Gordian distance and spectral sequences in Khovanov homology

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Gordian distance

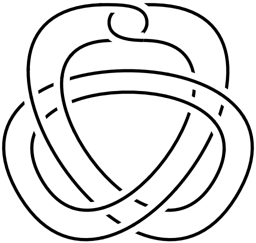
- Murakami defined the *Gordian distance* $d_G(K, J)$ between knots K and J to be the fewest number of crossing changes needed to transform any diagram of K into a diagram of J.
- Kawauchi defined the *alternation number* $\mathrm{alt}(K)$ of a knot K to be the minimum Gordian distance between K and the set of alternating knots, that is

$$\mathrm{alt}(K) = \min\{d_G(K, J) \mid J \text{ is alternating}\}.$$

- The *unknotting number* $u(K)$ of K is

$$u(K) = \min\{d_G(K, U) \mid U \text{ is the unknot}\}.$$

Example: $\mathrm{alt}(K) = u(K) = 1$



$\mathrm{alt}(K) = u(K) = 1$

Goal of this talk

Find a lower bound for $\text{alt}(K)$ using some spectral sequences arising from the Khovanov homology of K.

We look to work of Alishahi and Dowlin for inspiration (more on this later).

Khovanov homology

- Let R be $\mathbb{Z}$, $\mathbb{Q}$, or $\mathbb{Z}_p$ where p is a prime.
- The Khovanov homology $Kh(K;R)$ of K over R is a bigraded R-module that categorifies the Jones polynomial.
- There is a direct sum decomposition

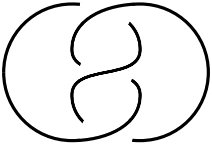
$$Kh(K;R) = \bigoplus_{i,j} Kh^{i,j}(K;R)$$

where $Kh^{i,j}(K;R)$ is the summand in homological grading i and polynomial grading j.

Lee spectral sequence

- Let $R = \mathbb{Q}$ or $\mathbb{Z}_p$ for an odd prime p.
- There is a spectral sequence $(E^r_{\text{Lee}}, d^r_{\text{Lee}})$ such that $E^1_{\text{Lee}} \cong Kh(K; R)$ and $E^\infty_{\text{Lee}} \cong R \oplus R$.
- The differential d^r_{Lee} has bidegree $(1, 4r)$.

Lee spectral sequence example 1



K

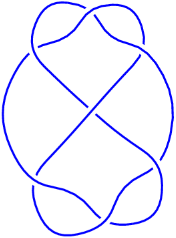
	0	1	2	3
9				1
7				
5			1	
3	1			
1	1			

$Kh(K;\mathbb{Q})$

	0	1	2	3
9				1
7				
5			1	
3	1			
1	1			

E^1_{Lee}

Lee spectral sequence example 2



7_4

	0	1	2	3	4	5	6	7
17								1
15								
13						2	1	
11					1			
9				1	2			
7			2	1				
5			1					
3	1	2						
1	1							

$Kh(7_4; \mathbb{Q})$

	0	1	2	3	4	5	6	7
17								1
15								
13						2	1	
11					1			
9				1	2			
7			2	1				
5			1					
3	1	2						
1	1							

E^1_{Lee}

Lee spectral sequence example 3

$T(5,6)$

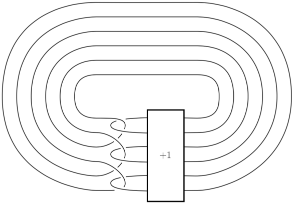
	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14
43														1	1
41													1	1	
39														1	
37												2	1		
35										2			1		
33								1		1	1				
31						1		1	2						
29						1	1		1						
27				1	1		1								
25					1										
23			1												
21	1														
19	1														

$Kh(T(5,6);\mathbb{Z}_3)$

	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14
43														1	1
41													1	1	
39														1	
37												2	1		
35										2			1		
33								1		1	1				
31						1		1	2						
29						1	1		1						
27				1	1		1								
25					1										
23			1												
21	1														
19	1														

E^1_{Lee}

Lee spectral sequence example 4



A positive full-twist goes in the $+1$ box.

	-18	-17	-16	-15	-14	-13	-12	-11	-10	-9	-8	-7	-6	-5	-4	-3	-2	-1	0	1	2	3	4
13																							1
11																						1	
9																				1	2	1	
7																			3	4	2		
5																		5	4	1			
3																1	6	6	4	1			
1															3	9	10	4	1	1			
-1														3	9	8	3	1	1				
-3													3	10	12	6	1						
-5											1	5	10	10	2		1						
-7									1	2	4	6	7	3	1								
-9								2	1	3	7	8	2										
-11							2	2	4	5	3												
-13						3	4	2	2	2	1												
-15					3	3	1	2	1														
-17				2	3	2	1																
-19			2	3	1																		
-21		1	2																				
-23		2																					
-25	1																						

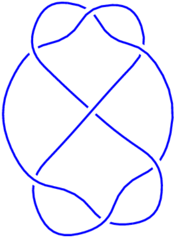
$Kh(K;\mathbb{Q})$

Manolescu and Marengon recently proved that there is a knot K where the Lee spectral sequence over $\mathbb{Q}$ collapses after the second page.

The Turner spectral sequence

- Bar-Natan defined a Khovanov-like homology theory with coefficients in $\mathbb{Z}_2$. For a knot K, the homology is always isomorphic to $\mathbb{Z}_2 \oplus \mathbb{Z}_2$.
- Turner showed there is a spectral sequence (E_T^r, d_T^r) such that $E_T^1 \cong Kh(K; \mathbb{Z}_2)$ and $E_T^\infty \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2$, the Bar-Natan homology of K.
- The differential d_T^r has bidegree $(1, 2r)$.

Turner spectral sequence example 1



7_4

	0	1	2	3	4	5	6	7
17								1
15							1	1
13						2	1	
11					3	2		
9				2	3			
7			3	2				
5		2	3					
3	1	2						
1	1							

$Kh(7_4; \mathbb{Z}_2)$

	0	1	2	3	4	5	6	7
17								1
15							1	1
13						2	1	
11					3	2		
9				2	3			
7			3	2				
5		2	3					
3	1	2						
1	1							

E_T^1

Turner spectral sequence example 2

$T(5,6)$

	0	1	2	3	4	5	6	7	8	9	10	11	12	13
41													1	1
39												1	1	1
37										1	1	2	1	
35										3	2	1	1	
33								1	1	2	1			
31						1	1	2	2					
29						1	2	1	1					
27				1	1		1							
25			1	1	1									
23			1											
21	1													
19	1													

$Kh(T(5,6);\mathbb{Z}_2)$

	0	1	2	3	4	5	6	7	8	9	10	11	12	13
41													1	1
39												1	1	1
37										1	1	2	1	
35										3	2	1	1	
33								1	1	2	1			
31						1	1	2	2					
29						1	2	1	1					
27				1	1		1							
25			1	1	1									
23			1											
21	1													
19	1													

E_T^1

Spectral sequences and the unknotting number

Let R be $\mathbb{Q}$ or $\mathbb{Z}_p$ where p is an odd prime. Define $p_{\text{Lee}}(K;R)$ and $p_T(K)$ be the pages where the Lee spectral sequence of K over R and the Turner spectral sequences of K collapse respectively. Let $u(K)$ be the unknotting number of K.

Theorem (Alishahi, Dowlin)

Let $R = \mathbb{Q}$ or $\mathbb{Z}_p$ where p is an odd prime. For any knot K,

$$p_{Lee}(K;R) \leq \left\lceil \frac{u(K)+2}{2} \right\rceil.$$

Theorem (Alishahi)

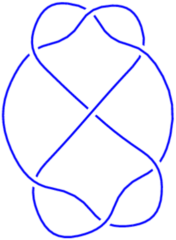
For any nontrivial knot K,

$$p_T(K) \leq u(K) + 1.$$

Kh-thin knots

Let R be $\mathbb{Z}$, $\mathbb{Q}$ or $\mathbb{Z}_p$ for a prime p. A knot is *Kh*-thin over R if there is an $s \in \mathbb{Z}$ such that $Kh^{i,j}(K;R) = 0$ for all i and j where $2i - j \neq s \pm 1$.

Lee proved that alternating knots are *Kh*-thin over all R.



7_4

	0	1	2	3	4	5	6	7
17								1
15								
13						2	1	
11					1			
9				1	2			
7			2	1				
5			1					
3	1	2						
1	1							

$Kh(7_4;\mathbb{Q})$

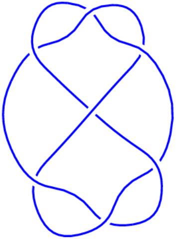
	0	1	2	3	4	5	6	7
17								1
15							1	1
13						2	1	
11					3	2		
9				2	3			
7			3	2				
5		2	3					
3	1	2						
1	1							

$Kh(7_4;\mathbb{Z}_2)$

Spectral sequences and *Kh*-thin knots

Suppose that K is a nontrivial Kh-thin knot. Then

$$p_{\text{Lee}}(K) = p_T(K) = 2.$$



7_4

	0	1	2	3	4	5	6	7
17								1
15								
13						2	1	
11					1			
9				1	2			
7			2	1				
5			1					
3	1	2						
1	1							

$Kh(7_4;\mathbb{Q})$

	0	1	2	3	4	5	6	7
17								1
15							1	1
13						2	1	
11					3	2		
9				2	3			
7			3	2				
5		2	3					
3	1	2						
1	1							

$Kh(7_4;\mathbb{Z}_2)$

Gordian distance and *Kh*-thin knots

Let $d_{\text{thin}}(K; R)$ be the minimum Gordian distance between K and any knot that is *Kh*-thin over R, that is

$$d_{\text{thin}}(K; R) = \min\{d_G(K, J) \mid J \text{ is } Kh\text{-thin over } R\}.$$

Since alternating knots are *Kh*-thin for all R, for every knot K

$$d_{\text{thin}}(K; R) \leq \text{alt}(K).$$

$p_{\mathrm{Lee}}(K;R)$ and $d_{\mathrm{thin}}(K;R)$

Theorem (L, Sazdanović)

Let R be $\mathbb{Q}$ or $\mathbb{Z}_p$ for an odd prime p. For any knot K, we have

$$p_{\mathrm{Lee}}(K;R) \leq \left\lceil \frac{d_{\mathrm{thin}}(K;R)+3}{2} \right\rceil \leq \left\lceil \frac{\mathrm{alt}(K)+3}{2} \right\rceil .$$

Corollary

If $p_{\mathrm{Lee}}(K;R) = 3$, then $2 \leq d_{\mathrm{thin}}(K;R) \leq \mathrm{alt}(K)$.

Alternation number at least two

$T(5,6)$

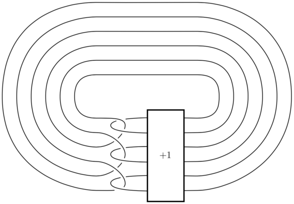
	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14
43														1	1
41													1	1	
39														1	
37												2	1		
35										2			1		
33								1		1	1				
31						1		1	2						
29						1	1		1						
27				1	1		1								
25					1										
23			1												
21	1														
19	1														

$Kh(T(5,6);\mathbb{Z}_3)$

	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14
43														1	1
41													1	1	
39														1	
37												2	1		
35										2			1		
33								1		1	1				
31						1		1	2						
29						1	1		1						
27				1	1		1								
25					1										
23			1												
21	1														
19	1														

E^1_{Lee}

Alternation number at least two



A positive full-twist goes in the $+1$ box.

	-18	-17	-16	-15	-14	-13	-12	-11	-10	-9	-8	-7	-6	-5	-4	-3	-2	-1	0	1	2	3	4
13																							1
11																						1	
9																				1	2	1	
7																			3	4	2		
5																		5	4	1			
3																1	6	6	4	1			
1															3	9	10	4	1	1			
-1														3	9	8	3	1	1				
-3													3	10	12	6	1						
-5											1	5	10	10	2		1						
-7									1	2	4	6	7	3	1								
-9								2	1	3	7	8	2										
-11							2	2	4	5	3												
-13						3	4	2	2	2	1												
-15					3	3	1	2	1														
-17				2	3	2	1																
-19			2	3	1																		
-21		1	2																				
-23		2																					
-25	1																						

$Kh(K;\mathbb{Q})$

$p_T(K)$ and $d_{\text{thin}}(K;\mathbb{Z}_2)$

Theorem (L, Sazdanović)

For any knot K, we have

$$p_T(K) \leq d_{\text{thin}}(K;\mathbb{Z}_2) + 2 \leq \text{alt}(K) + 2.$$

Corollary

If $p_T(K;R) = 4$, then $2 \leq d_{\text{thin}}(K;R) \leq \text{alt}(K)$.

$Kh(T(7,8);\mathbb{Z}_2)$

	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26
81																										1	1
79																										1	1
77																							1	1	1	1	
75																							2	2	1	1	
73																					2	3	2	2	1		
71																			1	2	4	5	2	1	1		
69																	1	1	2	5	3	2	1				
67																1	3	5	4	4	2						
65															1	4	3	5	4	1	1						
63													1	3	3	5	3	1	1								
61												1	2	6	4	2	2										
59										1	1	3	3	3	2												
57										3	3	3	3														
55								1	1	2	3	1	1														
53						1	1	2	2		1																
51						1	2	1	1																		
49				1	1		1																				
47			1	1	1																						
45			1																								
43	1																										
41	1																										

$Kh(T(7,8);\mathbb{Z}_2)$

	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26
81																										1	1
79																										1	1
77																							1	1	1	1	
75																							2	2	1	1	
73																					2	3	2	2	1		
71																			1	2	4	5	2	1	1		
69																	1	1	2	5	3	2	1				
67																1	3	5	4	4	2						
65															1	4	3	5	4	1	1						
63													1	3	3	5	3	1	1								
61												1	2	6	4	2	2										
59										1	1	3	3	3	2												
57										3	3	3	3														
55								1	1	2	3	1	1														
53						1	1	2	2		1																
51						1	2	1	1																		
49				1	1		1																				
47			1	1	1																						
45			1																								
43	1																										
41	1																										

Since $p_T(T(7,8)) = 4$, it follows that

$$2 \leq d_{\text{thin}}(T(7,8);\mathbb{Z}_2) \leq \text{alt}(T(7,8)).$$

The dealternating number

- Define $\mathrm{dalt}(D)$ to be the fewest number of crossing changes to change the knot diagram D into an alternating diagram.
- Adams et al. defines the *dealternating number* of K by

$$\mathrm{dalt}(K) = \min\{\mathrm{dalt}(D) \mid D \text{ is a diagram of } K\}.$$

- For each knot K,

$$\mathrm{alt}(K) \leq \mathrm{dalt}(K),$$

  and the gap between them can be arbitrarily large.

Turaev genus

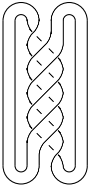
- The Turaev surface of a knot diagram D is an unknotted surface on which the knot has an alternating projection constructed from a cobordism between the all-A and all-B Kauffman states of D.
- For a knot diagram D, the genus of the Turaev surface is

$$g_T(D) = \frac{1}{2}\left(2 + c(D) - s_A(D) - s_B(D)\right)$$

  where $c(D)$ is the number of crossings in D and $s_A(D)$ and $s_B(D)$ are the number of components in the all-A and all-B Kauffman states of D respectively.
- The Turaev genus $g_T(K)$ of the knot K is

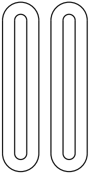
$$g_T(K) = \min\{g_T(D) \mid D \text{ is a diagram of } K\}.$$

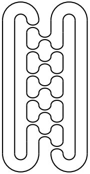
The Turaev surface



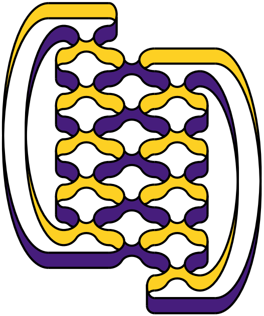
D

all-A state





all-B state



Turaev surface of D

$p_{\text{Lee}}(K; R)$, $p_T(K)$, $g_T(K)$, and $\text{dalt}(K)$

Theorem

Let R be $\mathbb{Q}$ or $\mathbb{Z}_p$ where p is an odd prime. For all nontrivial knots K,

$$p_{\text{Lee}}(K; R) - 2 \leq g_T(K) \leq \text{dalt}(K) \textit{ and}$$
$$p_T(K) - 2 \leq g_T(K) \leq \text{dalt}(K).$$

Sketch of proof.

This theorem follows from grading considerations, and the fact that the number of diagonals supporting $Kh(K)$ is bounded from above by $g_T(K) + 2$ and $\text{dalt}(K) + 2$.

□

Lee sequence and torsion in $Kh(K;\mathbb{Z})$

Theorem (L, Sazdanović)

Let K be a knot and p be an odd prime. If $p_{\text{Lee}}(K;\mathbb{Q}) < p_{\text{Lee}}(K;\mathbb{Z}_p)$, then the Khovanov homology $Kh(K;\mathbb{Z})$ has torsion of order p.

Idea of proof: If $Kh(K,\mathbb{Z})$ has no torsion of order p, then the differentials in the $\mathbb{Z}_p$ Lee spectral sequence are the differentials in the $\mathbb{Q}$ Lee spectral sequence tensored with $\mathbb{Z}_p$.

The converse fails

The Khovanov homology $Kh(T(5,6),\mathbb{Z})$ has torsion of order 5, but $p_{\text{Lee}}(T(5,6);\mathbb{Q}) = p_{\text{Lee}}(T(5,6);\mathbb{Z}_5) = 2$.

$T(5,6)$

	0	1	2	3	4	5	6	7	8	9	10	11	12	13
41													1	1
39														1
37												2	1	
35										2			1	
33								1		1	1			
31						1		1	2					
29						1	1		1					
27				1	1		1							
25					1									
23			1											
21	1													
19	1													

$Kh(T(5,6);\mathbb{Q})$

	0	1	2	3	4	5	6	7	8	9	10	11	12	13
41													1	1
39												1	1	1
37												2	1	
35										2	1	1	1	
33								1		1	1			
31						1		1	2					
29						1	1		1					
27				1	1		1							
25					1									
23			1											
21	1													
19	1													

$Kh(T(5,6);\mathbb{Z}_5)$

Questions

- Does $p_{\text{Lee}}(K; R)$ say anything about torsion of order p^k for $k > 1$?
- Does $p_T(K)$ say anything about 2^k torsion?
- Can the alternation lower bounds be improved by using more information from the Khovanov homology of K?

Thank you!