Knots of Turaev genus one

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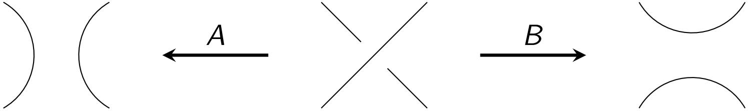
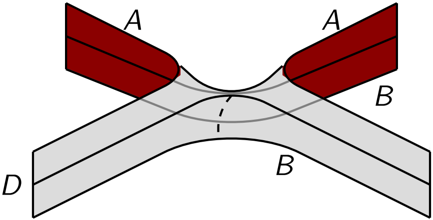
Construction of the Turaev surface $F(D)$

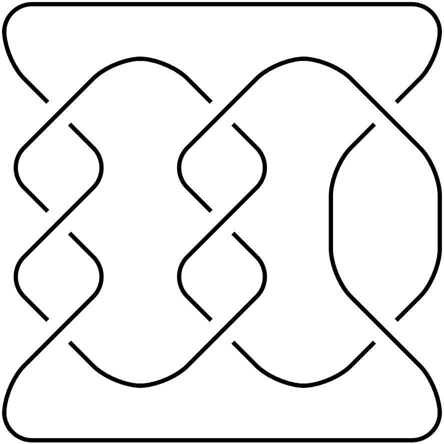
1. Replace arcs of D not near crossings with bands transverse to the projection plane.
2. Replace crossings of D with saddles interpolating between the all-B and all-A states of D
3. Cap off the boundary components with disks to obtain $F(D)$.

The Turaev surface in pictures

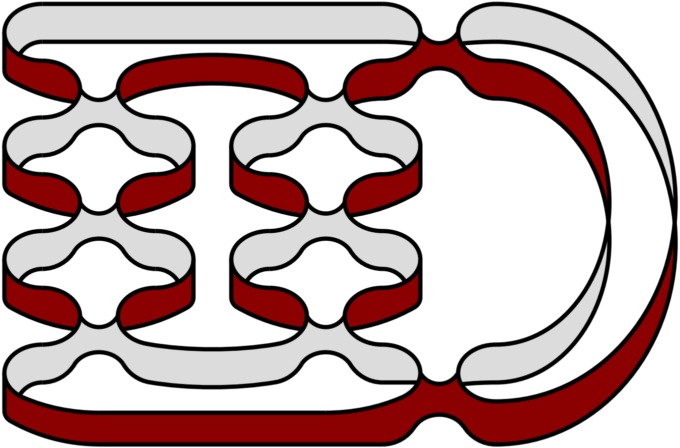




The pretzel knot $P(3, 3, -2)$



The Turaev surface $F(P(3,3,-2))$



Turaev genus

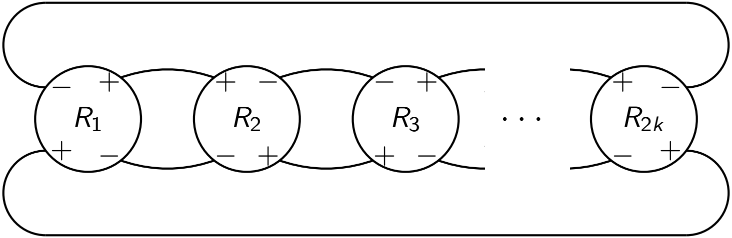
- For a diagram D of a link L, let $g_T(D)$ denote the genus of the Turaev surface $F(D)$.
- The Turaev genus $g_T(L)$ of the link L is

$$g_T(L) = \min\{g_T(D) \mid D \text{ is a diagram of } L\}.$$

Some history of the Turaev surface

- Turaev (1987) related $F(D)$ to the difference between the span of the Jones polynomial and the crossing number.
- DFKLS (2006) - Connections with the Jones polynomial.
- Champanerkar, Kofman, and Stoltzfus (2007) - Connections with spanning tree version of Khovanov homology.
- L. (2007) - Connections with spanning tree version knot Floer homology.
- Dasbach, L. (2009) - Connections with signature, and the s and τ invariants.
- Dasbach, L. (2015) - A Turaev surface model of Khovanov homology.

Classification of links of Turaev genus one



Theorem (Armond, L.; Kim)

1. *Let $R_1, \ldots, R_{2k}$ be alternating two-tangles, and let D be a link diagram connecting $R_1, \ldots, R_{2k}$ as depicted above. Then $g_T(D) = 1$.*
2. *Moreover, if L is a non-split link with $g_T(L) = 1$, then L has a diagram as above.*

Almost-alternating links

- A link diagram D is almost-alternating if one crossing change transforms D into an alternating diagram (Adams, Brock, Bugbee - 1992).
- A link L is almost-alternating if it is non-alternating and has an almost-alternating diagram.
- If L is almost-alternating, then $g_T(L) = 1$.

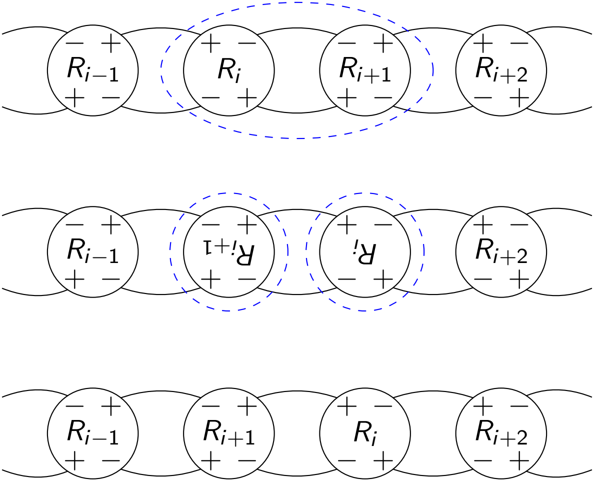
Mutation

- Let B be a 3-ball whose boundary intersects the link L in exactly 4-points. A mutation of L is a link obtained by removing B from S^3, rotating it 180° about a principle axis, and then gluing B back into S^3.
- Any two links related to one another via a sequence of mutations are said to be mutant to one another.

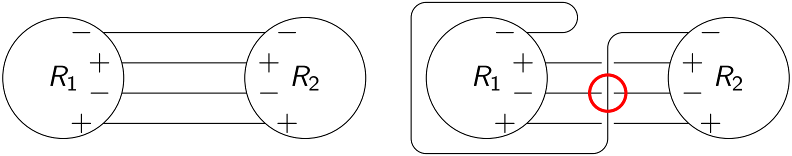
Theorem (Armond, L.)

If $g_T(L) = 1$, then L is mutant to an almost-alternating link.

Mutation proof



Mutation proof continued



Jones polynomial results

Let $V_L(t) = a_m t^m + a_{m+1} t^{m+1} + \cdots + a_{n-1} t^{n-1} + a_n t^n$ be the Jones polynomial of L where a_m and a_n are nonzero.

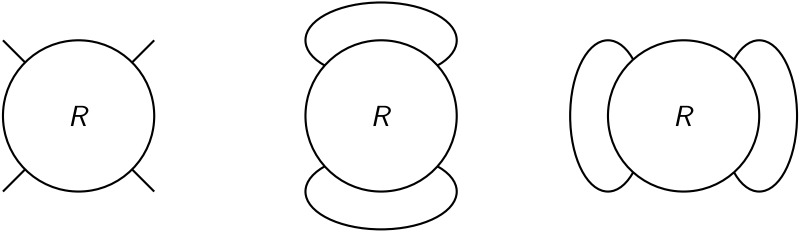
Theorem (Dasbach, L.)

If L is almost-alternating, then either $|a_m| = 1$ or $|a_n| = 1$ (or both equal 1).

Corollary

If $g_T(L) = 1$, then either $|a_m| = 1$ or $|a_n| = 1$ (or both equal 1).

Numerator and denominator

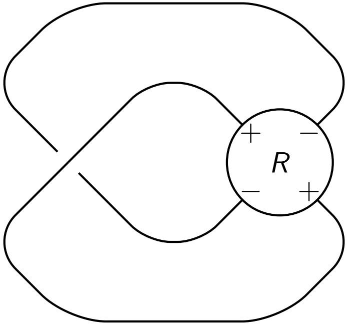


$N(R)$ $D(R)$

A tangle R, its numerator closure $N(R)$ and its denominator closure $D(R)$.

Jones polynomial proof

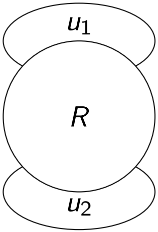
If D is almost-alternating, then it has a diagram as below.



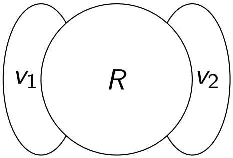
Jones polynomial proof

- $\langle D \rangle = A\langle D(R) \rangle + A^{-1}\langle N(R) \rangle$.
- Both $D(R)$ and $N(R)$ are alternating diagrams.
- For any alternating diagram D_{alt}, the extreme coefficients of $\langle D_{\text{alt}} \rangle$ are ± 1.
- For any alternating diagram D_{alt}, the penultimate coefficients of $\langle D_{\text{alt}} \rangle$ are determined by counting the vertices and certain edges of the checkerboard graphs of D_{alt}.

Adjacent faces



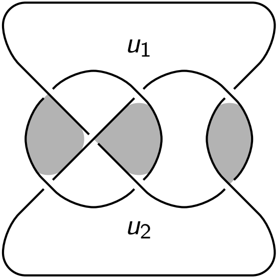
$N(R)$



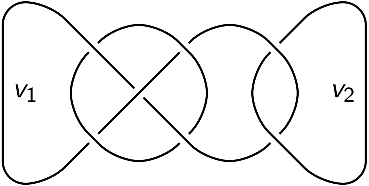
$D(R)$

- Let $\text{adj}(u_1, u_2)$ be the number of faces of $N(R)$ adjacent to both u_1 and u_2.
- Let $\text{adj}(v_1, v_2)$ be the number of faces of $D(R)$ adjacent to both v_1 and v_2.

Example



$\mathrm{adj}(u_1, u_2) = 3$



$\mathrm{adj}(v_1, v_2) = 0$

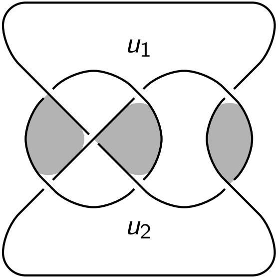
Kauffman bracket computation

- Suppose D is almost-alternating.
- $$\begin{aligned}\langle D\rangle =& A\langle D(R)\rangle + A^{-1}\langle N(R)\rangle \\ =& A\left(\pm A^{p-1} \mp b_1 A^{p+3} \pm \cdots \pm b_2 A^{q-5} \mp A^{q-1}\right) \\ & + A^{-1}\left(\mp A^{p+1} \pm c_1 A^{p+5} \mp \cdots \mp c_2 A^{q-3} \pm A^{q+1}\right) \\ =& \pm (c_1 - b_1) A^{p+4} \mp \cdots \mp (b_2 - c_2) A^{q-4}.\end{aligned}$$
- Dasbach, Lin (2006) implies that $|c_1 - b_1| = |\operatorname{adj}(u_1, u_2) - 1|$ and $|b_2 - c_2| = |\operatorname{adj}(v_1, v_2) - 1|$.

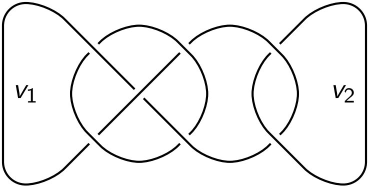
The penultimate coefficients

- We want either $|\operatorname{adj}(u_1, u_2) - 1| = 1$ or $|\operatorname{adj}(v_1, v_2) - 1| = 1$.
- If $\operatorname{adj}(u_1, u_2) \geq 3$, then $\operatorname{adj}(v_1, v_2) = 0$, and if $\operatorname{adj}(v_1, v_2) \geq 3$, then $\operatorname{adj}(u_1, u_2) = 0$.
- Unless $\operatorname{adj}(u_1, u_2) = \operatorname{adj}(v_1, v_2) = 1$, then we have achieved our goal.

The example ... again



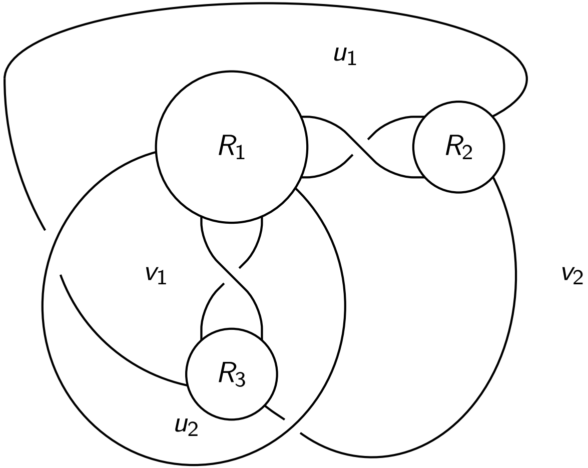
$\mathrm{adj}(u_1, u_2) = 3$



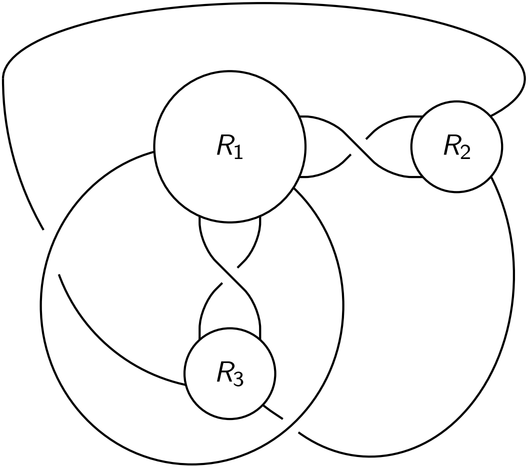
$\mathrm{adj}(v_1, v_2) = 0$

The remaining case

If $\mathrm{adj}(u_1, u_2) = \mathrm{adj}(v_1, v_2) = 1$, then D has diagram as below.



Isotopy



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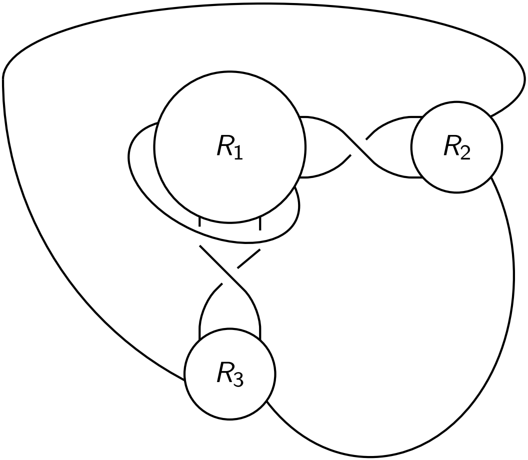

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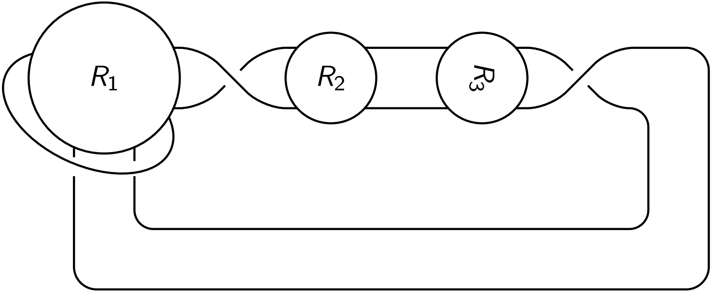

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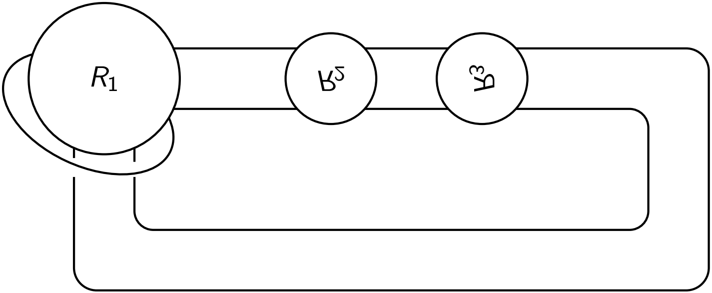

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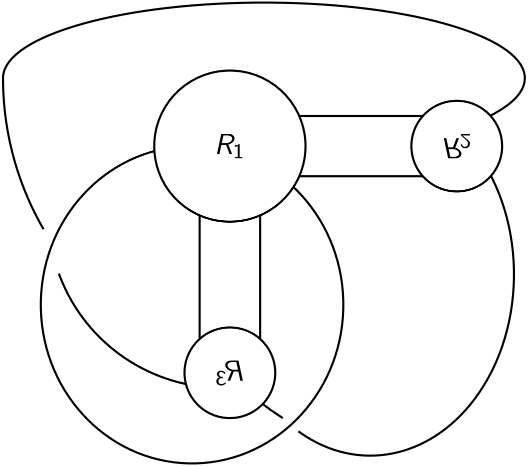
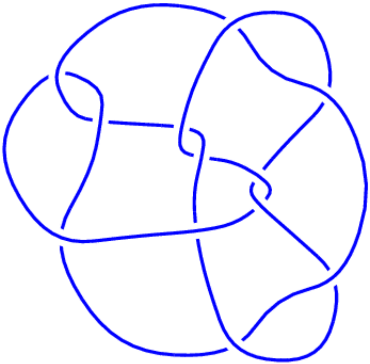


Summary of the $\text{adj}(u_1, u_2) = \text{adj}(v_1, v_2) = 1$ case

- If $\text{adj}(u_1, u_2) = \text{adj}(v_1, v_2) = 1$, then we can describe the tangle structure of D.
- The diagram D can be isotoped to another almost-alternating diagram D' with two fewer crossings.
- Thus by induction (on the number of crossings in an almost-alternating diagram), the Jones polynomial of L satisfies the desired condition.

An example: $12n375$



$$V_{12n375}(t) = 2t^2 - 4t^3 + 8t^4 - 9t^5 + 10t^6 - 10t^7 + 7t^8 - 5t^9 + 2t^{10}$$

Low crossing results

- Among all knots with 12 or fewer crossings, it is unknown whether 37 of them are almost-alternating or have Turaev genus one (according to Knot Info).
- Our work shows that 11 of these 37 knots are not almost-alternating and do not have Turaev genus one.

Thank you!