus_{i\in\mathbb{Z}} Kh^{i,j_{\max}}(L) \cong \mathbb{Z}.$$

Moreover, the summand of $\mathbb{Z}$ above appears on the penultimate diagonal.

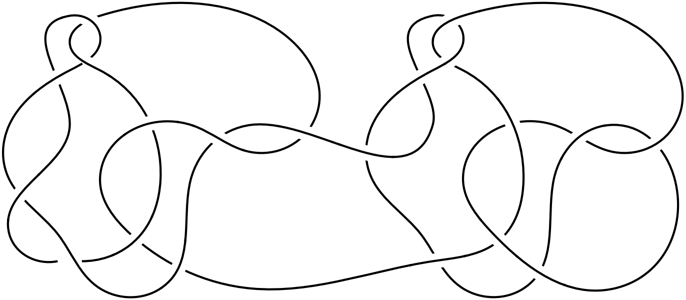
Example: the $(3,4)$-torus knot

$j \backslash i$	0	1	2	3	4	5
17						1
15						1
13				1	1	
11				1_2	1	
9			1			
7	1					
5	1					

Example: the $(3,4)$-torus knot

$j \backslash i$	0	1	2	3	4	5
17						1
15						1
13				1	1	
11				1_2	1	
9			1			
7	1					
5	1					

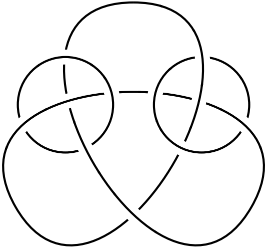
Example: $10_{132}\#\overline{10_{132}}$



Example: $Kh(10_{132}\#\overline{10_{132}})$

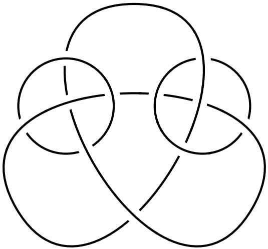
$j \backslash i$	-7	-6	-5	-4	-3	-2	-1	0	1	2	3	4	5	6	7
13														1	1
11													1	1_2	1_2
9												2	$3,1_2$	1	
7										1	5	$3,2_2$	2_2		
5									1	$3,1_2$	$3,5_2$	$2,2_2$			
3								1	$7,1_2$	$8,3_2$	$2,1_2$				
1							3	$8,1_2$	$4,6_2$	$1,3_2$					
-1						1	$4,3_2$	$8,6_2$	$3,1_2$						
-3					2	$8,1_2$	$7,3_2$	$1,1_2$							
-5				2	$3,2_2$	$3,5_2$	$1,1_2$								
-7				$3,2_2$	$5,2_2$	1									
-9		1	3	$2,1_2$											
-11		1_2	$1,1_2$												
-13	1	1													

Example: $11n_{376}$



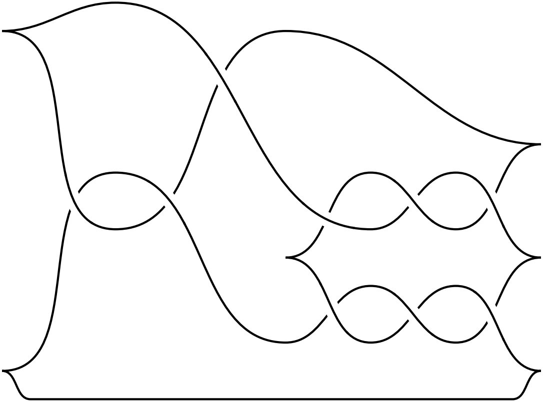
$j \backslash i$	-7	-6	-5	-4	-3	-2	-1	0
-3								2
-5							1	2
-7						2	1_2	
-9				1	1	$1,2_2$		
-11				4	$2,1_2$			
-13				$2,2_2$				
-15		1	2					
-17		1_2						
-19	1							

Example: $11n_{376}$



$j\backslash i$	-7	-6	-5	-4	-3	-2	-1	0
-3								2
-5							1	2
-7						2	1_2	
-9				1	1	$1, 2_2$		
-11				4	$2, 1_2$			
-13				$2, 2_2$				
-15		1	2					
-17		1_2						
-19	1							

Legendrian front diagrams: the $(3, 4)$ torus knot



Thurston Bennequin numbers

Let $\mathcal{L}$ be a Legendrian link with front diagram F. The *Thurston Bennequin number* $\text{tb}(\mathcal{L})$ of $\mathcal{L}$ is the difference between the writhe of F and half the number of cusps of F. The maximal Thurston Bennequin number of a classical link L is

$$\overline{\text{tb}}(L) = \max\{\text{tb}(\mathcal{L}) \mid \mathcal{L} \text{ has topological type } L\}.$$

$\overline{\text{tb}}(L)$ and Khovanov homology

Theorem (Ng)

Let L be a non-split link. Then

$$\overline{\text{tb}}(L) \leq \min\{j - i \mid Kh^{i,j}(L) \neq 0\}.$$

Theorem (Ng)

Let L be a non-split link with reduced alternating diagram D. Then

$$\overline{\text{tb}}(L) = w(D) - s_A(D).$$

$\overline{\text{tb}}(L)$ for Turaev genus one links

Theorem (Dasbach, L.)

Let L be a link with Turaev genus one diagram D. At least one of the following inequalities hold:

$$\begin{aligned} w(D) - s_A(D) &\leq \overline{\text{tb}}(L) \leq w(D) - s_A(D) + 1, \text{ or} \\ -w(D) - s_B(D) &\leq \overline{\text{tb}}(\overline{L}) \leq -w(D) - s_B(D) + 1. \end{aligned}$$

Relationship to semi-adequate links

Every theorem above about the Jones polynomial or Khovanov homology of a Turaev genus one link also holds for semi-adequate links (Lickorish, Thistlethwaite; Stoimenov; Khovanov; Abe; Kálmán).

Open Question. Is every Turaev genus one link semi-adequate?

Kim proved every inadequate Turaev genus one link is almost-alternating. So it suffices to answer the above question for almost-alternating links.

Thank you!