Signatures and Turaev genera of knots

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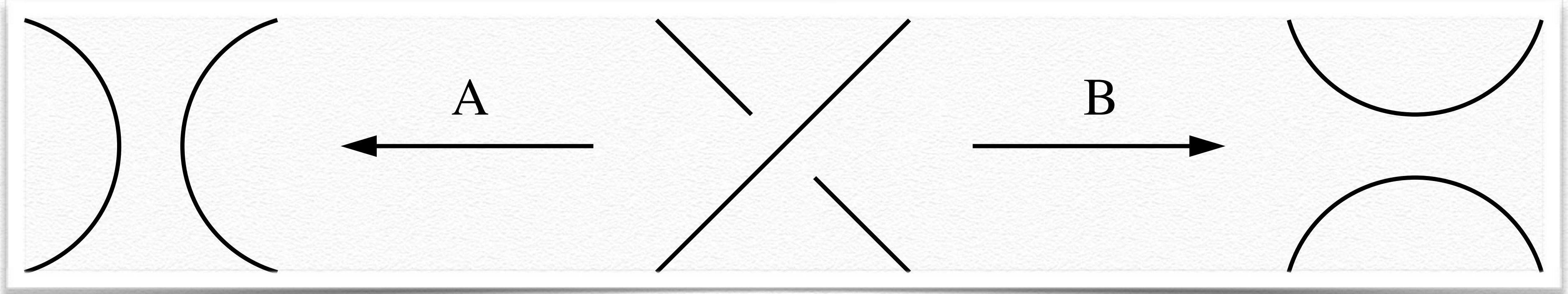
Joint with Cody Armond and Oliver Dasbach

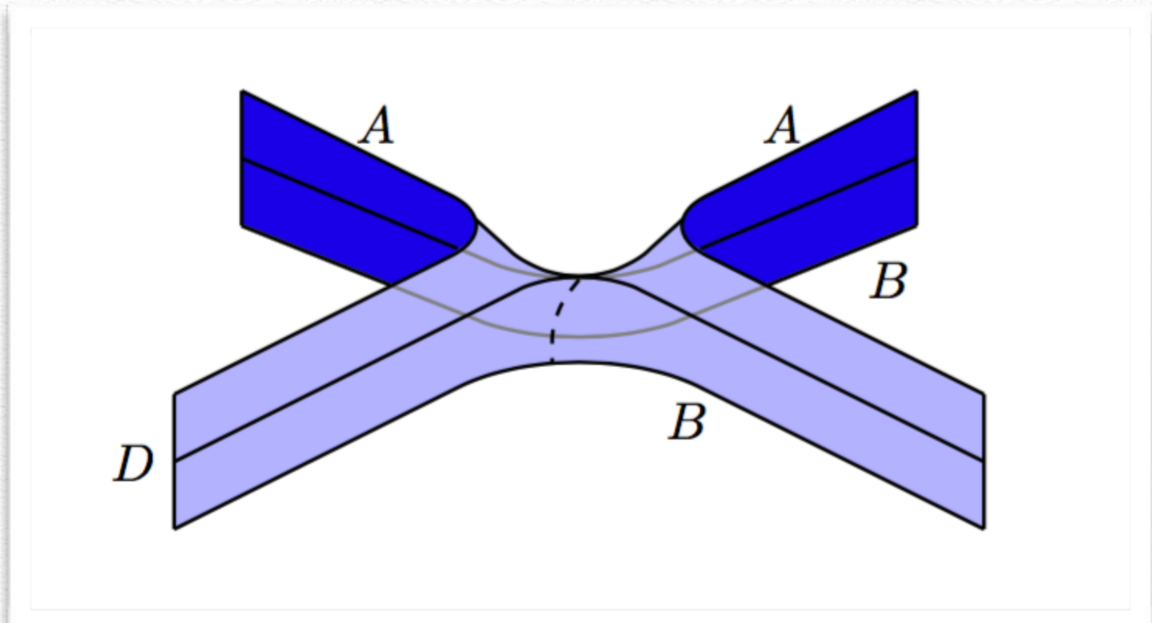
Knots in Washington XLI, December 5, 2015,
George Washington University

Turaev surfaces

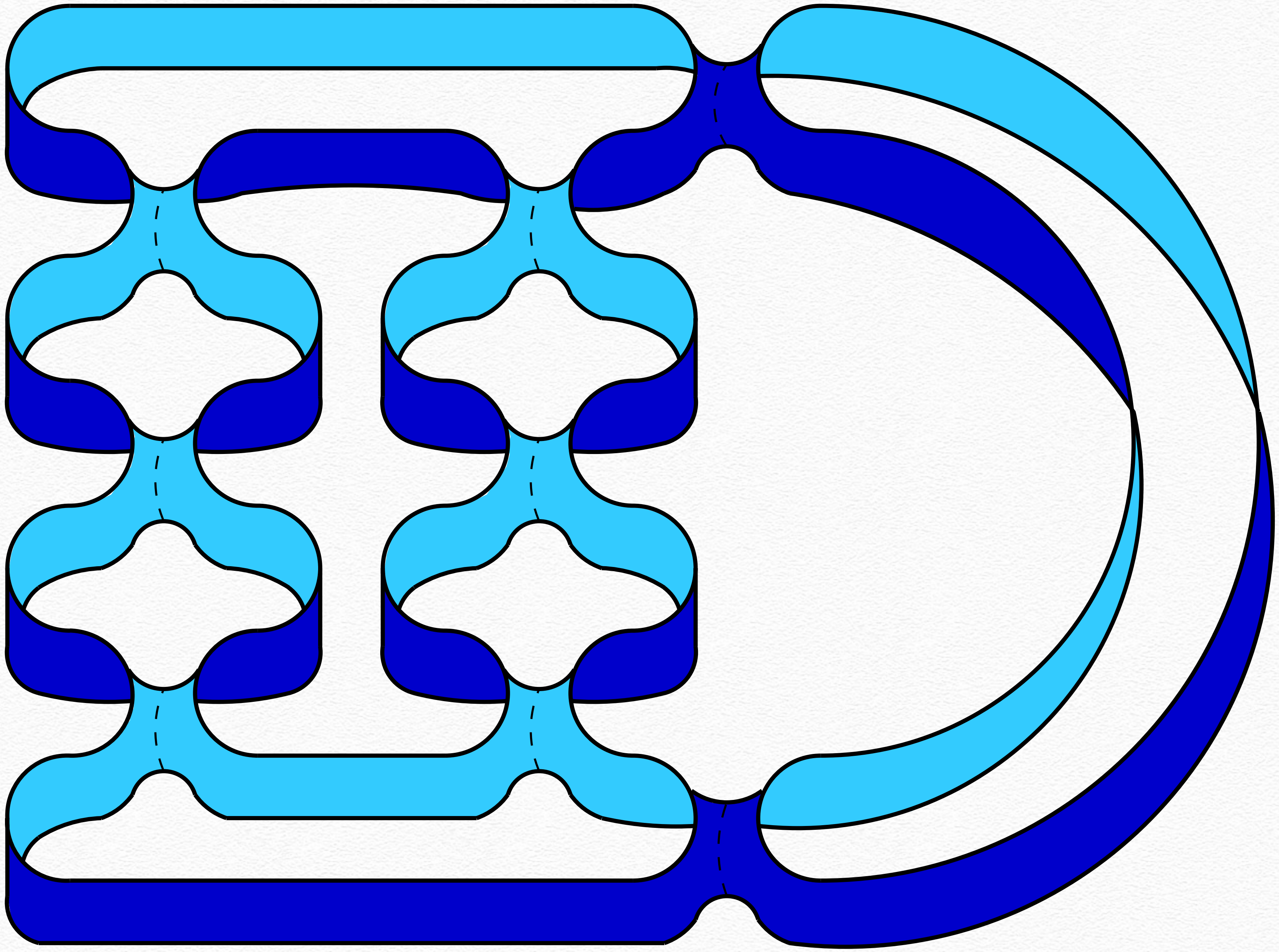
The ***A***-resolution and ***B***-resolution of a crossing:



Create Turaev surface from link diagram:



Turaev surface: *P(3,3,-2)*

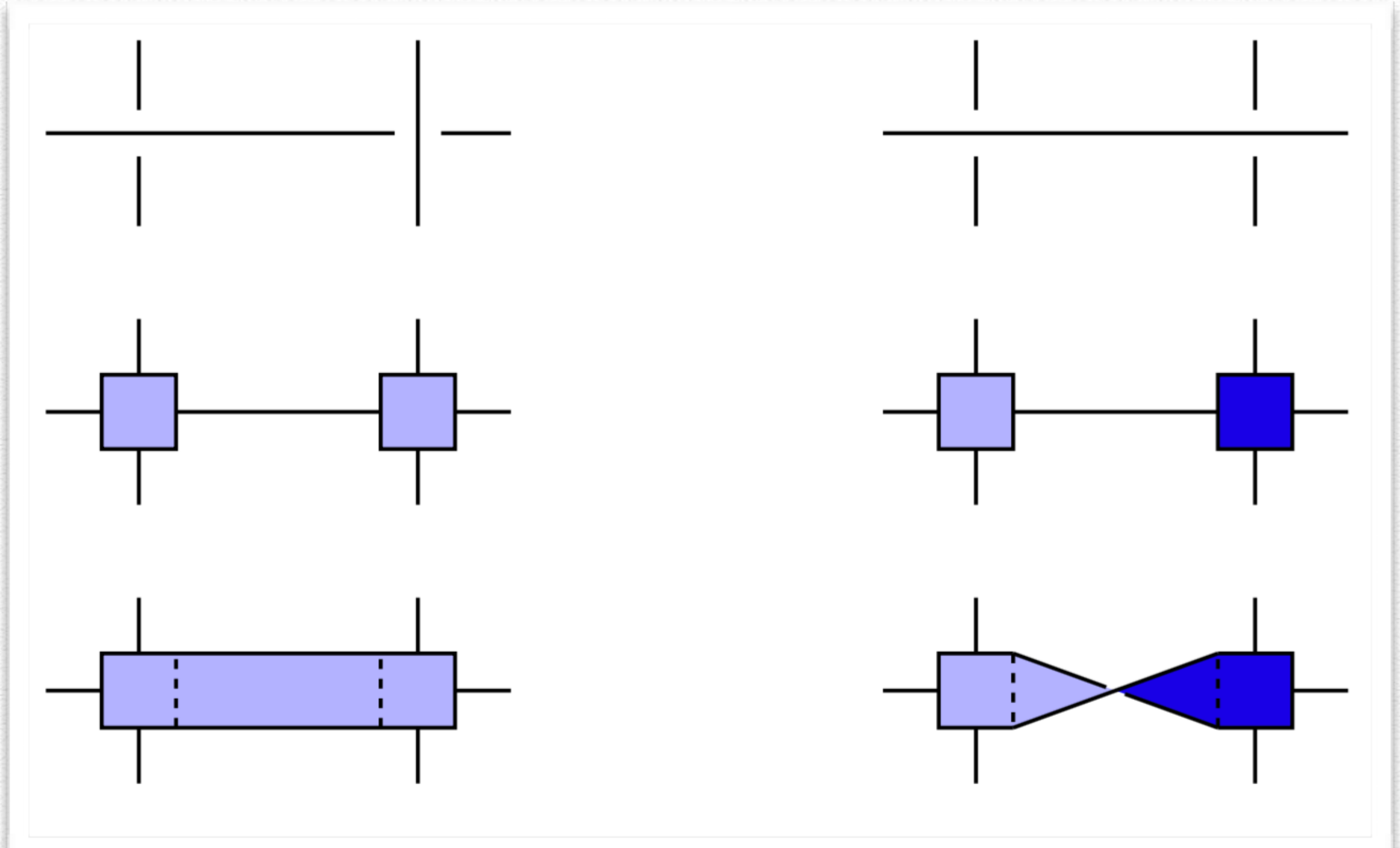


Turaev surfaces

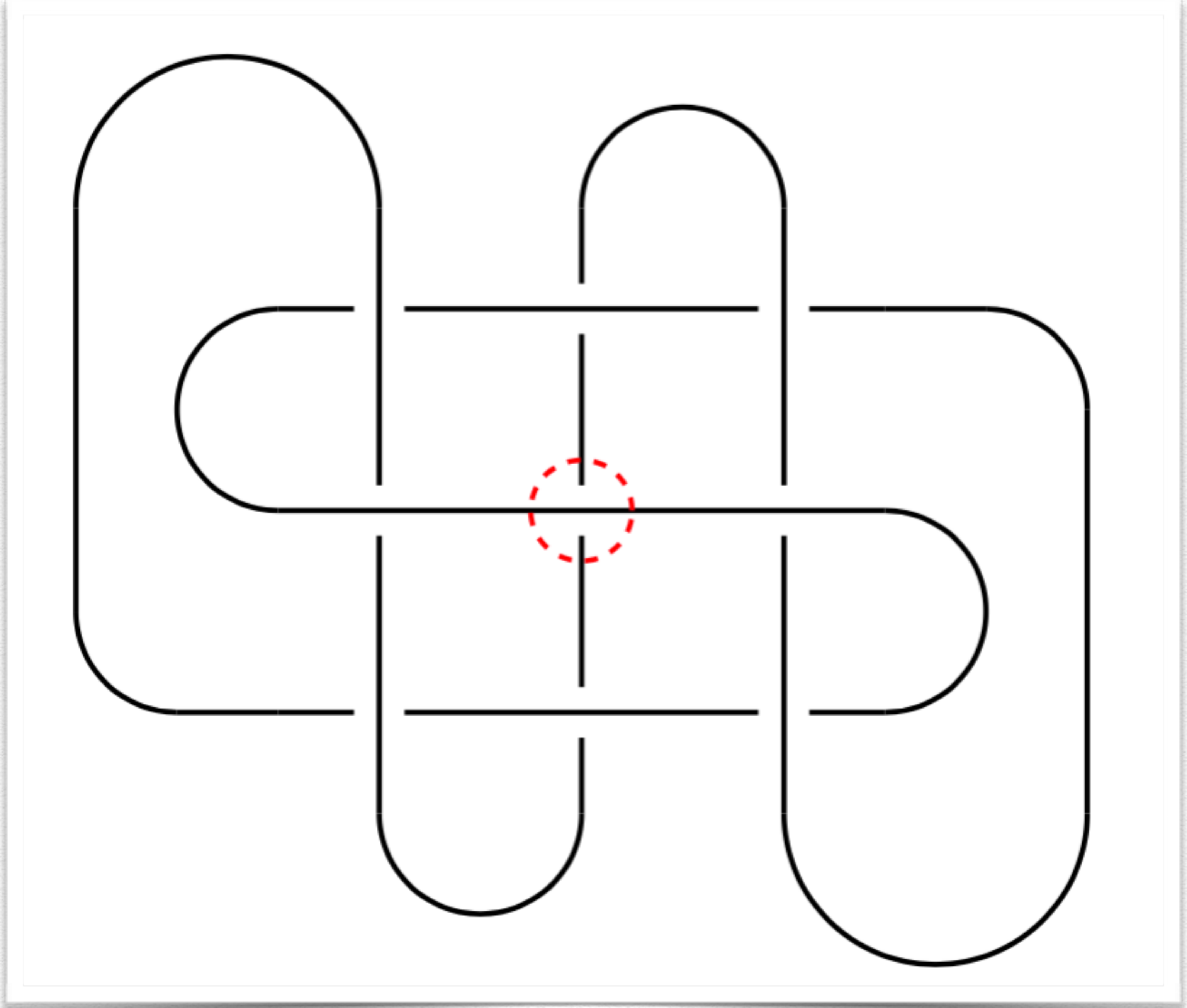
- Start with cobordism between all-***A*** and all-***B*** state.
- Cap off boundary components with disks.
- Turaev genus $g_T(L)$: Minimal genus among all Turaev surfaces.
- The link projection on the Turaev surface is alternating.
- $g_T(L) = 0$ if and only if the link is alternating.

Turaev surfaces

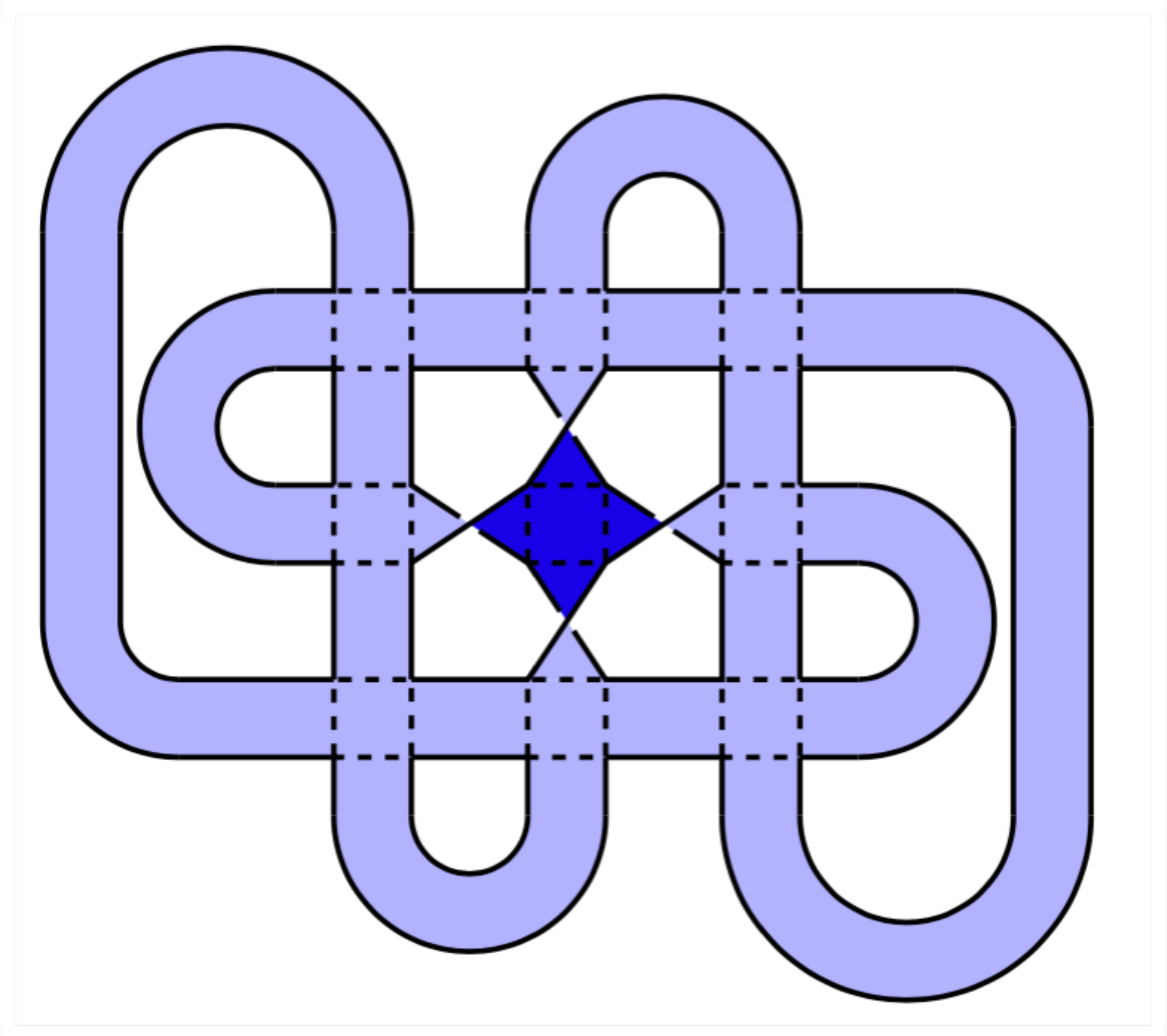
- Replace crossings with disks.
- Replace edges with possibly twisted bands.
- Cap off boundary components with disks.



Turaev surfaces

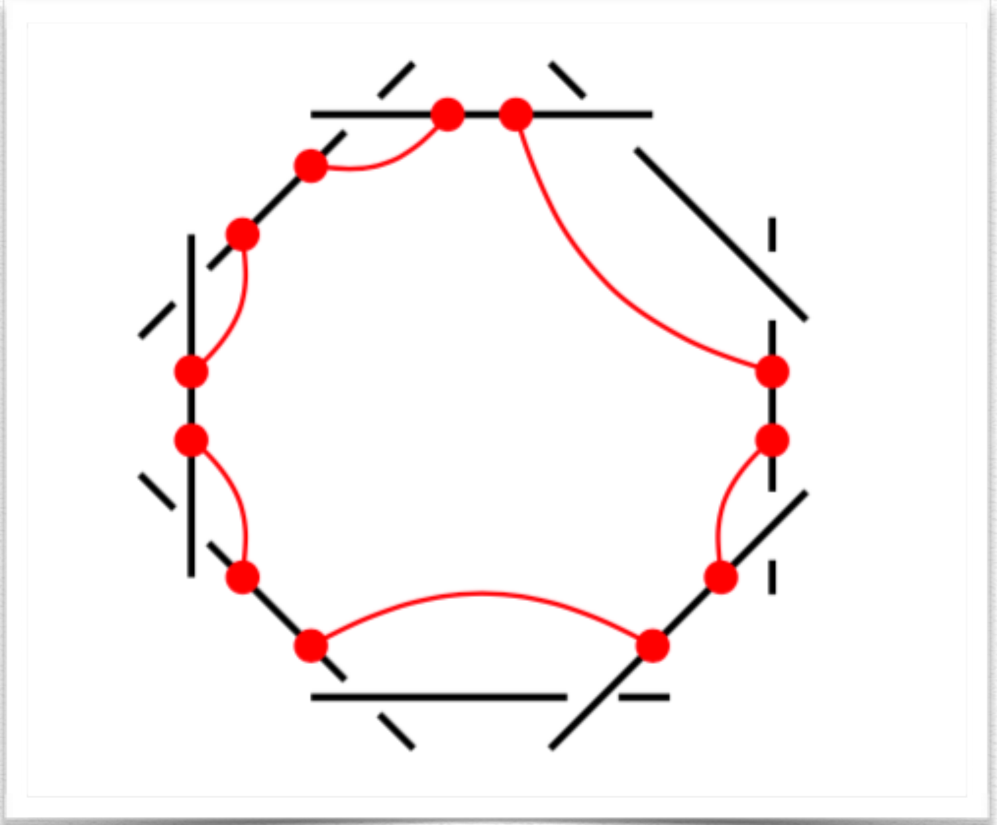


Turaev surfaces

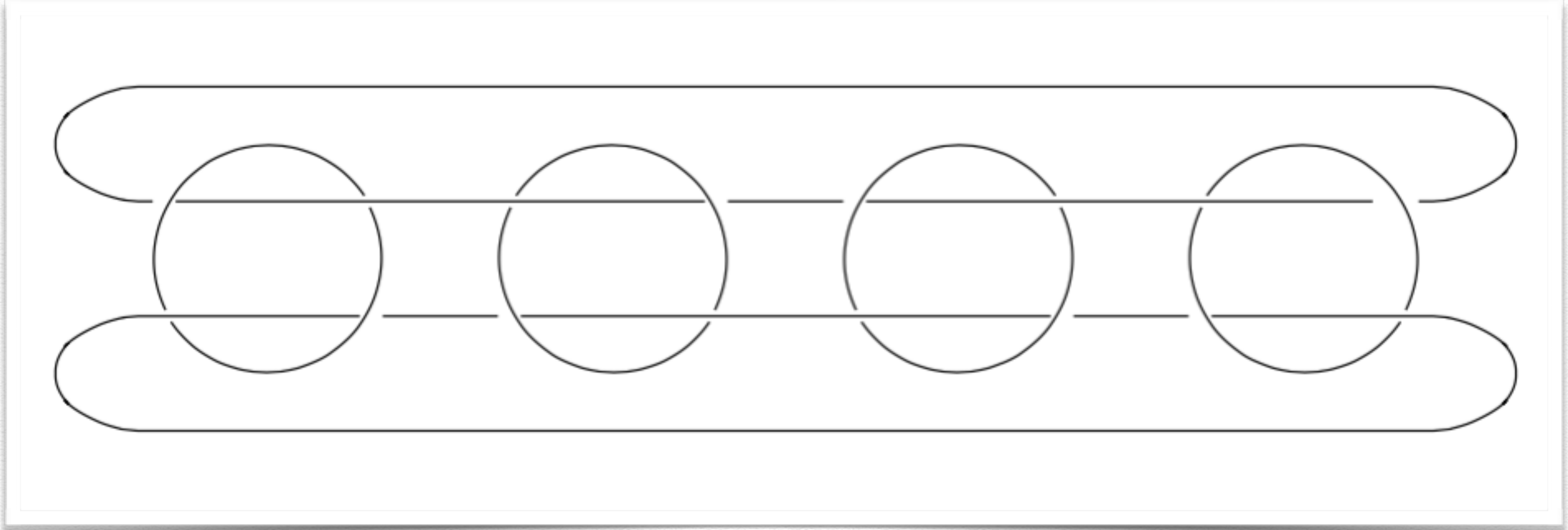


Alternating decompositions

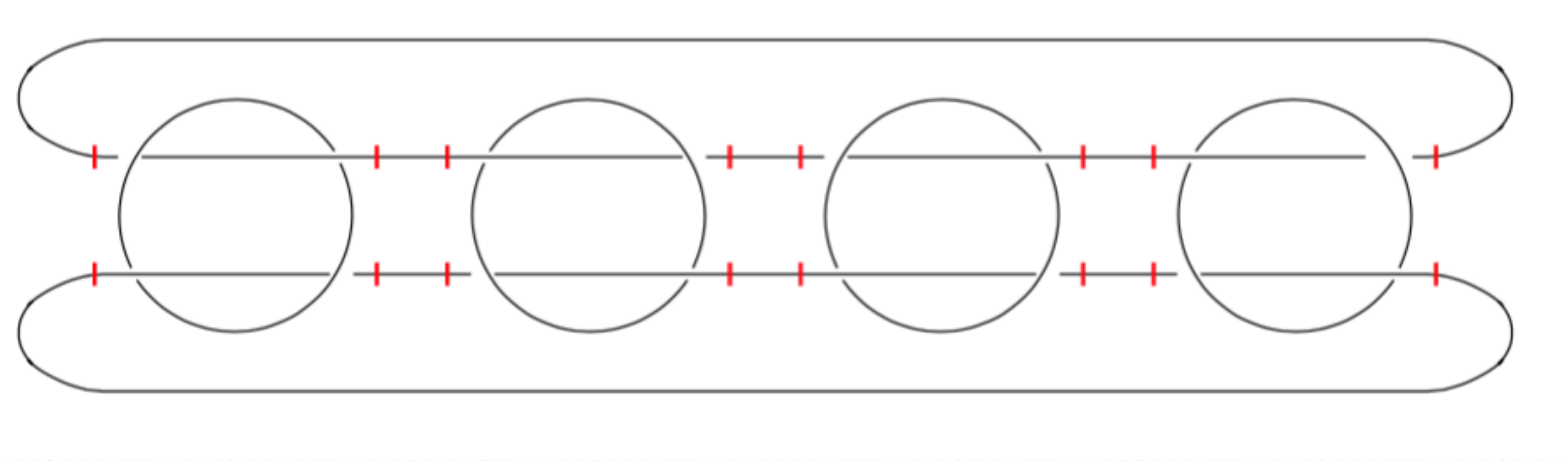
- Separate a link diagram into its maximal alternating regions.
- Mark each non-alternating edge with two points.
- Inside each face, connect the marked points with arcs as below.



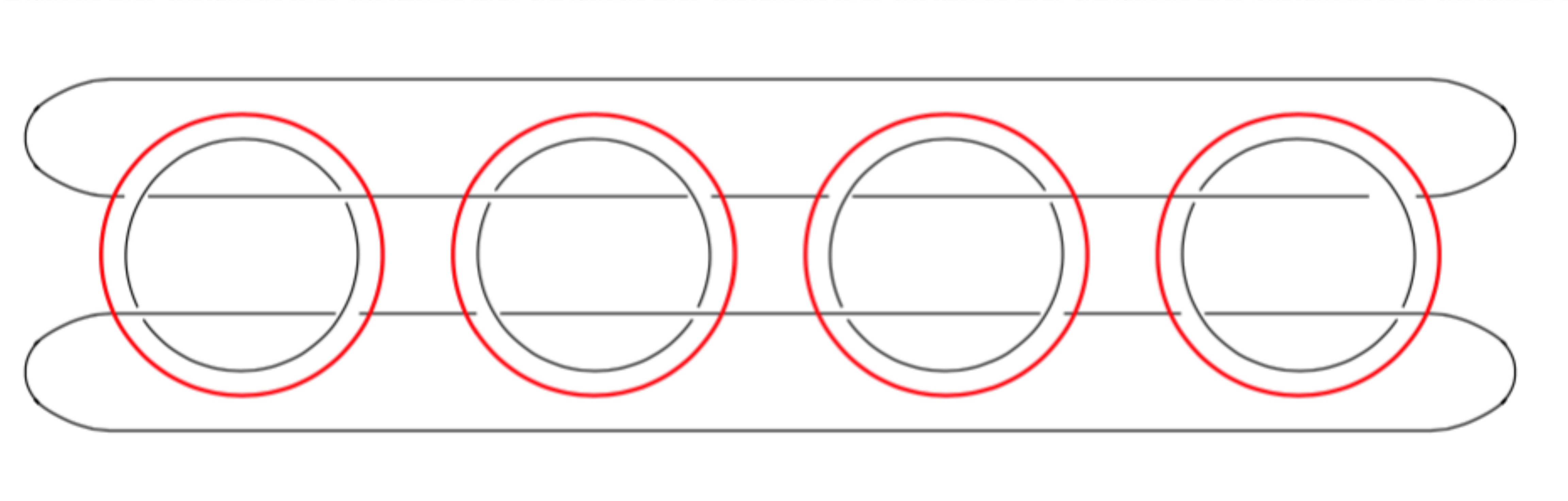
Alternating decompositions



Alternating decompositions



Alternating decompositions



Alternating decompositions

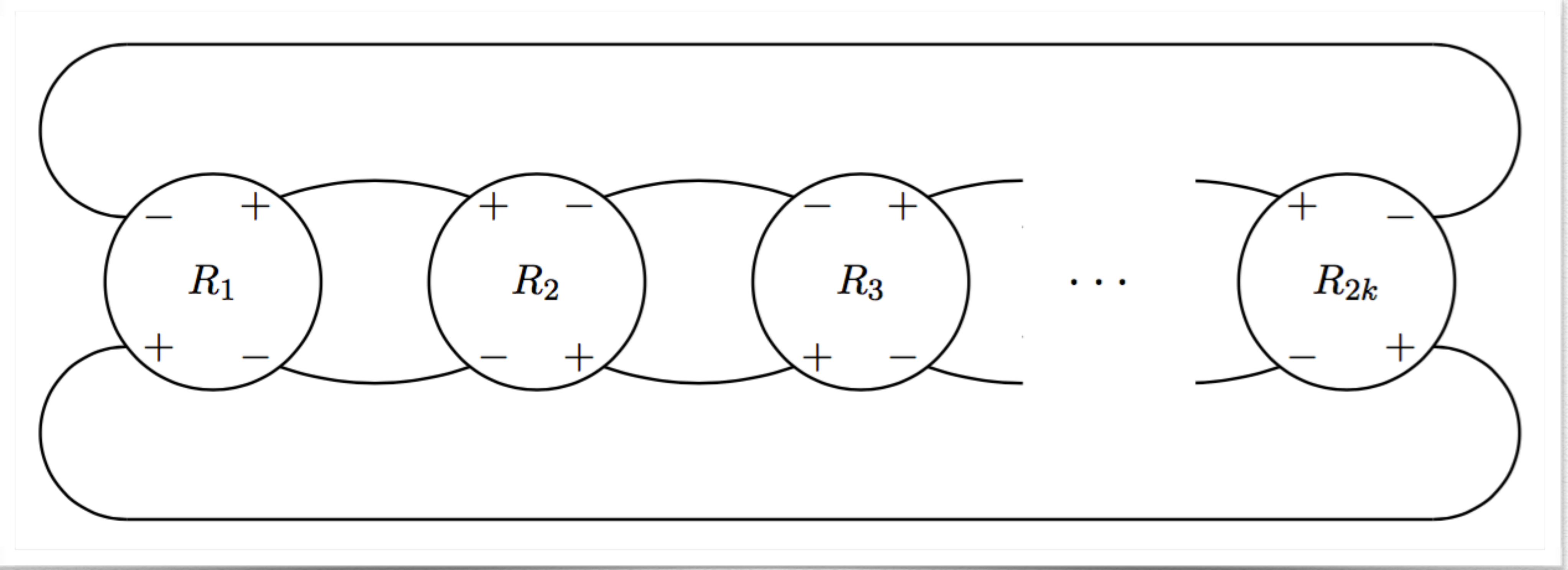
- The alternating decomposition determines a graph: curves → vertices; non-alternating arcs → edges.

- The graph determines the genus of the Turaev surface.

- There is a recursive algorithm on graphs to compute the genus of the Turaev surface.

Turaev genus one diagrams

(Armond, L.; Kim) Let ***D*** a prime link diagram. The Turaev surface of ***D*** is genus one if and only if the alternating decomposition graph is a doubled cycle of even length.



Classification results

- Lengthening/shortening doubled paths preserves the genus of the Turaev surface.

- There are a finite number of double-path equivalence classes of alternating decomposition graphs of any fixed Turaev genus.

- There are five equivalence classes for Turaev genus two.

Determinant and *Signature* of a knot

V Seifert matrix for knot

Determinant:

$$|\det(V+V^T)| = |\Delta_K(-1)| = |V_K(-1)|$$

Signature:

$$\sigma(V+V^T)$$

Determinant and Signature of a knot

Determinant of a knot is odd and
the signature is even

Known: Changing a positive to
a negative crossing:

$$\sigma(K_-) - 2 \leq \sigma(K_+) \leq \sigma(K_-)$$

Signature for Turaev genus 0 links (alternating links)

(Gordon-Litherland, Traczyk) For an alternating knot ***K*** with alternating diagram ***D***

$\sigma(K) = s_A(D) - c_+(D) -1 = -s_B(D) + c_-(D)+1$

Signature and Turaev genus

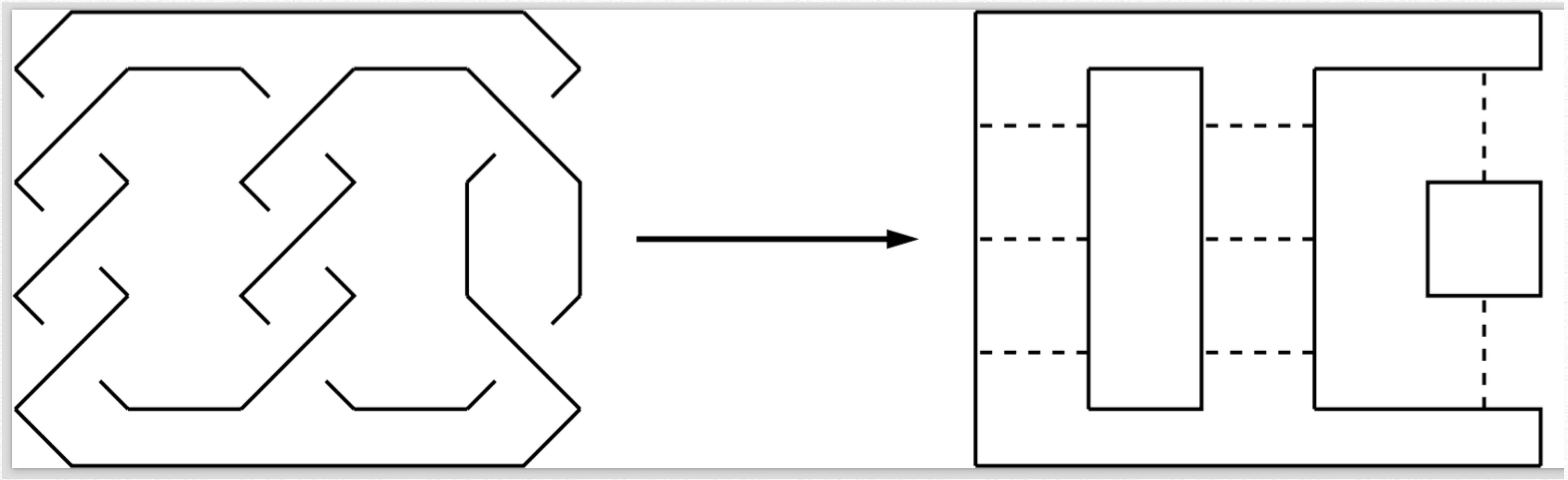
The genus $g_T(D)$ of the Turaev surface is given by

$$2g_T(D) = 2 + c(D) - s_A(D) - s_B(D)$$
$$= (-s_B(D) + c_-(D) + 1) - (s_A(D) - c_+(D) - 1)$$

(Dasbach, L.) For a knot ***K*** with diagram ***D***

$$s_A(D) - c_+(D) - 1 \le \sigma(K) \le -s_B(D) + c_-(D) + 1$$

Example: Signature of *P(3,3,-2)*



$s_A(D) - c_+(D) - 1 = 3 - 8 - 1 = -6$

$g_T(D) = 1$

$-6 \leq \sigma(K) \leq -4$

Signature and Turaev genus

(Murasugi) For a knot ***K***

$$\sigma(K) = \det(K) - 1 \mod 4.$$

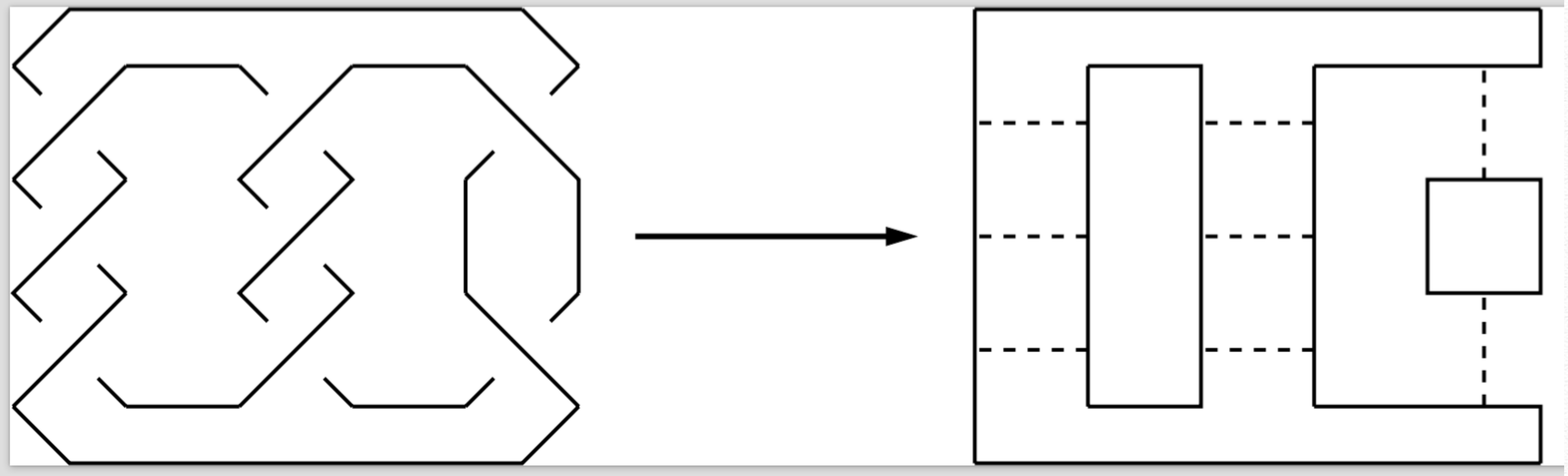
(Dasbach, L.) For a knot ***K*** with diagram ***D*** of Turaev genus 1 the signature is determined by:

$$\sigma(K) = s_A(D) - c_+(D) \pm 1$$

and

$$\sigma(K) = \det(K) - 1 \mod 4.$$

Example: Signature of *P(3,3,-2)*



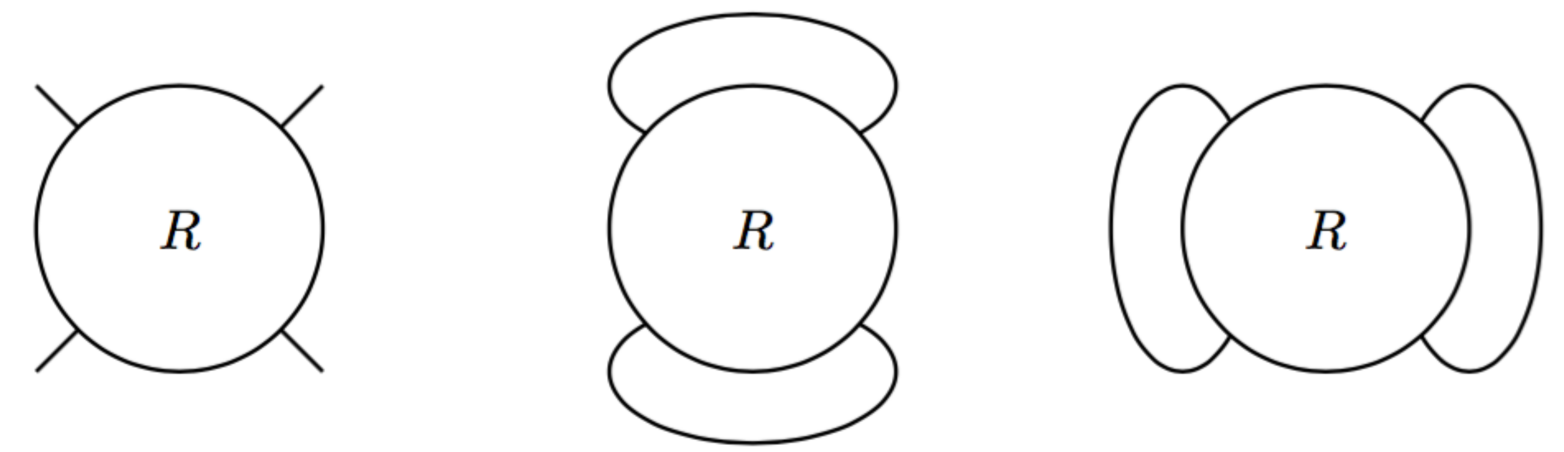
$$\sigma(K) = -6 \text{ or } \sigma(K) = -4$$

$$\det(K) = 3$$

$$\sigma(K) \equiv 2 \mod 4$$

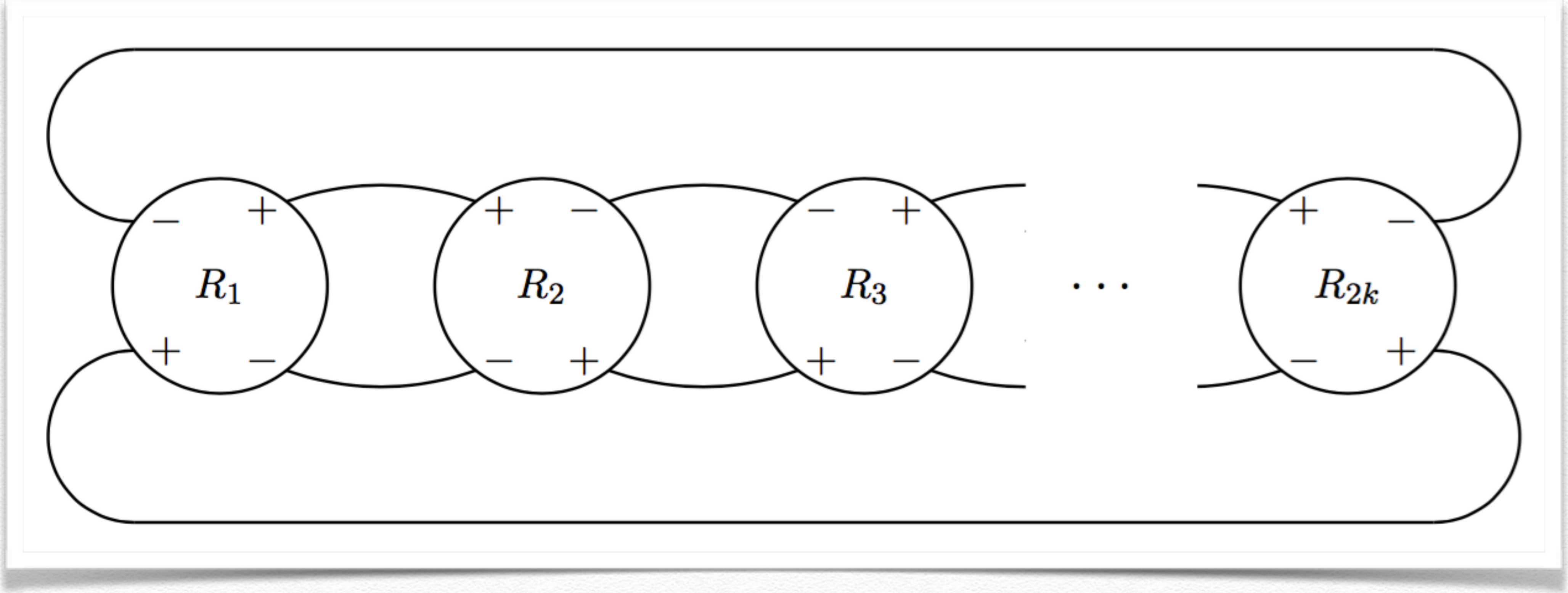
$$\Rightarrow \sigma(K) = -6$$

Closures of tangles



$N(R)$ $D(R)$

Signature and Turaev genus



(Dasbach, L.) Let ***K*** be a knot with diagram as above. Then

$$\sigma(K) = \pm 1 + \sum \sigma(N(R_i)),$$

or

$$\sigma(K) = \pm 1 + \sum \sigma(D(R_i)).$$

Signatures of Montesinos knots

- Murasugi computes the signature of the rational knot K(p/q) from p and q.
- Qazaqzeh, Yasein, and Abu-Qamar compute the signature of K(p/q) from the continued fraction expansion of p/q.
- Champanerkar and Ording give a formula for the determinant of a Montesinos link.
- The theorem on the previous slide then tells us how to compute the signature of a Montesinos knot.

Thank you