Measuring a knot's distance from alternating

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Distance from alternating

A real valued knot invariant $d(K)$ measures a knot's distance from alternating if

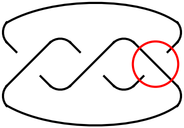
- $d(K) \geq 0$ for every knot K,
- $d(K) = 0$ if and only if K is alternating, and
- $d(K_1 \# K_2) \leq d(K_1) + d(K_2)$ for all knots K_1 and K_2.

Let's meet the invariants

1. The dealternating number of K, denoted $\mathrm{dalt}(K)$, defined by Adams.
2. The alternation number of K, denoted $\mathrm{alt}(K)$, defined by Kawauchi.
3. The alternating genus of K, denoted $g_{\mathrm{alt}}(K)$, defined by Adams.
4. The Turaev genus of K, denoted $g_T(K)$, defined by Turaev and Dasbach et. al.

The dealternating number

Let D be a diagram of K. The dealternating number of D, denoted $\mathrm{dalt}(D)$, is the minimum number of crossing changes required to transform D into an alternating diagram.



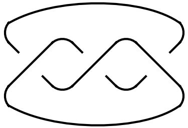
$\mathrm{dalt}(D) = 1$

The dealternating number of K is defined as

$$\mathrm{dalt}(K) = \min\{\mathrm{dalt}(D) \mid D \text{ is a diagram of } K\}.$$

The alternation number

Let D be a diagram of K. The alternation number of D, denoted $\text{alt}(D)$, is the minimum number of crossing changes required to transform D into a (possibly non-alternating) diagram of an alternating knot.



$\text{alt}(D) = 0$

The alternation number of K is defined as

$$\text{alt}(K) = \min\{\text{alt}(D) \mid D \text{ is a diagram of } K\}.$$

Our first inequality

$$\operatorname{alt}(K) \leq \operatorname{dalt}(K)$$

Question. Are there knots where $\operatorname{dalt}(K) - \operatorname{alt}(K)$ is arbitrarily large?
Answer. Yes, more on this later.

The alternating genus

Let Σ be a Heegaard surface in S^3. Suppose that K lies in a neighborhood $\Sigma \times [-\varepsilon, \varepsilon]$ of Σ, and let $\pi : \Sigma \times [-\varepsilon, \varepsilon] \to \Sigma$ be the projection map. Suppose that

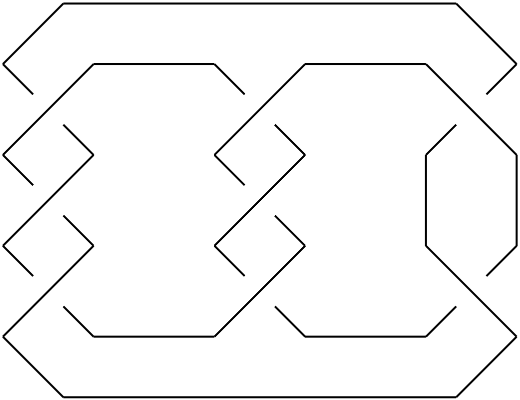
- $\pi(K)$ is alternating on Σ, and
- $\Sigma - \pi(K)$ is a collection of disks.

The alternating genus of K, denoted $g_{\text{alt}}(K)$, is the minimum genus surface onto which K has such a projection.

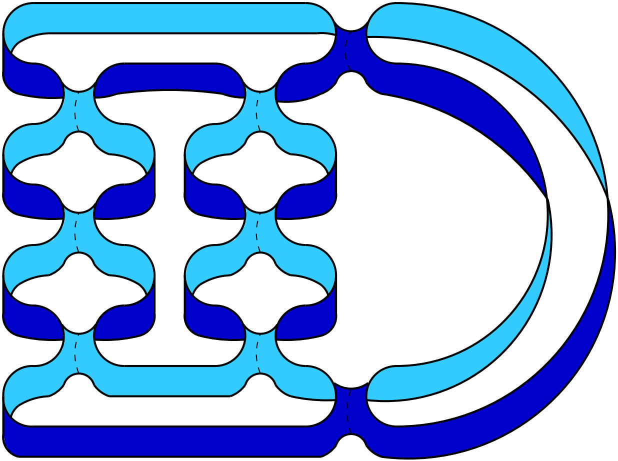
The Turaev surface

Every knot diagram D has an associated Turaev surface Σ_D constructed as follows. Think of the diagram D as a subset of S^2, which itself is a subset of S^3. Replace the crossings of D with saddles so that the A-smoothing lies on one side of S^2 and the B-smoothing lies on the other side of S^2. Replace the arcs of D not near crossings with bands orthogonal to S^2. The resulting surface is a cobordism between the all-A Kauffman state and the all-B Kauffman state. The Turaev surface Σ_D is obtained by capping off the boundary components of this cobordism with disks.

A picture is worth a thousand (really 96) words



A picture is worth a thousand (really 96) words



Turaev genus

The Turaev genus of K, denoted $g_T(K)$, is defined as

$$g_T(K) = \min\{g(\Sigma_D) \mid D \text{ is a diagram of } K\}.$$

Another inequality

Dasbach, Futer, Kalfagianni, Lin, and Stoltzfus proved that

- the Turaev surface Σ_D is a Heegaard surface in S^3,
- K has an alternating projection to Σ_D, and
- the complement of the projection of K to Σ_D is a collection of disks.

Resulting inequality.

$$g_{\text{alt}}(K) \leq g_T(K).$$

Question. Are there knots where $g_T(K) - g_{\text{alt}}(K)$ is arbitrarily large?
Answer. Yes, again more on this later.

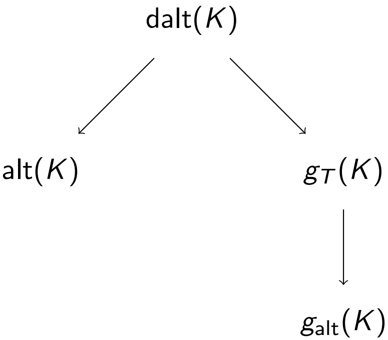
Abe's inequality

By studying the behavior of the Turaev surface under crossing changes, Abe proved that

$$g_T(K) \leq \mathrm{dalt}(K).$$

Question. Are there knots where $\mathrm{dalt}(K) - g_T(K)$ is arbitrarily large?
Easier Question. Are there knots where $g_T(K) < \mathrm{dalt}(K)$?
Answer to both. I don't know.

Our inequalities

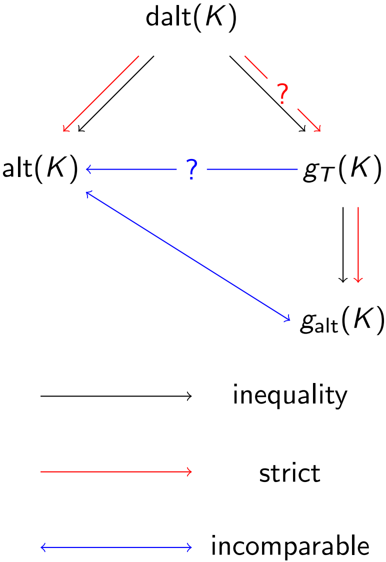


What we will show

Let d_1 and d_2 be two different measures of a knot's distance from alternating. We decorate the previous slide with a line from d_1 to d_2 as follows.

- Strict ($\longrightarrow$): $d_1(K) \geq d_2(K)$ for all K and there exists a knot K' with $d_1(K') > d_2(K')$.
- Possibly strict ($\overset{?}{\longrightarrow}$): $d_1(K) \geq d_2(K)$ for all K and it is unknown whether there exists a knot K' with $d_1(K') > d_2(K')$.
- Incomparable ($\longleftrightarrow$): There exists knots K_1 and K_2 such that $d_1(K_1) < d_2(K_1)$ and $d_1(K_2) > d_2(K_2)$.
- Possibly comparable ($\overset{?}{\longrightarrow}$): It is unknown whether $d_1(K) \geq d_2(K)$ for all knots.

What we will show



Our invariants and iterated Whitehead doubles

Theorem

Let W_k be the untwisted k-th iterated Whitehead double of the Figure-8 knot. Then $g_{alt}(W_k) > 1$ for all k. Also

$$g_T(W_k) - alt(W_k) \to \infty \text{ as } k \to \infty$$

and consequently

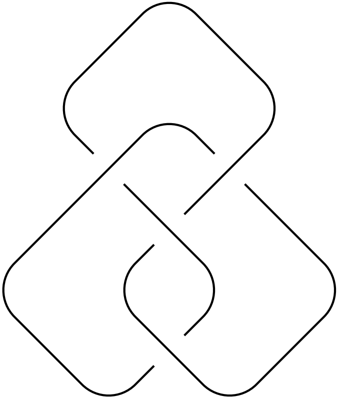
$$dalt(W_k) - alt(W_k) \to \infty \text{ as } k \to \infty.$$

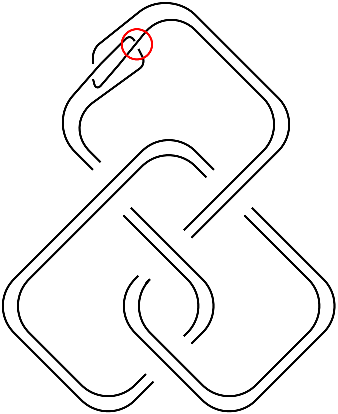
Proof

Adams showed that if K is prime and $g_{\text{alt}}(K) = 1$, then K is a hyperbolic knot. Since W_k is a prime, non-alternating, satellite knot, it follows that $g_{\text{alt}}(W_k) > 1$ for all k.

Proof, continued

For each k, we have $\operatorname{alt}(W_k) = 1$ since the unknotting number of W_k is one for all k.





Proof, continued

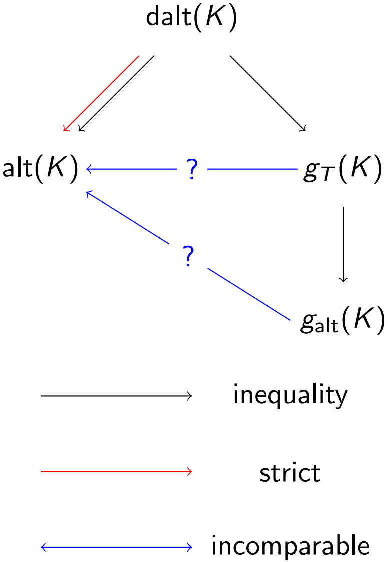
- Hedden proved that the width of the knot Floer homology of W_k, denoted width$(\widehat{HFK}(W_k))$, is k.
- For any knot K,

$$\text{width}(\widehat{HFK}(K)) - 1 \leq g_T(K).$$

- Thus $g_T(W_k) \geq k - 1$, and so $g_T(W_k) - \text{alt}(W_k) \to \infty$ as $k \to \infty$.
- Abe's inequality ($g_T(K) \leq \text{dalt}(K)$) implies that $\text{dalt}(W_k) - \text{alt}(W_k) \to \infty$ as $k \to \infty$.

□

What we've shown so far



A modified torus knot $\widetilde{T}_{p,q}$

Let B_p be the braid group on p-strands, and let $\Delta_p \in B_p$ denote the braid

$$\Delta_p = \sigma_1\sigma_2\sigma_3\cdots\sigma_{p-1}.$$

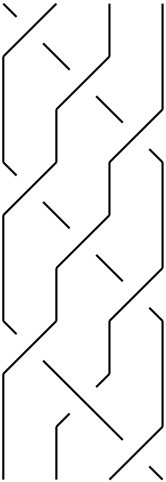
Let $\widetilde{\Delta}_p$ denote the braid

$$\widetilde{\Delta}_p = \sigma_1\sigma_2^{-1}\sigma_3\cdots\sigma_{p-1}^{(-1)^p}.$$

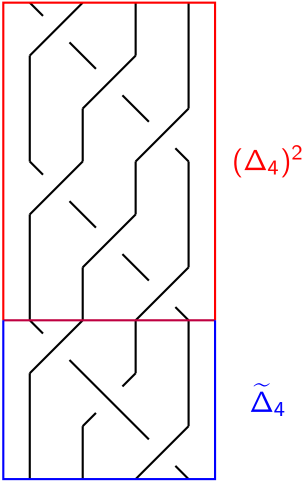
Assume p and q are positive and coprime. Then the (p,q)-torus knot $T_{p,q}$ is the closure of the braid $(\Delta_p)^q$. Define $\widetilde{T}_{p,q}$ to be the closure of the braid

$$(\Delta_p)^{(q-1)}\widetilde{\Delta}_p.$$

Example: $\widetilde{T}_{4,3}$



Example: $\widetilde{T}_{4,3}$



$\widetilde{T}_{p,q}$ and our invariants

Theorem

Let p and q be coprime integers both greater than 2. For any such fixed p,

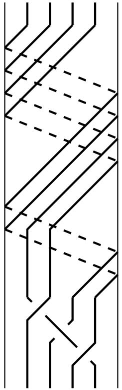
$$g_T(\widetilde{T}_{p,q}) - g_{alt}(\widetilde{T}_{p,q}) \to \infty \text{ as } q \to \infty$$

and

$$alt(\widetilde{T}_{p,q}) - g_{alt}(\widetilde{T}_{p,q}) \to \infty \text{ as } q \to \infty.$$

Sketch of proof

We see in the following picture that $g_{\mathrm{alt}}(\widetilde{T}_{4,7}) = 1$.



One can show that $g_{\mathrm{alt}}(\widetilde{T}_{p,q}) = 1$ similarly.

Sketch of proof, continued

Alternation number and Turaev genus have a common lower bound:

$$\frac{|s(K)+\sigma(K)|}{2} \leq \text{alt}(K) \text{ (due to Abe)}$$
$$\frac{|s(K)+\sigma(K)|}{2} \leq g_T(K) \text{ (due to Dasbach-L.)}$$

where $s(K)$ is the Rasmussen invariant and $\sigma(K)$ is the signature of the knot.
One can show that this lower bound goes to infinity as $q \to \infty$. $\square$

Summary