Are almost alternating links semi-adequate?

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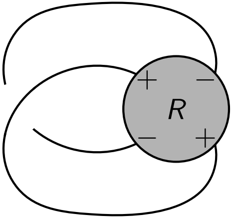
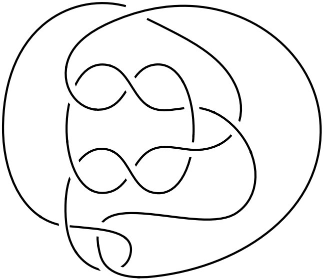
Almost alternating links

An *almost alternating link* is a non-alternating link with a diagram that can be transformed into an alternating diagram via a single crossing change. They were first defined by Colin Adams and a group of undergraduates as part of the SMALL summer research program.

Diagrams of almost alternating links



R is an alternating tangle.



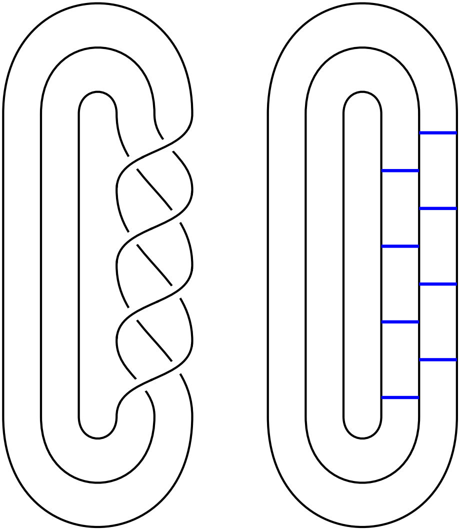
The $(3, 4)$-torus knot.

Semi-adequate links

A link diagram is *A-adequate* if no two arcs in the resolution of the same crossing lie on the same component of the of the all-A Kauffman state. Similarly, a link diagram is *B-adequate* if no two arcs in the resolution of the same crossing lie on the same component of the of the all-B Kauffman state. A link with either at least one A-adequate or B-adequate diagram is *semi-adequate*. A link with no A-adequate or B-adequate diagrams is *inadequate*.

Semi-adequate example

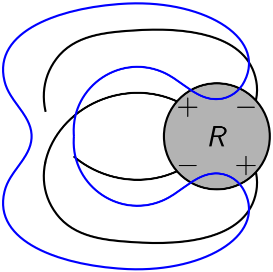
$T_{3,4}$



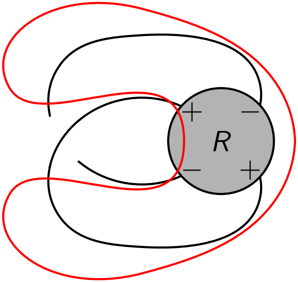
all-A state

An observation

An almost alternating diagram is never a semi-adequate diagram.



all-A state



all-B state

Another observation

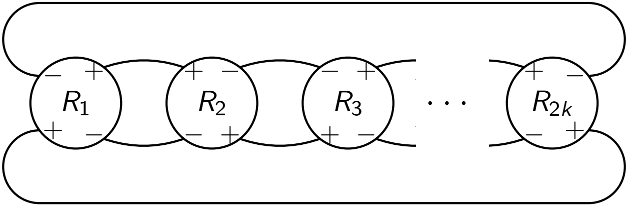
There are non-alternating, semi-adequate links that are not almost alternating.
The $(3, 7)$-torus knot is semi-adequate, but is not almost alternating. Its Khovanov homology is supported on four adjacent diagonals, but Asaeda and Przytycki proved that the Khovanov homology of an almost alternating link is supported on at most three adjacent diagonals.

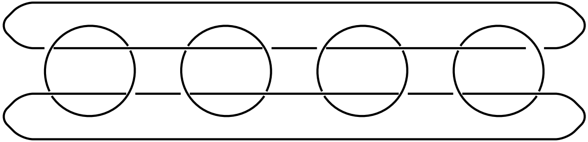
The titular question

Question. Is every almost alternating link semi-adequate?

Links of Turaev genus one

Turaev genus one links are a generalization of Montesinos links.





Turaev genus one links

- Every almost alternating link is Turaev genus one.
- It is unknown whether every Turaev genus one link is almost alternating.
- Kim proved that every inadequate Turaev genus one is almost alternating.
- The question becomes: is Kim's result about the empty set?

Evidence

- Many theorems satisfied by semi-adequate links are also satisfied by almost alternating links.
- The remainder of the talk will discuss several such theorems.
- Use the above theorems to show certain links are neither semi-adequate nor almost alternating.

Extreme terms of the Jones polynomial

Theorem (Lickorish, Thistlethwaite)

Let L be a semi-adequate link whose Jones polynomial is

$$V_L(t) = a_m t^m + a_{m+1} t^{m+1} + \cdots + a_{n-1} t^{n-1} + a_n t^n,$$

where a_m and a_n are nonzero. Then at least one of $|a_m|$ or $|a_n|$ is one.

Extreme terms of the Jones polynomial

Theorem (Dasbach, L.)

Let L be an almost alternating link whose Jones polynomial is

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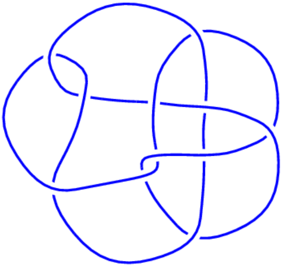
where a_m and a_n are nonzero. Then at least one of $|a_m|$ or $|a_n|$ is one.

Example: $11n_{95}$



$$V_{11n_{95}}(t) = 2t^2 - 3t^3 + 5t^4 - 6t^5 + 6t^6 - 5t^7 + 4t^8 - 2t^9$$

Penultimate terms of the Jones polynomial

Theorem (Stoimenow)

Let L be a semi-adequate link whose Jones polynomial is

$$V_L(t) = a_m t^m + a_{m+1} t^{m+1} + \cdots + a_{n-1} t^{n-1} + a_n t^n,$$

where a_m and a_n are nonzero. At least one of the following holds

- $|a_m| = 1$ *and* $a_m a_{m+1} \leq 0$ *or*
- $|a_n| = 1$ *and* $a_n a_{n-1} \leq 0$.

Penultimate terms of the Jones polynomial

Theorem (L., Spyropoulos)

Let L be an almost alternating link whose Jones polynomial is

$$V_L(t) = a_m t^m + a_{m+1} t^{m+1} + \cdots + a_{n-1} t^{n-1} + a_n t^n,$$

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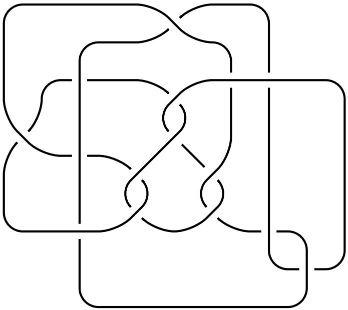
Example $K = 15n_{41133}$



$$V_K(t) = t^4 + t^5 - 3t^6 + 8t^7 - 12t^8 + 14t^9 \\ - 15t^{10} + 13t^{11} - 10t^{12} + 6t^{13} - 2t^{14}.$$

Non-triviality of the Jones polynomial

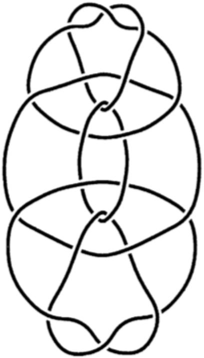
Theorem (Stoimenow)

If L is a c-component semi-adequate link, then $V_L(t)$ is not equal to the Jones polynomial of the c-component unlink.

Theorem (L., Spyropoulos)

If L is a c-component almost alternating link, then $V_L(t)$ is not equal to the Jones polynomial of the c-component unlink.

Example: Eliahou, Kauffman, Thistlethwaite



$$V_L(t) = \left(-t^{1/2} - t^{-1/2}\right)^3$$

Extremal Khovanov homology

Theorem (Khovanov, Abe)

Let L be a non-split semi-adequate link. Either

$$\bigoplus_{i\in\mathbb{Z}} Kh^{i,j_{\min}}(L) \cong \mathbb{Z} \text{ or } \bigoplus_{i\in\mathbb{Z}} Kh^{i,j_{\max}}(L) \cong \mathbb{Z}.$$

Moreover, the summand of $\mathbb{Z}$ above appears on the penultimate diagonal.

Extremal Khovanov homology

Theorem (Dasbach, L.)

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Moreover, the summand of $\mathbb{Z}$ above appears on the penultimate diagonal.

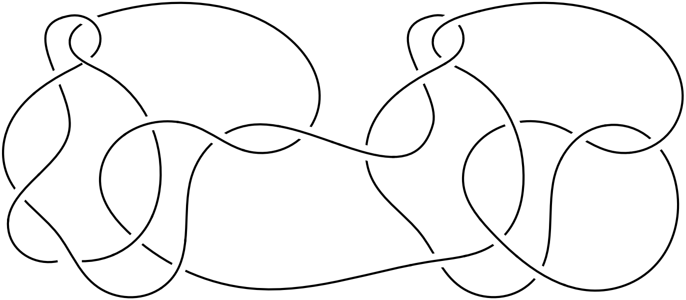
Example: the $(3, 4)$-torus knot

$j \backslash i$	0	1	2	3	4	5
17						1
15						1
13				1	1	
11				1_2	1	
9			1			
7	1					
5	1					

Example: the $(3, 4)$-torus knot

$j \backslash i$	0	1	2	3	4	5
17						1
15						1
13				1	1	
11				1_2	1	
9			1			
7	1					
5	1					

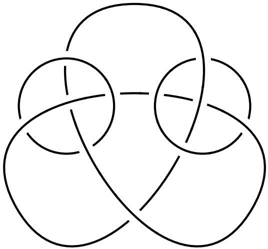
Example: $10_{132} \# \overline{10_{132}}$



Example: $Kh(10_{132}\#\overline{10_{132}})$

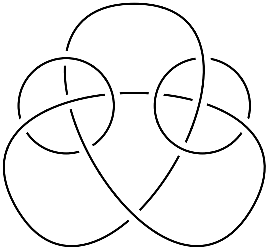
$j\backslash i$	-7	-6	-5	-4	-3	-2	-1	0	1	2	3	4	5	6	7
13														1	1
11													1	1_2	1_2
9												2	$3,1_2$	1	
7										1	5	$3,2_2$	2_2		
5									1	$3,1_2$	$3,5_2$	$2,2_2$			
3								1	$7,1_2$	$8,3_2$	$2,1_2$				
1							3	$8,1_2$	$4,6_2$	$1,3_2$					
-1						1	$4,3_2$	$8,6_2$	$3,1_2$						
-3					2	$8,1_2$	$7,3_2$	$1,1_2$							
-5				2	$3,2_2$	$3,5_2$	$1,1_2$								
-7				$3,2_2$	$5,2_2$	1									
-9		1	3	$2,1_2$											
-11		1_2	$1,1_2$												
-13	1	1													

Example: $11n_{376}$



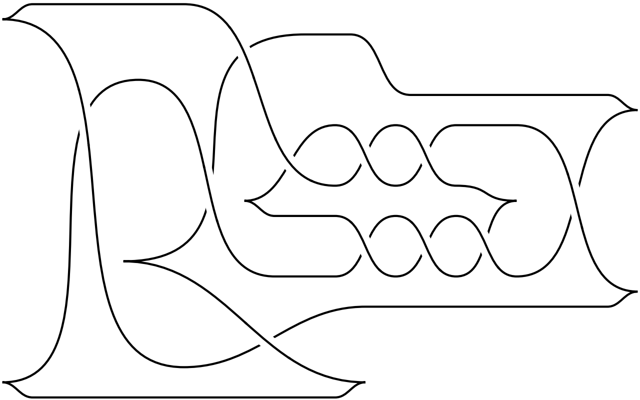
$j \backslash i$	-7	-6	-5	-4	-3	-2	-1	0
-3								2
-5							1	2
-7						2	1_2	
-9				1	1	$1, 2_2$		
-11				4	$2, 1_2$			
-13				$2, 2_2$				
-15		1	2					
-17		1_2						
-19	1							

Example: $11n_{376}$



$j \backslash i$	-7	-6	-5	-4	-3	-2	-1	0
-3								2
-5							1	2
-7						2	1_2	
-9				1	1	$1, 2_2$		
-11				4	$2, 1_2$			
-13				$2, 2_2$				
-15		1	2					
-17		1_2						
-19	1							

Legendrian front diagram: $(3,4)$-torus knot



Thurston Bennequin numbers

Let $\mathcal{L}$ be a Legendrian link with front diagram F. The *Thurston Bennequin number* $\mathrm{tb}(\mathcal{L})$ of $\mathcal{L}$ is the difference between the writhe of F and half the number of cusps of F. The maximal Thurston Bennequin number of a classical link L is

$$\overline{\mathrm{tb}}(L) = \max\{\mathrm{tb}(\mathcal{L}) \mid \mathcal{L} \text{ has topological type } L\}.$$

$\overline{\mathrm{tb}}(L)$ for alternating and semi-adequate

Theorem (Ng)

Let L be a link with reduced alternating diagram D. Then

$$\overline{\mathrm{tb}}(L) = w(D) - s_A(D).$$

Theorem (Kálmán)

Let L be a link with diagram D. If D is A-adequate, then

$$\overline{\mathrm{tb}}(L) = w(D) - s_A(D).$$

If D is B-adequate, then

$$\overline{\mathrm{tb}}(\overline{L}) = -w(D) - s_B(D).$$

$\overline{\text{tb}}(L)$ for almost alternating

Theorem (L., Dasbach)

Let L be a link with strongly reduced almost alternating diagram D. At least one of the following inequalities hold:

$$\begin{aligned} w(D) - s_A(D) \;&\leq\; \overline{\text{tb}}(L) \;\leq\; w(D) - s_A(D) + 1, \ \textit{or} \\ -w(D) - s_B(D) \;&\leq\; \overline{\text{tb}}(\overline{L}) \;\leq\; -w(D) - s_B(D) + 1. \end{aligned}$$

Closing thoughts

1. Versions of the theorems discussed also hold for Turaev genus one links.
2. The Kauffman polynomial and colored Jones polynomials of semi-adequate links have special forms. So far this is unknown for almost alternating links.
3. Many of the invariants discussed are mutation invariant. Perhaps every almost alternating link is mutant to a semi-adequate link.

Thank you!