# The Jones polynomial of almost alternating links

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# Facts about alternating links

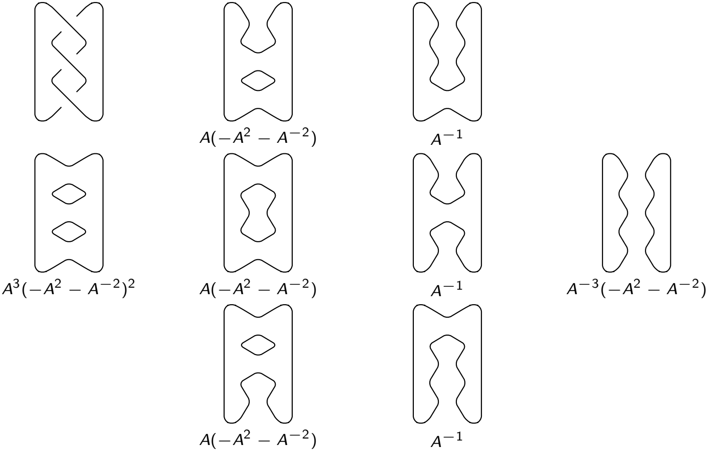
Let $D$ be an alternating diagram of the link $L$ with $n$ crossings.

- (Kauffman, Murasugi, Thistlethwaite)

$$\text{span}\, V_L(t) = c(L) = n.$$

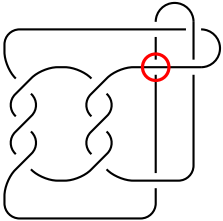
- $L$ is the unknot if and only if $V_L(t) = 1$.
- (Kauffman) The first and last coefficients of $V_L(t)$ are $\pm 1$.
- (Dasbach, Lin) The first and last three coefficients of $V_L(t)$ can be expressed in terms of the checkerboard graph of $D$.

# Trefoil states



# Almost alternating links

A link diagram $D$ is *almost alternating* if $D$ can be transformed into an alternating diagram via one crossing change. A link $L$ is *almost alternating* if it is non-alternating and if it has an almost alternating diagram.



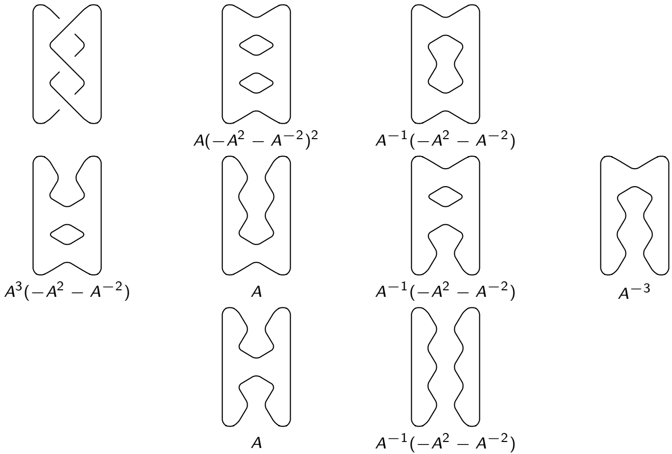
$T_{3,4}$ is almost alternating.

# Facts about almost alternating links

Let $D$ be an almost alternating diagram of the link $L$ with $n$ crossings.

- (Adams et. al)
$$\text{span } V_L(t) \leq n - 3$$
- (Dasbach, L, Spyropoulos) The first and last two potential coefficients of $V_L(t)$ can be expressed in terms of the checkerboard graph of $D$.
- (Dasbach, L) At least one of the first and last coefficient of $V_L(t)$ is $\pm 1$.
- (L, Spyropoulos) $L$ is the unknot if and only if $V_L(t) = 1$.

# An almost alternating unknot



# Extremal coefficients

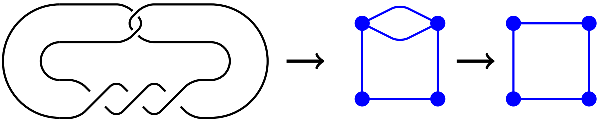
## Theorem (Dasbach, Lin)

*Let $L$ be a link with reduced alternating diagram $D$. Let $\alpha_0$, $\alpha_1$, and $\alpha_3$ be the first three coefficients of $V_L(t)$. Then (up to sign)*

$$\begin{aligned}
\alpha_0 &= 1, \\
\alpha_1 &= v - e - 1, \text{ and} \\
\alpha_2 &= \binom{v-1}{2} - e(v-2) + \mu + \binom{e}{2} - \tau,
\end{aligned}$$

*where all variables come from the checkerboard graph of $D$.*

# Example



$$
\begin{aligned}
\alpha_0 &= 1, \\
\alpha_1 &= v - e - 1 = 4 - 4 - 1 = -1, \\
\alpha_2 &= \binom{v-1}{2} - e(v-2) + \mu + \binom{e}{2} - \tau \\
&= 3 - 4(4-2) + 1 + 6 - 0 = 2, \\
V_K(t) &= t - t^2 + 2t^3 - t^4 + t^5 - t^6.
\end{aligned}
$$

# Extremal coefficients for almost alternating

Theorem (Dasbach, L, Spyropoulos)

*Let L be a link with almost alternating diagram D. Let $\alpha_0$ and $\alpha_1$ be the potential first two coefficients of $V_L(t)$. Then (up to sign)*

$$\alpha_0 = 1 - P_{N,2}, \text{ and}$$

$$\alpha_1 = (\beta_1 + 1)(P_{N,2} - 1) - \binom{P_{N,2}}{2} + P_{N,2,2} - P_{N,2,0} + P_{N,3} - S_N.$$

# The other side

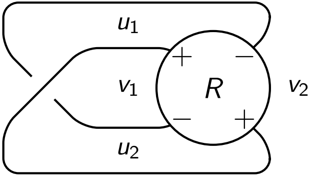
By a symmetric argument, we have that the potential first and last coefficients of $V_L(t)$ are (up to sign)

$$\alpha_0 = 1 - P_{N,2}, \text{ and}$$
$$\alpha_{n-3} = 1 - P_{D,2}.$$

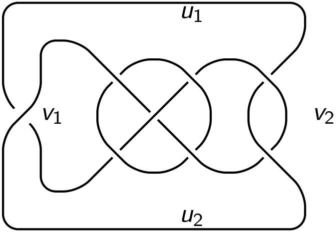
There is also an expression for $\alpha_{n-4}$ similar to the expression for $\alpha_1$.

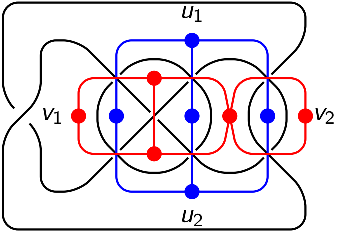
# $P_{N,2}$ and $P_{D,2}$

Let $G$ and $G^*$ be the checkerboard graphs of an almost alternating diagram with vertices $u_1, u_2, v_1$, and $v_2$ as below. Define $P_{N,2}$ to be the number of paths of length two between $u_1$ and $u_2$ in the simplification of $G$, and define $P_{D,2}$ to be the number of paths of length two between $v_1$ and $v_2$ in the simplification of $G^*$.



# An example




$$P_{N,2} = 3 \text{ and } P_{D,2} = 0.$$

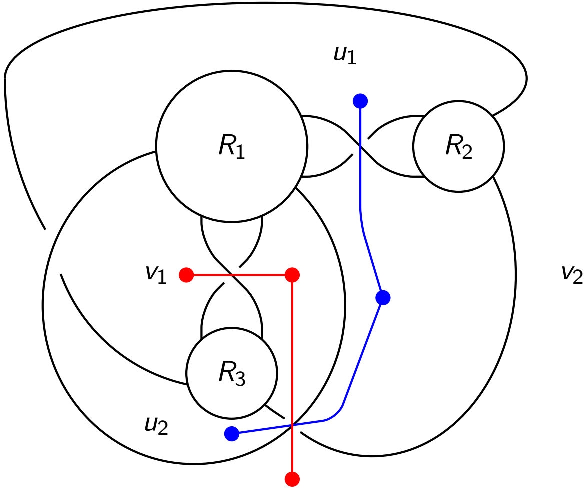
# Zero coefficients

**Lemma**

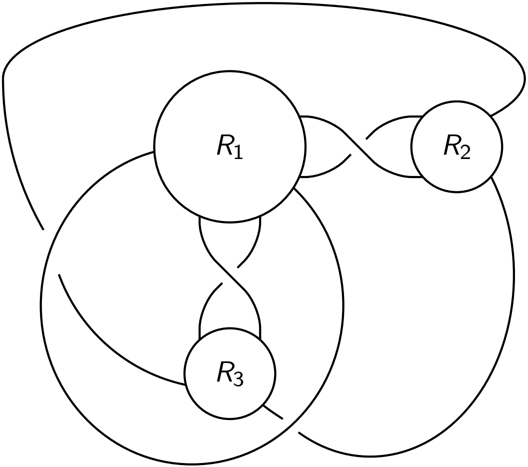
*Let D be an almost alternating diagram of L such that D has the fewest possible crossings among all almost alternating diagrams of L. Then at least one of $\alpha_0$ and $\alpha_{n-3}$ is nonzero.*

# Proof of the zero coefficient lemma

If $\alpha_0 = \alpha_{n-3} = 0$, then $P_{N,2} = P_{D,2} = 1$ and $D$ has diagram as below.



# Proof of the zero coefficient lemma



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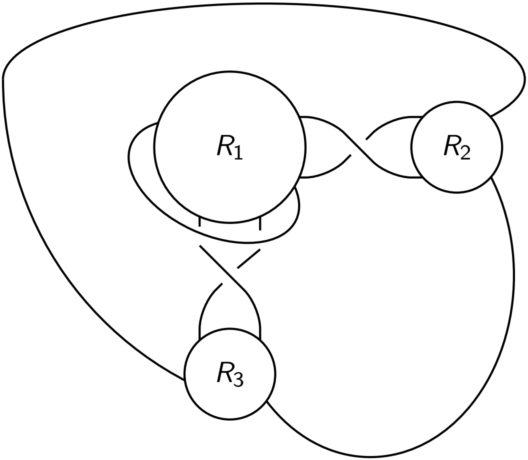

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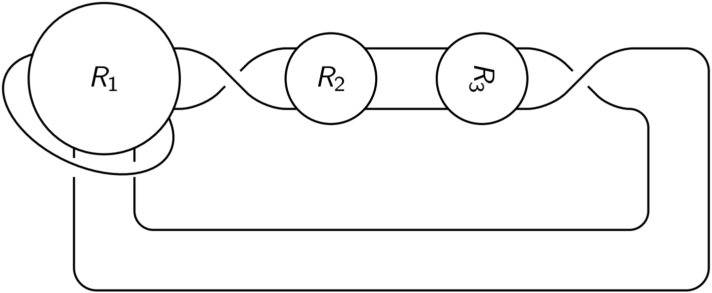

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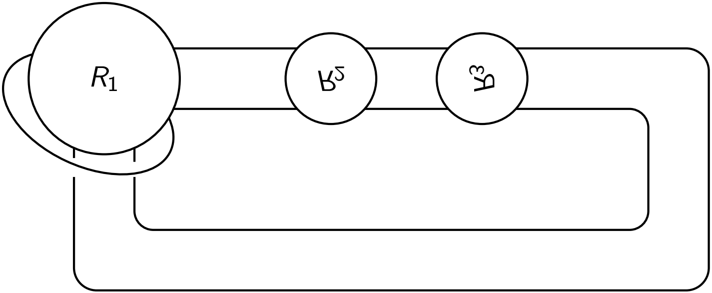
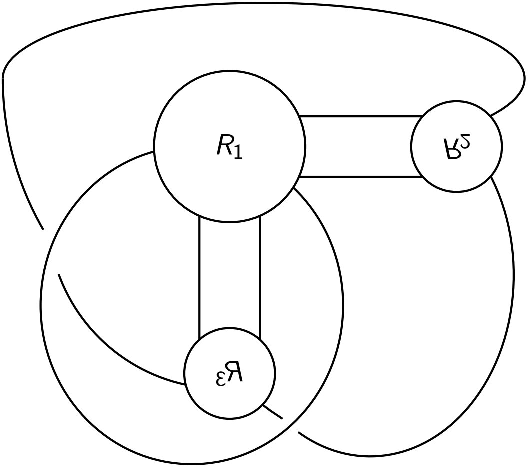


# Proof of the zero coefficient lemma



# Minimal crossing almost alternating diagrams

## Theorem (L, Spyropoulos)

*Let $D$ be an almost alternating diagram of $L$.*

- *If $P_{N,2} \neq 1$ and $P_{D,2} \neq 1$, then $D$ has the fewest crossings among all almost alternating diagrams of $L$.*
- *If $P_{N,2} = 1$ and $P_{D,2} = 1$, then there exists an almost alternating diagram $D'$ of $L$ with fewer crossings than $D$.*

# Monic Jones polynomial for almost alternating

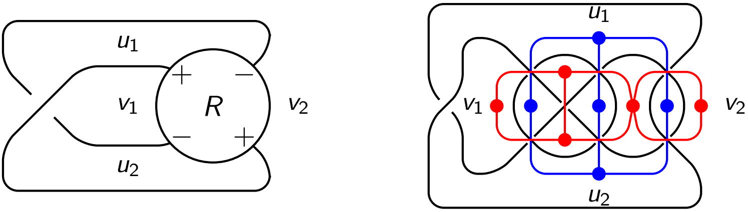
Theorem (Dasbach, L)

*Let $L$ be an almost alternating link, then at least one of the first or the last coefficient of $V_L(t)$ is $\pm 1$.*

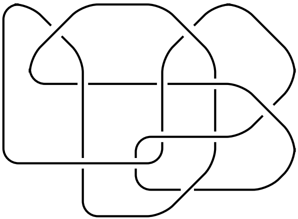
# Sketch of the proof

- $\alpha_0 = 1 - P_{N,2}$ and $\alpha_{n-3} = 1 - P_{D,2}$.
- At least one of $\alpha_0$ and $\alpha_1$ is nonzero.
- If $P_{N,2} \geq 3$, then $P_{D,2} = 0$ (and vice versa).
- So either $\alpha_0 = 1$ or $\alpha_{n-3} = 1$.

# $P_{N,2} \geq 2$ implies $P_{D,2} = 0$



# Is $11n_{95}$ almost alternating?



Since

$$V_{11n_{95}}(t) = 2t^2 - 3t^3 + 5t^4 - 6t^5 + 6t^6 - 5t^7 + 4t^8 - 2t^9,$$

the knot $11n_{95}$ above is not almost alternating.

# Detecting the unknot

Theorem (L, Spyropoulos)

*Let $D$ be an almost-alternating diagram of the link $L$. Then $V_L(t) = 1$ if and only if $L$ is the unknot.*

# Sketch of the proof

- The potential first and last two coefficients of $V_L(t)$ are $\alpha_0$, $\alpha_1$, $\alpha_{n-4}$, and $\alpha_{n-3}$.
- Assuming minimal crossings, the zero coefficient lemma tells us at least one of the four coefficients is nonzero.
- If two of the coefficients are nonzero, then $V_L(t) \neq 1$.
- If only one coefficient is nonzero, then we handle it case by case.

# Questions

1. Can we determine the minimal crossing almost alternating diagram of a link?
2. Do the coefficients of the Jones polynomial of an almost alternating link give bounds on its hyperbolic volume?
3. Does the Jones polynomial detect the unknot on the class of $k$-almost alternating links for $k > 1$?

Thank you!