Odd torsion in the Khovanov homology of semi-adequate links

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Overview

- The Khovanov homology $Kh(L)$ of a link L is a categorification of the Jones polynomial of L (Khovanov - 1999).
- $Kh(L) = \bigoplus_{i,j\in\mathbb{Z}} Kh^{i,j}(L)$ is a bi-graded $\mathbb{Z}$-module.
- Experimentally, $Kh(L)$ has an abundance of torsion, only some of which we can explain.
- Among all 1,701,936 prime knots with at most 16 crossings
  1. all non-trivial knots up to 14 crossings have only 2-torsion in their Khovanov homology,
  2. 38 knots with 15 crossings and 129 knots with 16 crossings have 4-torsion in their Khovanov homology, and
  3. the first known knot with odd torsion in $Kh(K)$ is the $(5, 6)$-torus knot.

Motivating conjecture

Conjecture (Shumakovitch)

Let L be any prime link other than the unknot or the Hopf link. Then $Kh(L)$ contains 2-torsion.

- The conjecture implies that Khovanov homology is an unknot detector.
- Kronheimer and Mrowka (2010) proved Khovanov homology is an unknot detector using gauge theory.
- The conjecture is known to be true in many cases.

Some more conjectures

Conjecture (Przytycki - Sazdanović)

- *The Khovanov homology of a closed* 3*-braid can have only* 2*-torsion.*
- *The Khovanov homology of a closed* 4*-braid can have only* 2*-torsion or* 4*-torsion.*
- *The Khovanov homology of a closed n-braid cannot have p-torsion for $p > n$, where p is prime.*

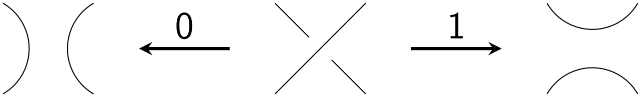
Methods

Some approaches for proving things about torsion in Khovanov homology are:

- explicit construction,
- connections with Hochschild homology,
- connections with chromatic polynomial cohomology, and
- spectral sequence arguments.

Kauffman states

- Each crossing has a 0 and a 1 resolution.



- The collection of simple closed curves in the plane obtained by taking a 0 or 1 resolution at each crossing is a *Kauffman state*.
- The number of components in a Kauffman state s is denoted $|s|$.

Spaces

- Define $\mathcal{A} = \mathbb{Z}[x]/(x^2)$.
- To each Kauffman state s, associate the space $\mathcal{A}^{\otimes |s|}$.
- The space $\mathcal{A}^{\otimes |s|}$ has basis given by all possible labelings of the components of s with 1 and x.
- Arrange the Kauffman states into a cube of resolutions.

Maps

Define $\mathbb{Z}$-linear maps

$$m : \mathcal{A} \otimes \mathcal{A} \to \mathcal{A} \qquad m : \begin{cases} 1 \otimes 1 \mapsto 1 & 1 \otimes x \mapsto x \\ x \otimes 1 \mapsto x & x \otimes x \mapsto 0 \end{cases}$$

$$\Delta : \mathcal{A} \to \mathcal{A} \otimes \mathcal{A} \qquad \Delta : \begin{cases} 1 \mapsto 1 \otimes x + x \otimes 1 \\ x \mapsto x \otimes x. \end{cases}$$

Maps and the hypercube

- If an edge ξ in the hypercube merges two components into one, then define d_ξ to be multiplication m on the two factors of $\mathcal{A}$ associated to the merging components, and define d_ξ to be the identity on all other factors of $\mathcal{A}$.
- If an edge ξ in the hypercube splits a component into two, then define d_ξ to be comultiplication Δ on the factor of $\mathcal{A}$ associated to the splitting component, and define d_ξ to be the identity on all other factors of $\mathcal{A}$.
- Assign each edge ξ a plus or minus sign $(-1)^\xi$ according to whether the number of 1-resolutions before the changed index is even or odd.

The Khovanov complex $CKh(D)$

- Define $CKh(D) = \bigoplus_s \mathcal{A}^{\otimes |s|}$.
- $d : CKh(D) \to CKh(D)$ by $\sum_\xi (-1)^\xi d_\xi$.
- $CKh(D) = \bigoplus_{i,j} CKh^{i,j}(D)$ is bigraded.
- d has bidegree $(1, 0)$, that is $d^{i,j} : CKh^{i,j}(D) \to CKh^{i+1,j}(D)$.
- The homology of the complex is a link invariant called Khovanov homology (Khovanov - 1999), and is denoted

$$Kh(D) = \bigoplus_{i,j} Kh^{i,j}(D).$$

An explicit computation

Let's explicitly compute some torsion in the Khovanov homology of the left-handed trefoil.

Computations of odd torsion

- Torus knots $(5,6)$, $(5,7)$, $(5,8)$, and $(5,9)$ have 5-torsion in their Khovanov homology.
- Przytycki and Sazdanović predicted that the closure K of

$$\sigma_1^2\sigma_2^2\sigma_1^3\sigma_2^2\sigma_1\sigma_3\sigma_2^2\sigma_4^2\sigma_3\sigma_1^2\sigma_2^2\sigma_1^3\sigma_2^3\sigma_1^2\sigma_3\sigma_2^2\sigma_4^2\sigma_3^2$$

  has 5-torsion in its Khovanov homology.
- Shumakovitch (2012) confirmed that K has 5-torsion in its Khovanov homology by showing that the difference of the Poincare polynomials of $Kh(K;\mathbb{Z}_5)$ and $Kh(K;\mathbb{Z}_7)$ is

$$(t^{12}+t^{11})q^{51}+(t^{11}+t^{10})q^{47}.$$

$Kh(K;\mathbb{Z}_5)$

$$KH_5(K) = q^{31}t^0 + q^{33}t^0 + q^{35}t^2 + q^{39}t^3 + 2q^{37}t^4 + q^{39}t^4 + 2 + q^{41}t^5 + q^{43}t^5 + q^{39}t^6 + 2q^{41}t^6 + 2q^{43}t^7 + 2q^{45}t^7 + 4q^{41}t^8 + 3q^{43}t^8 + q^{47}t^8 + 13q^{43}t^9 + 4q^{45}t^9 + 4q^{47}t^9 + 2q^{43}t^{10} + 29q^{45}t^{10} + 14q^{47}t^{10} + q^{51}t^{10} + 9q^{45}t^{11} + 44q^{47}t^{11} + 31q^{49}t^{11} + q^{51}t^{11} + 2q^{45}t^{12} + 34q^{47}t^{12} + 68q^{49}t^{12} + 42q^{51}t^{12} + 2q^{53}t^{12} + 11q^{47}t^{13} + 85q^{49}t^{13} + 97q^{51}t^{13} + 59q^{53}t^{13} + 45q^{49}t^{14} + 159q^{51}t^{14} + 142q^{53}t^{14} + 63q^{55}t^{14} + 137q^{51t^15} + 245q^{53}t^{15} + 202q^{55}t^{15} + 9q^{57}t^{15} + 345q^{53}t^{16} + 5376q^{55}t^{16} + 237q^{57}t^{16} + 54q^{59}t^{16} + 735q^{55}t^{17} + 589q^{57}t^{17} + 260q^{59}t^{17} + 37q^{61}t^{17} + 1328q^{57}t^{18} + 953q^{59}t^{18} + 253q^{61}t^{18} + 21q^{63}t^{18} + 2040q^{59}t^{19} + 1501q^{61}t^{19} + 220q^{63}t^{19} + 9q^{65}t^{19} + 2729q^{61}t^{20} + 2149q^{63}t^{20} + 173q^{65}t^{20} + 2q^{67}t^{20} + 2q^{61}t^{21} + 3203q^{63}t^{21} + 2779q^{65}t^{21} + 109q^{67}t^{21} + 11q^{63}t^{22} + 3344q^{65}t^{22} + 3219q^{67}t^{22} + 50q^{69}t^{22} + 36q^{65}t^{23} + 3127q^{67}t^{23} + 3345q^{69}t^{23} + 16q^{71}t^{23} + 81q^{67}t^{24} + 2608q^{69}t^{24} + 3116q^{71}t^{24} + 3q^{73}t^{24} + 137q^{69}t^{25} + 1934q^{71}t^{25} + 2572q^{73}t^{25} + 191q^{71}t^{26} + 1271q^{73}t^{26} + 1853q^{75}t^{26} + 228q^{73}t^{27} + 759q^{75}t^{27} + 1134q^{77}t^{27} + 238q^{75}t^{28} + 446q^{77}t^{28} + 568q^{79}t^{28} + 219q^{77}t^{29} + 294q^{79}t^{29} + 218q^{81}t^{29} + 175q^{79}t^{30} + 226q^{81}t^{30} + 56q^{83}t^{30} + 119q^{81}t^{31} + 175q^{83}t^{31} + 7q^{85}t^{31} + 65q^{83}t^{32} + 119q^{85}t^{32} + 26q^{85}t^{33} + 65q^{87}t^{33} + 7q^{87}t^{34} + 26q^{89}t^{34} + q^{89}t^{35} + 7q^{91}t^{35} + q^{93}t^{36}$$

$Kh(K;\mathbb{Z}_7)$

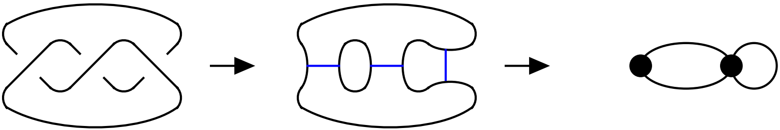
$$\begin{aligned}
KH_7(K) \quad = \quad & q^{31}t^0+q^{33}t^0+q^{35}t^2+q^{39}t^3+2q^{37}t^4+q^{39}t^4+2q^{41}t^5+q^{43}t^5+q^{39}t^6+2q^{41}t^6+\\
& 2q^{43}t^7+2q^{45}t^7+4q^{41}t^8+3q^{43}t^8+q^{47}t^8+13q^{43}t^9+4q^{45}t^9+4q^{47}t^9+\\
& 2q^{43}t^{10}+29q^{45}t^{10}+13q^{47}t^{10}+q^{51}t^{10}+9q^{45}t^{11}+43q^{47}t^{11}+31q^{49}t^{11}+\\
& 2q^{45}t^{12}+34q^{47}t^{12}+68q^{49}t^{12}+41q^{51}t^{12}+2q^{53}t^{12}+11q^{47}t^{13}+85q^{49}t^{13}+\\
& 97q^{51}t^{13}+59q^{53}t^{13}+45q^{49}t^{14}+159q^{51}t^{14}+142q^{53}t^{14}+63q^{55}t^{14}+137q^{51}t^{15}+\\
& 245q^{53}t^{15}+202q^{55}t^{15}+59q^{57}t^{15}+345q^{53}t^{16}+376q^{55}t^{16}+237q^{57}t^{16}+\\
& 54q^{59}t^{16}+735q^{55}t^{17}+589q^{57}t^{17}+260q^{59}t^{17}+37q^{61}t^{17}+1328q^{57}t^{18}+\\
& 953q^{59}t^{18}+253q^{61}t^{18}+21q^{63}t^{18}+2040q^{59}t^{19}+1501q^{61}t^{19}+220q^{63}t^{19}+\\
& 9q^{65}t^{19}+2729q^{61}t^{20}+2149q^{63}t^{20}+173q^{65}t^{20}+2q^{67}t^{20}+2q^{61}t^{21}+3203q^{63}t^{21}+\\
& 2779q^{65}t^{21}+109q^{67}t^{21}+11q^{63}t^{22}+3344q^{65}t^{22}+3219q^{67}t^{22}+50q^{69}t^{22}+\\
& 36q^{65}t^{23}+3127q^{67}t^{23}+3345q^{69}t^{23}+16q^{71}t^{23}+81q^{67}t^{24}+2608q^{69}t^{24}+\\
& 3116q^{71}t^{24}+3q^{73}t^{24}+137q^{69}t^{25}+1934q^{71}t^{25}+2572q^{73}t^{25}+191q^{71}t^{26}+\\
& 1271q^{73}t^{26}+1853q^{75}t^{26}+228q^{73}t^{27}+1134q^{77}t^{27}+238q^{75}t^{28}+446q^{77}t^{28}+\\
& 568q^{79}t^{28}+219q^{77}t^{29}+294q^{79}t^{29}+218q^{81}t^{29}+759q^{75}t^{27}+175q^{79}t^{30}+226q^{81}t^{30}+\\
& 56q^{83}t^{30}+119q^{81}t^{31}+175q^{83}t^{31}+7q^{85}t^{31}+65q^{83}t^{32}+119q^{85}t^{32}+26q^{85}t^{33}+\\
& 65q^{87}t^{33}+7q^{87}t^{34}+26q^{89}t^{34}+q^{89}t^{35}+7q^{91}t^{35}+q^{93}t^{36}
\end{aligned}$$

From links to graphs

From a link diagram D construct the all-0 state graph $G_0(D)$ as follows.

1. Choose the 0-resolution for each crossing of D, leaving a small segment that connects the two arcs of the resolution.
2. The components of the all positive resolution become vertices of $G_0(D)$.
3. The small segments at the crossings become the edges of $G_0(D)$.

A 3-crossing unknot



Adequate and semi-adequate links

- A link L is *adequate* if it has a diagram where both $G_0(D)$ and $G_0(\overline{D})$ have no loops.
- A link L is *semi-adequate* if either $G_0(D)$ or $G_0(\overline{D})$ have no loops.
- Alternating links are adequate.
- Many links are semi-adequate. For example, Stoimenow (2012) computed at least 249,649 of the 253,293 knots with crossing number 15 are semi-adequate.

Fancier results coming from explicit computation

Theorem (Asaeda - Przytycki)

1. *If* $G_0(D)$ *is loop-less and contains a cycle of odd length, then* $Kh(D)$ *contains* 2*-torsion.*
2. *If* $G_0(D)$ *is loop-less and contains a cycle of even length with an edge that is not part of a bigon, then* $Kh(D)$ *contains* 2*-torsion.*
3. *If* D *is prime and alternating and* D *is not the unknot or Hopf link, then either* $G_0(D)$ *or* $G_0(\overline{D})$ *contains an edge that is not part of a bigon. Thus* $Kh(D)$ *contains* 2*-torsion.*

Remark. Shumakovitch's conjecture is true for alternating links and "many" semi-adequate links.

Hochschild homology and $Kh(T_{2,n})$

- Let P_n be the polygon with n vertices.
- Let $C_n(\mathcal{A})$ be the space generated by labelings of the vertices of P_n with elements of $\mathcal{A}$.
- Define a map $C_n(\mathcal{A}) \to C_{n-1}(\mathcal{A})$ obtained by contracting edges and multiplying the labels on the identified vertices.
- Przytycki (2005) showed this complex gives the Hochschild homology $HH(\mathcal{A})$ and the Khovanov homology of $Kh(T_{2,n})$ in certain gradings.
- Allows for explicit computations of 2-torsion inside of $Kh(T_{2,n})$.

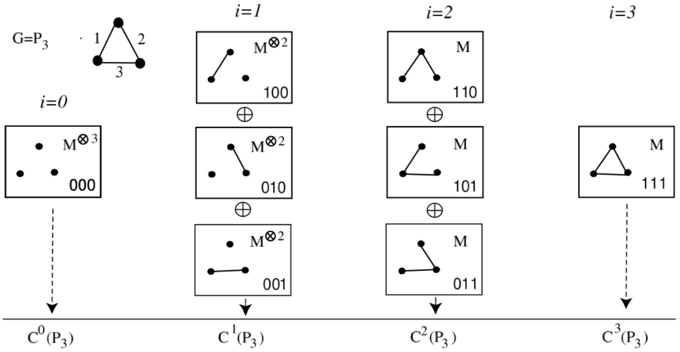
From Hochschild to chromatic polynomial cohomology

- Hochschild homology gives a sort of comultiplication free version of Khovanov homology for a polygon.
- Helme-Guizon and Rong (2004) define the chromatic polynomial cohomology. It can be simultaneously thought of as a comultiplication free version of Khovanov homology for any graph or as an extension of Hochschild homology for any graph.
- Its definition follows a similar recipe as the construction of Khovanov homology.

Chromatic polynomial cohomology: spaces

- Let G be a graph with edges $E = \{e_1, \ldots, e_n\}$.
- Let $s \subseteq E$ and define $[G : s]$ to be the spanning subgraph of G with edge set s.
- Let $|G : s|$ denote the number of components of $[G : s]$.
- Associate the space $\mathcal{A}^{\otimes |G:s|}$ to each subset $s \subseteq E$.

Spaces and the hypercube



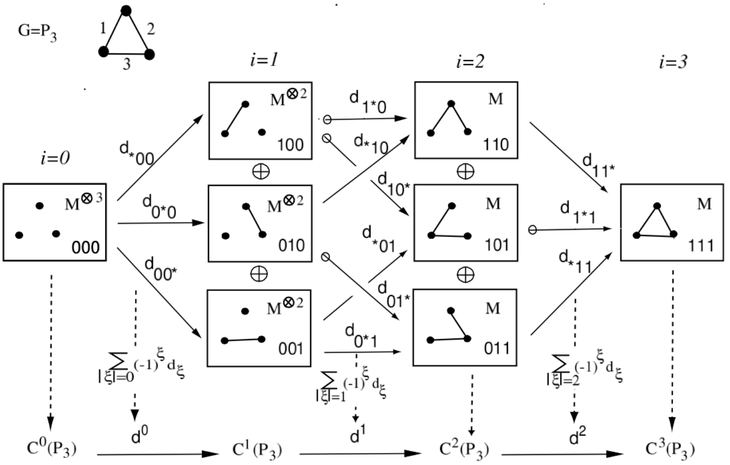
Chromatic polynomial cohomology: maps

- Suppose that $s_0, s_1 \subseteq E$ such that $s_1 = s_0 \cup \{e_i\}$. Then the number of components of $[G : s_1]$ is either the number of components of $[G : s_0]$ or one less than the number of components of $[G : s_0]$.

- If adding e_i to $[G : s_0]$ merges two components, then the edge map d_ξ is multiplication m on the two factors of $\mathcal{A}$ associated to the merging components and is the identity on all other factors of $\mathcal{A}$.

- If adding e_i to $[G : s_0]$ results in the same number of components, then the edge map d_ξ is the identity.

Maps



G=P3
1
2
3
i=0
i=1
i=2
i=3
M⊗3
000
M⊗2
100
010
001
M
110
101
011
111
d*00
d0*0
d00*
d1*0
d*10
d10*
d*01
d01*
d0*1
d11*
d1*1
d*11
Σ|ξ|=0 (-1)ξ dξ
Σ|ξ|=1 (-1)ξ dξ
Σ|ξ|=2 (-1)ξ dξ
d0
d1
d2
C0(P3)
C1(P3)
C2(P3)
C3(P3)

The chromatic polynomial complex $C(G)$

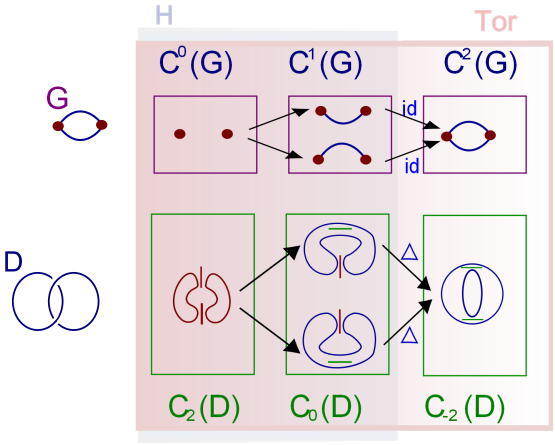
- Define $C(G) = \bigoplus_{s \subseteq E} \mathcal{A}^{\otimes |G:s|}$.
- $d : C(G) \to C(G)$ by $\sum_{\xi} (-1)^{\xi} d_{\xi}$.
- $C(G) = \bigoplus_{i,j} C^{i,j}(G)$ is bigraded.
- d has bidegree $(1,0)$, that is $d^{i,j} : C^{i,j}(G) \to C^{i+1,j}(G)$.
- The homology of the complex is a categorification of the chromatic polynomial (Helme-Guizon, Rong - 2004), and is denoted
$$H(G) = \bigoplus_{i,j \in \mathbb{Z}} H^{i,j}(G).$$

Comparing $Kh(D)$ and $H(G_0(D))$

Theorem (Helme-Guizon, Przytycki, Rong - 2006)

Let D be a link diagram with all-0 state graph $G_0(D)$. Let ℓ be the girth of $G_0(D)$. There is an isomorphism between $Kh(D)$ and $H(G_0(D))$ in the first $\ell - 1$ supported i-gradings and an isomorphism of $\text{Tor}\, Kh(D)$ and $\text{Tor}\, H(G_0(D))$ in the ℓth i-grading.

$CKh(D)$ and $C(G_0(D))$ at the same time

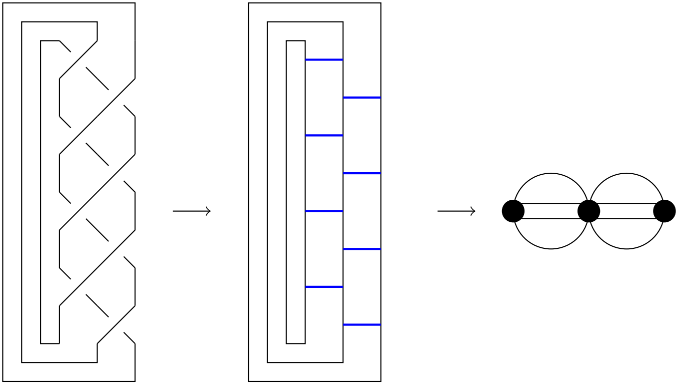


Corollaries on Khovanov homology

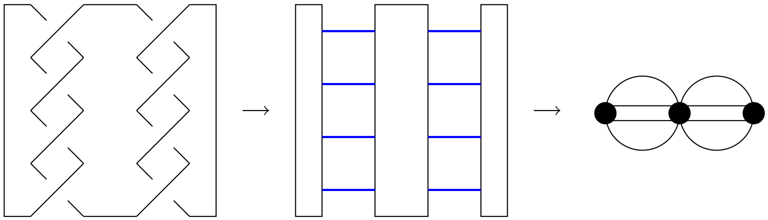
Corollary

Let D and D' be link diagrams such that $G_0(D) = G_0(D')$, and let $\ell > 1$ be the girth of $G_0(D)$. Then there is an isomorphism between $Kh(D)$ and $Kh(D')$ in the first $\ell - 1$ supported i-gradings and an isomorphism between $\operatorname{Tor} Kh(D)$ and $\operatorname{Tor} Kh(D')$ in the ℓth i-grading.

Example $T_{3,4}$



Example $T_{2,4} \# T_{2,4}$



Odd Khovanov analog

Odd Khovanov homology $Kh'(D)$ is a categorification of the Jones polynomial due to Ozsváth, Rasmussen (2007), and Szabó.

Theorem (L., Sazdanović)

Let D and D' be link diagrams such that $G_0(D) = G_0(D')$, and let ℓ be the girth of $G_0(D)$. Then there is an isomorphism between $Kh'(D)$ and $Kh'(D')$ in the first $\ell - 1$ supported i-gradings and an isomorphism between $\mathrm{Tor}\, Kh(D)$ and $\mathrm{Tor}\, Kh(D')$ in the ℓth i-grading.

Proof idea

- Until the first cycle closes in $G_0(D)$, there is a one-to-one correspondence between the components of the Kauffman states of D and D'.
- The extra combinatorial data is also preserved until the first cycle closes.
- The sign assignments can be handled by considering a truncated cube complex.

Chromatic cohomology results

- Pabiniak, Przytycki, and Sazadnović (2006) use Hochschild homology and chromatic cohomology to explicitly compute Khovanov homology (including torsion) of semi-adequate links in certain gradings.
- Przytycki and Sazdanović (2012) strengthen the relationship between chromatic polynomial cohomology and Khovanov homology by modifying the comultiplication map in the chromatic complex.

More on semi-adequate links

- **So far.** The Khovanov homology of semi-adequate links where $G_0(D)$ contains an odd cycle or an even cycle with an edge that is not part of a bigon contains 2-torison.
- **New from Przytycki, Sazdanović.** The Khovanov homology of any semi-adequate link where $G_0(D)$ has girth at least 3 contains 2-torsion.
- **Result.** Shumakovitch's conjecture is true for all semi-adequate links except possibly those where $G_0(D)$ only has 2-cycles.

Spectral sequences

- Khovanov homology and related invariants arise in many spectral sequences.
- These spectral sequences are often only defined over certain coefficient rings (e.g. $\mathbb{Q}$, $\mathbb{Z}_2$, or $\mathbb{Z}_p$ for odd p).
- Use the behavior of these sequences to prove or disprove the existence of torsion.

Lee's differential

- Work over $\mathbb{Q}$ instead of $\mathbb{Z}$.
- Define $\mathbb{Q}$-linear maps

$$m_\Phi : \mathcal{A} \otimes \mathcal{A} \to \mathcal{A} \qquad m_\Phi : \begin{cases} 1 \otimes 1 \mapsto 0 & 1 \otimes x \mapsto 0 \\ x \otimes 1 \mapsto 0 & x \otimes x \mapsto 1 \end{cases}$$

$$\Delta_\Phi : \mathcal{A} \to \mathcal{A} \otimes \mathcal{A} \qquad \Delta_\Phi : \begin{cases} 1 \mapsto 0 \\ x \mapsto 1 \otimes 1. \end{cases}$$

- Using the same conventions as in the definition of Khovanov homology, define a differential Φ on $CKh(D)$.

Lee's spectral sequence

- $(CKh(D), d, \Phi)$ form a double complex, and so there is an associated spectral sequence
- For a knot, the spectral sequence converges to $\mathbb{Q} \oplus \mathbb{Q}$.
- Shumakovitch (2004) showed that this spectral sequence also exists over $\mathbb{Z}_p$ for p an odd prime.

Gradings and Lee's spectral sequence

- Lee's differential is of bidegree $(1, 4)$.
- The bidegree of the map on the r-th page of the spectral sequence is $(1, 4r)$.
- In all known examples (over $\mathbb{Q}$) the spectral sequence collapses after the bidegree $(1, 4)$ differential.
- In such cases, $Kh(D; \mathbb{Q})$ can be arranged into "knight move" pairs.

Knight move example 7_7

	a
a	

$\mathbf{7_7}$	-3	-2	-1	0	1	2	3	4
9								**1**
7							**1**	
5						**2**	**1**	
3					**2**	**1**		
1				**1** + **1**	**2**			
-1			**2**	**1** + **2**				
-3		**1**	**1**					
-5		**2**						
-7	**1**							

Homologically thin links

- A link is homologically thin if its Khovanov homology is entirely supported in two adjacent $j - 2i$ gradings.
- Lee (2002) proved that non-split alternating links are homologically thin.
- Manolescu and Ozsváth (2007) proved that quasi-alternating links are homologically thin.
- If L is homologically thin, then it's Lee spectral sequence collapses after the $(1, 4)$ differential.

An odd torsion theorem

Theorem (Shumakovitch)

If L is homologically thin, then $Kh(L)$ contains no odd torsion.

Sketch.

Suppose that $Kh(L)$ has p-torsion for some odd prime p. Then for some i and j we have

$$\dim Kh^{i,j}(L;\mathbb{Q}) \leq \dim Kh^{i,j}(L;\mathbb{Z}_p).$$

The spectral sequence implies that

$$\dim Kh^{i+1,j+4}(L;\mathbb{Q}) \leq \dim Kh^{i+1,j+4}(L;\mathbb{Z}_p).$$

One can use this to derive a contradiction. □

Application to semi-adequate links

Corollary

Let D be link diagram whose all-0 state graph $G_0(D)$ is **planar** *and has girth $\ell > 1$. Then $Kh(D)$ has no odd torsion in its first ℓ supported i-gradings.*

Proof.

Since $G_0(D)$ is planar, there is an alternating diagram D' with $G_0(D') = G_0(D)$. Because D' is alternating, its Khovanov homology has no odd torsion. The partial isomorphism theorem from earlier implies the result. □

An odd Khovanov version

Corollary

Let D be link diagram whose all-0 state graph $G_0(D)$ is planar and has girth ℓ. Then $Kh'(D)$ is torsion free in its first ℓ supported i-gradings.

Proof.

Since $G_0(D)$ is planar, there is an alternating diagram D' with $G_0(D') = G_0(D)$. Because D' is alternating, its odd Khovanov homology is torsion free (ORS-2007). The result follows from the partial isomorphism proved earlier. □

Analog in chromatic homology

- Chmutov, Chmutov, and Rong (2005) proved that the chromatic complex over $\mathbb{Q}$ has a differential similar to Φ.
- Leads to a spectral sequence analogous to the Lee spectral sequence.
- The chromatic polynomial cohomology is homologically thin.
- One can show this sequence also works over $\mathbb{Z}_p$ for p and odd prime.

An odd torsion theorem for chromatic homology

Theorem (L., Sazdanović)

The chromatic polynomial cohomology of any graph contains no odd torsion.

Remark. The proof is similar to Shumakovitch's proof for Khovanov homology of homologically thin links.

A Khovanov homology corollary

Corollary

Let D be link diagram whose all-0 state graph $G_0(D)$ and has girth $\ell > 1$. Then $Kh(D)$ has no odd torsion in its first ℓ supported i-gradings.

Remark 1. The chromatic polynomial cohomology result allows us to relax the planarity condition.

Remark 2. The above corollary also applies to the Khovanov homology of certain virtual links.

More spectral sequences

- Shumakovitch defines maps ν^r of bidegree $(0, 2r)$ on $CKh(D; \mathbb{Z}_2)$.
- The maps ν^r are compatible with other differentials.
- The Euler characteristic of $Kh(D; \mathbb{Z}_2)$ with respect to ν^r is 0.

More torsion theorems

Theorem (Shumakovitch)

If L is homologically thin, then its Khovanov homology only has 2*-torsion.*

Corollary

If K is a nontrivial, homologically thin knot, then its Khovanov homology contains 2*-torsion.*

Speculation

- Does chromatic polynomial cohomology contain v^r-like differentials?
- If so, does this imply that chromatic polynomial cohomology contains only 2-torsion?

Maximum girth

- Define the 0-girth of L to be

$$g_0(L) = \max\{\text{girth}(G_0(D)) \mid D \text{ is a diagram of } L\}.$$

- Define 1-girth of L to be $g_0(\overline{L})$.
- L is adequate if both $g_0(L)$ and $g_1(L)$ are greater than one, and L is semi-adequate if one of $g_0(L)$ or $g_1(L)$ is greater than one.
- Large girth simplifies formulas for the head and tail of the colored Jones polynomial.

Khovanov upper bound on girth

Suppose that the Khovanov homology of L lies on more than 2 diagonals. Let m be the difference between the first i-grading on which a third diagonal supports $Kh(L)$ and the first i-grading on which that $Kh(L)$ is supported. Then $g_0(L) \leq m$.

Example 9_{42}

9_{42}	-4	-3	-2	-1	0	1	2
7							1
5							1_2
3					1	1	
1				1	1_2 1		
-1				1_2 1	1		
-3		1	1				
-5		1_2					
-7	1						

$$g_0(L) \leq 4$$

Questions

- How can we generate odd torsion in $Kh(L)$?
- What does torsion in $Kh(L)$ tell us about the link?
- Can we explain the difference in torsion in $Kh(L)$ and $Kh'(L)$?
- Exploiting torsion in functorality.

Thank you!