Khovanov homology of oriented ribbon graphs

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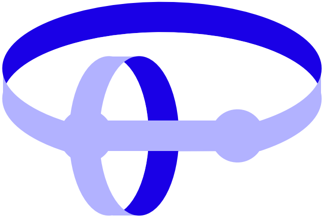
Ribbon graphs

A ribbon graph $\mathbb{G}$ is a surface with boundary represented as the union of two sets of topological disks, the set of vertices $V(\mathbb{G})$ and the set of edges $E(\mathbb{G})$ such that

- the vertices and edges intersect in disjoint arcs,
- each such arc lies on the boundary of one vertex and one edge, and
- every edge contains exactly two such arcs.

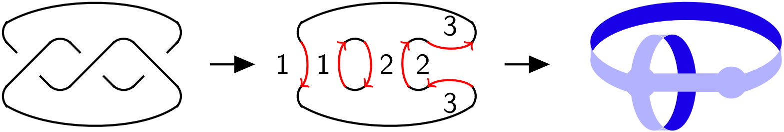


From links to ribbon graphs

From a link diagram L construct the *all-A ribbon graph* $\mathbb{G}$ as follows.

1. Choose the A-resolution for each crossing of L, leaving red arcs that point in opposite directions.
2. The components of the all-A resolution become the boundaries of the vertex disks.
3. The edge bands (disks) are attached to the vertices so that the red arcs either both agree or both disagree with the boundary orientation of the edge band.

A 3-crossing unknot



Links to ribbon graphs (and back)

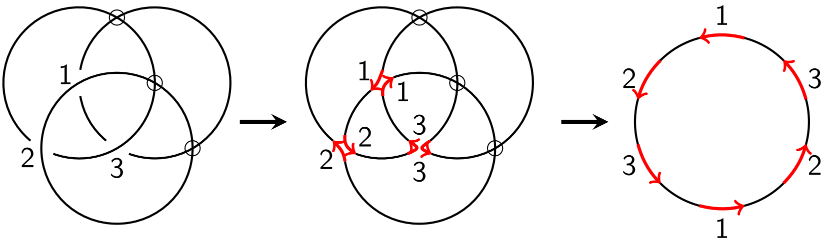
- The all-A ribbon graph $\mathbb{G}$ of a link diagram is oriented.
- There exists oriented ribbon graphs that are not that all-A ribbon graphs of any link diagram.
- (Moffatt) If $L \neq L'$ have the same all-A ribbon graph, then L and L' are diagrams of the same composite link.

Virtual links to ribbon graphs

From a virtual link diagram L, Chmutov and Pak construct the *all-A ribbon graph* $\mathbb{G}$ as follows.

1. Choose the A-resolution for each classical crossing of L, leaving red arcs that point in opposite directions.
2. Untwist all virtual crossings of the all-A resolution. The resulting untwisted components become the boundaries of the vertex disks .
3. The edge bands (disks) are attached to the vertices so that the red arcs either both agree or both disagree with the boundary orientation of the edge band.

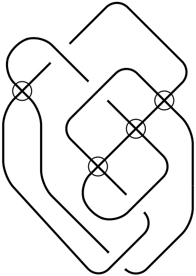
An example for a virtual diagram

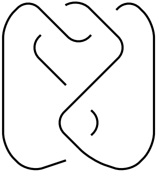


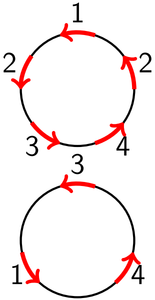
Virtual links to ribbon graphs (and back)

- The all-A ribbon graph of a virtual diagram may be orientable or non-orientable.
- The all-A ribbon graph of a virtual diagram L is orientable if and only if L is *alternatable*, i.e. L can be transformed into an alternating diagram via some number of crossing changes.
- Every ribbon graph $\mathbb{G}$ has (many) virtual diagrams whose all-A ribbon graphs are $\mathbb{G}$. There exists distinct virtual links with diagrams that have the same all-A ribbon graph.

Two distinct diagrams with the same all-A ribbon graph







Khovanov homology for oriented ribbon graphs

Khovanov homology for oriented ribbon graphs is similar to Khovanov homology for links.

- Replace Kauffman states with spanning ribbon subgraphs.
- Instead of changing smoothings at a single crossing, we add a single edge to the spanning ribbon subgraph.
- Instead of counting components in each Kauffman state, we count the number of boundary components of spanning ribbon subgraph.

Khovanov homology for the oriented ribbon graph $\mathbb{G}$

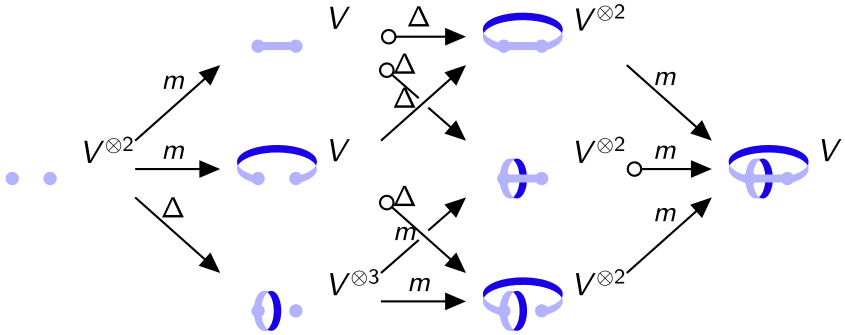
- Let $V = \mathbb{Z}[x]/(x^2)$, and associate to each spanning ribbon subgraph $\mathbb{H}$ of $\mathbb{G}$ the $\mathbb{Z}$-module $V^{\otimes bc(\mathbb{H})}$, where $bc(\mathbb{H})$ is the number of boundary components of $\mathbb{H}$.

- Arrange the spanning ribbon subgraphs into a hypercube with a directed edge in the hypercube corresponding to adding a single edge of the ribbon graph.

- Let
$$CKh(\mathbb{G}) = \bigoplus_{\mathbb{H}} V^{\otimes bc(\mathbb{H})}.$$
Define a differential d on $CKh(\mathbb{G})$ based on edges of the hypercube in the same manner as Khovanov homology for links.

- Define $Kh(\mathbb{G})$ to be the homology of $(CKh(\mathbb{G}), d)$.

An example of $Kh(\mathbb{G})$



Comparison with $Kh(L)$

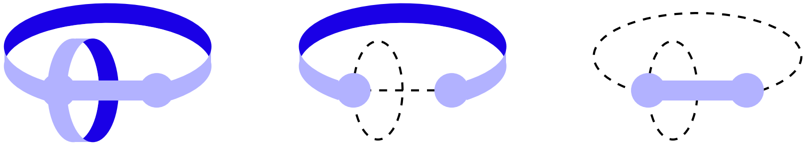
Theorem (Dasbach - L.)

Let $\mathbb{G}$ be an oriented ribbon graph such that $\mathbb{G}$ is the all-A ribbon graph of some classical or virtual link diagram L. After some overall grading shift, there is an isormorphism

$$Kh(\mathbb{G}) \cong Kh(L).$$

Spanning quasi-trees

A *spanning quasi-tree* $\mathbb{T}$ of $\mathbb{G}$ is a spanning ribbon subgraph such that $bc(\mathbb{T}) = 1$.



Spanning trees and spanning quasi-trees

$$\begin{tikzcd}[column sep=huge, row sep=huge]
\begin{array}{c}\text{Reduced Khovanov}\\ \text{complex of } L\end{array} \arrow[r, leftrightarrow] \arrow[d, leftrightarrow, "\text{(C-K, W)}"'] & \begin{array}{c}\text{Reduced Khovanov}\\ \text{complex of } \mathbb{G}\end{array} \arrow[d, leftrightarrow] \\
\begin{array}{c}\text{Spanning tree}\\ \text{complex of } L\end{array} \arrow[r, leftrightarrow, "\text{(C-K-S)}"] & \begin{array}{c}\text{Spanning quasi-tree}\\ \text{complex of } \mathbb{G}\end{array}
\end{tikzcd}$$

Spanning quasi-trees and $Kh(\mathbb{G})$

Theorem (Dasbach - L.)

There is a complex whose generators are in one-to-one correspondence with the spanning quasi-trees of $\mathbb{G}$ *and whose homology is the reduced Khovanov homology of* $\mathbb{G}$*. Moreover, the diagonal grading in this complex corresponds to the genus of the spanning quasi-tree.*

Corollary

The Jones polynomial of L *can be expressed as*

$$V_L(q) = \sum_{\mathbb{T}} (-1)^{i(\mathbb{T})} q^{j(\mathbb{T})}.$$

Width of $Kh(\mathbb{G})$

The *width* of $Kh(\mathbb{G})$ is the number of slope 2 diagonals in the bi-grading that support $Kh(\mathbb{G})$. The *Turaev genus* $g_T(L)$ of L is the minimum genus of any all-A ribbon graph of L.

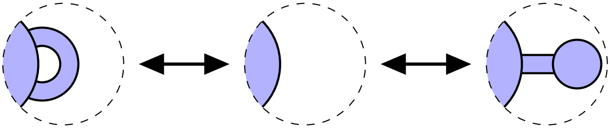
Theorem (Dasbach - L.)

Let L be a (classical or virtual) link with a diagram whose all-A ribbon graph is $\mathbb{G}$. Then

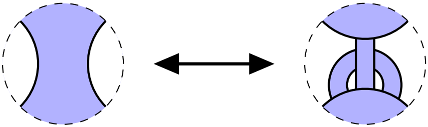
$$\begin{aligned} width(Kh(\mathbb{G})) - 2 &\leq genus(\mathbb{G}) \text{ and} \\ width(Kh(\mathbb{G})) - 2 &\leq g_T(L). \end{aligned}$$

Reidemeister moves for ribbon graphs

RI:



RII:



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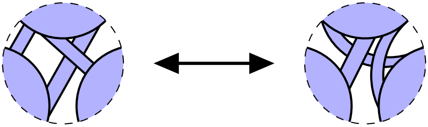

Reidemeister moves and links

- If L and L' are two diagrams of the same classical link, then there is a sequence of ribbon graph Reidemeister moves transforming the all-A ribbon graph of L into the all-A ribbon graph of L'.
- If L and L' are virtual link diagrams whose all-A ribbon graphs $\mathbb{G}$ and $\mathbb{G}'$ are oriented, then there is a sequence of ribbon graph Reidemeister moves transforming $\mathbb{G}$ into $\mathbb{G}'$.

Reidemeister moves and $Kh(\mathbb{G})$

Theorem (Dasbach - L.)

If $\mathbb{G}$ and $\mathbb{G}'$ are oriented ribbon graphs such that $\mathbb{G}$ can be transformed into $\mathbb{G}'$ by a ribbon graph Reidemeister move, then after a grading shift

$$Kh(\mathbb{G}) \cong Kh(\mathbb{G}').$$

Questions

1. Does there exist classical links $L \neq L'$ with all-A ribbon graphs $\mathbb{G}$ and $\mathbb{G}'$ such that $\mathbb{G}$ can be transformed into $\mathbb{G}'$ by a sequence of ribbon graph Reidemeister moves?
2. What can be done with non-orientable ribbon graphs?