Odd torsion in the Khovanov homology of semi-adequate links

Adam Lowrance - Vassar College
Joint with Radmila Sazdanović - North Carolina State University

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Overview

- Experimentally, $Kh(L)$ has an abundance of torsion, only some of which we can explain.
- In almost all examples, $Kh(L)$ contains 2-torsion.
- Torsion of order other than 2 appears, but is much more rare.

Motivating conjecture

Conjecture (Shumakovitch)

Let L be any prime link other than the unknot or the Hopf link. Then $Kh(L)$ contains 2-torsion.

- The conjecture implies that Khovanov homology is an unknot detector.
- Kronheimer and Mrowka (2010) proved Khovanov homology is an unknot detector using gauge theory.
- The conjecture is known to be true in many cases.

Some more conjectures

Conjecture (Przytycki - Sazdanović)

- *The Khovanov homology of a closed* 3*-braid can have only* 2*-torsion.*
- *The Khovanov homology of a closed* 4*-braid can have only* 2*-torsion or* 4*-torsion.*
- *The Khovanov homology of a closed* n*-braid cannot have* p*-torsion for* $p > n$*, where* p *is prime.*

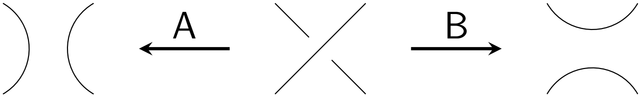
Methods

Some approaches for proving things about torsion in Khovanov homology are:

- explicit construction (Asaeda, Przytycki),
- connections with Hochschild homology (Przytycki),
- connections with chromatic polynomial cohomology (Helme-Guizon, Pabiniak, Przytycki, Rong, Sazdanović), and
- spectral sequence arguments (Shumakovitch).

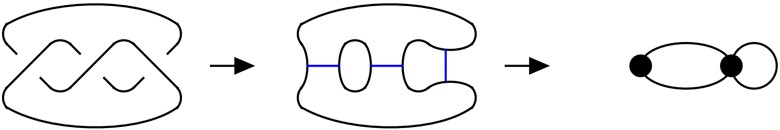
Kauffman states

- Each crossing has an A and a B resolution.



- The collection of simple closed curves in the plane obtained by taking an A or B resolution at each crossing is a *Kauffman state*.
- The all-A state graph G of D has vertices corresponding to the components of the all-A state of D and edges corresponding to the crossings of D.

A 3-crossing unknot



Khovanov and chromatic homology

- Let D be a link diagram, and let G be its all-A state graph.
- The Khovanov homology of D is $Kh(D)$.
- The chromatic homology of G is $H(G)$.
- Both are bigraded:

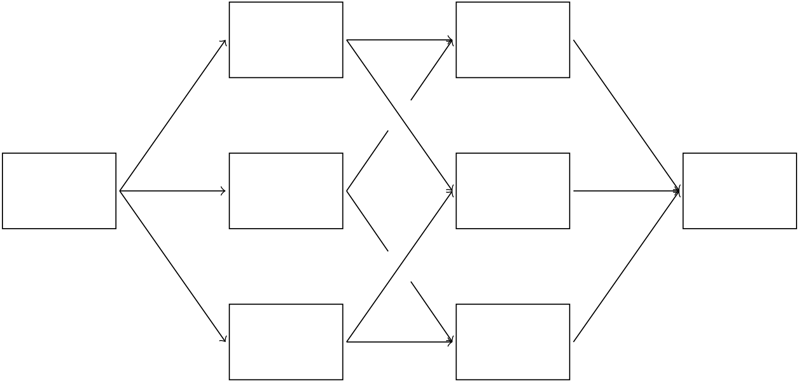
$$Kh(D) = \bigoplus_{i,j} Kh^{i,j}(D) \text{ and } H(G) = \bigoplus_{i,j} H^{i,j}(G).$$

Comparing $Kh(D)$ and $H(G)$

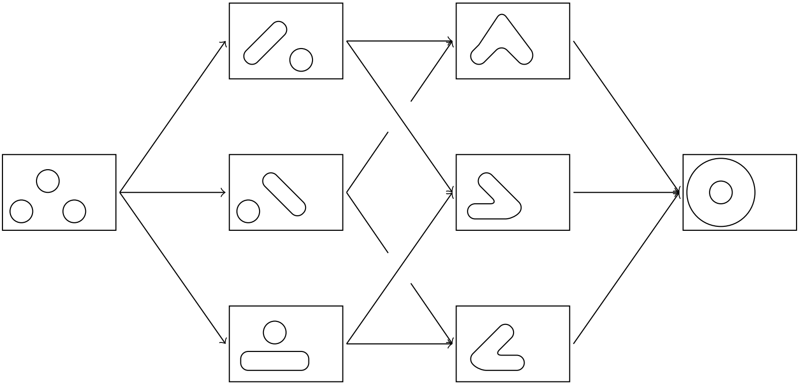
Theorem (Helme-Guizon, Przytycki, Rong - 2006)

If the length ℓ of the shortest cycle in G is greater than one, then there is an isomorphism between $Kh(D)$ and $H(G)$ in the first $\ell - 1$ supported i-gradings and an isomorphism of $\mathrm{Tor}\, Kh(D)$ *and* $\mathrm{Tor}\, H(G)$ *in the ℓth i-grading.*

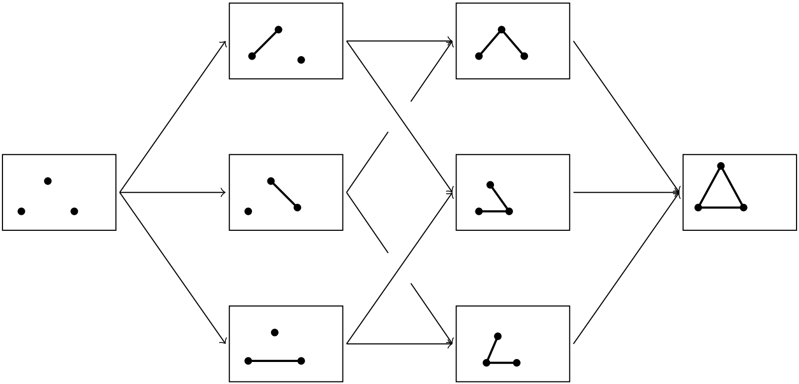
Hypercube



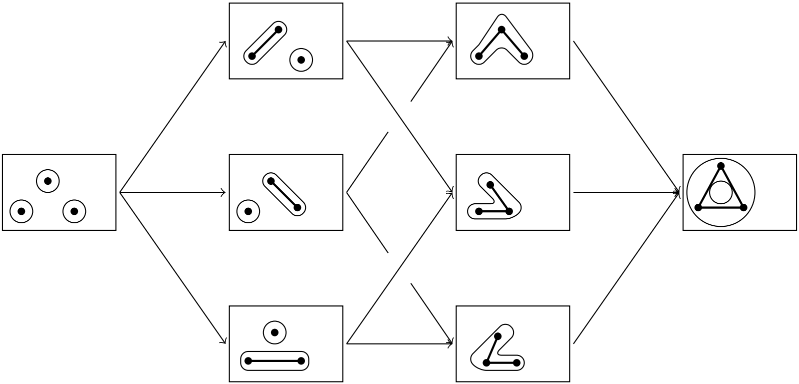
Kauffman states



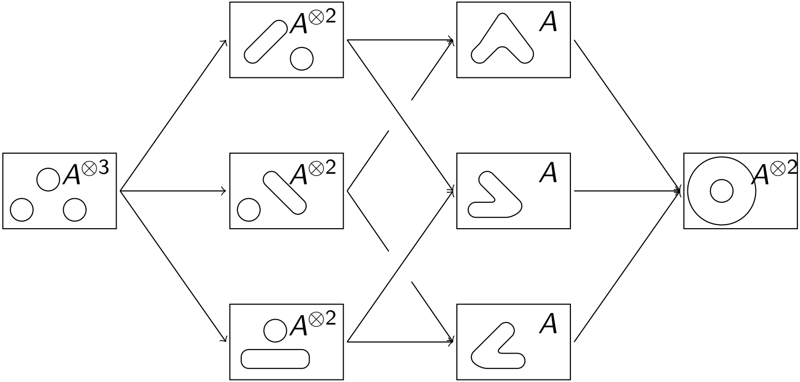
Spanning subgraphs



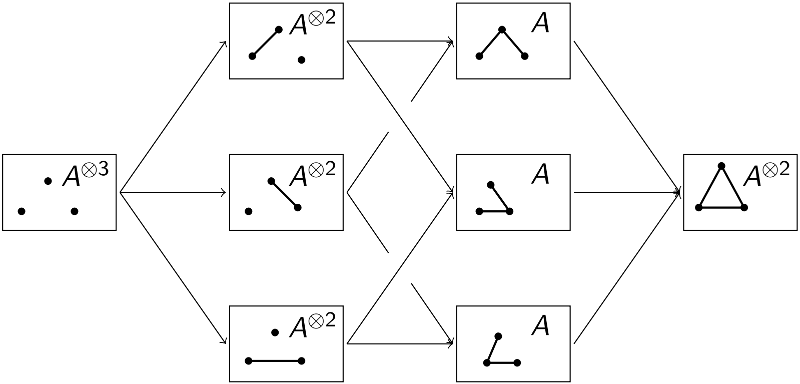
Kauffman states and spanning subgraphs



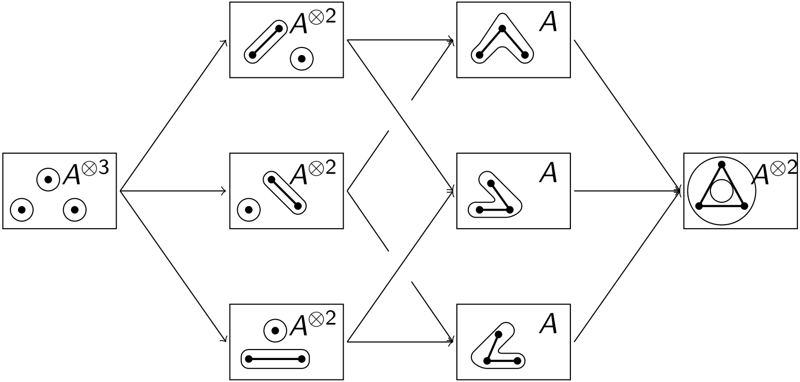
Kauffman states and spaces



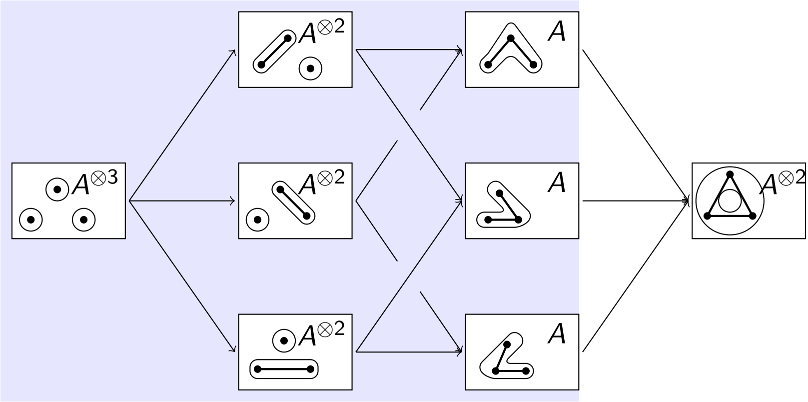
Spanning subgraphs and spaces



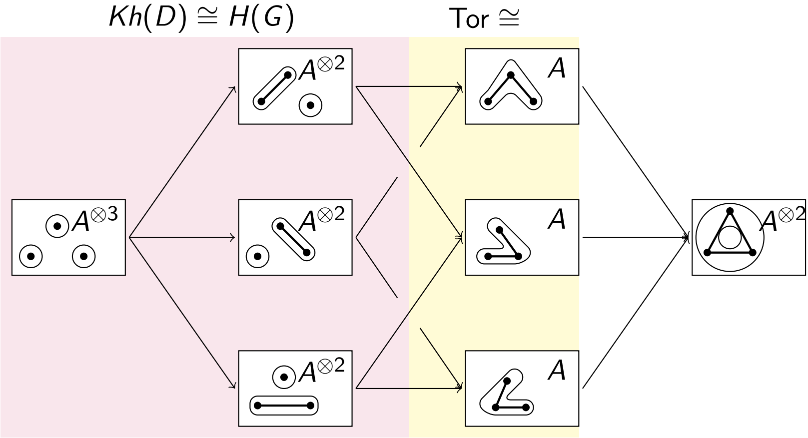
Spaces for both



Multiplication until a cycle closes



Partial Isomorphism Picture



Khovanov homology of nonsplit alternating links

- Let L be a nonsplit alternating link.
- (Lee) $Kh(L;\mathbb{Q})$ lies on two adjacent $j-2i$ diagonals.
- (Shumakovitch) $Kh(L)$ has no torsion other than 2-torsion.
- $Kh(L)$ is determined by the Jones polynomial of L.

The chromatic homology

- (Chmutov, Chmutov, Rong) The chromatic homology $H(G; \mathbb{Q})$ lies on two adjacent $i + j$ diagonals.
- (Helme-Guizon, Przytycki, Rong) All torsion in $H(G)$ lies on $i + j = v(G)$ diagonal.
- (L., Sazdanović) The chromatic homology $H(G)$ contains no odd torsion.

A Khovanov homology corollary

Corollary

Let D be a diagram with all-A state graph G. Suppose that the length ℓ of the shortest cycle of G is greater than one. Then $Kh(D)$ has no odd torsion in its first ℓ supported i-gradings.

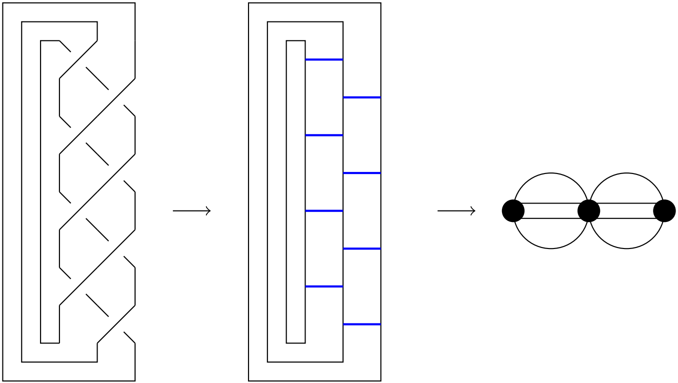
Odd Khovanov analog

Odd Khovanov homology $Kh'(D)$ is a categorification of the Jones polynomial due to Ozsváth, Rasmussen, and Szabó (2007).

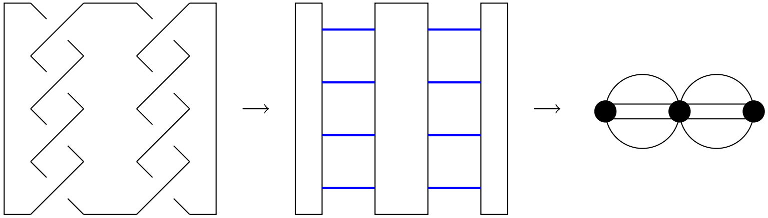
Theorem (L., Sazdanović)

Let D and D' be link diagrams with the same all-A state graph G. Suppose that the length ℓ of the shortest cycle of G is greater than one. Then there is an isomorphism between $Kh'(D)$ and $Kh'(D')$ in the first $\ell - 1$ supported i-gradings and an isomorphism between Tor $Kh(D)$ *and* Tor $Kh(D')$ *in the ℓth i-grading.*

Example $T_{3,4}$



Example $T_{2,4} \# T_{2,4}$



An odd Khovanov corollary

Corollary

Let D be link diagram whose all-A state graph G is planar and has shortest cycle of length ℓ. Then $Kh'(D)$ is torsion free in its first ℓ supported i-gradings.

Proof.

Since G is planar, there is an alternating diagram D' whose all-A state graph is G. Because D' is alternating, its odd Khovanov homology is torsion free (ORS-2007). The result follows from the partial isomorphism proved earlier. □

Maximum girth

- Define the A-girth of L to be

$$g_A(L) = \max\{\text{girth}(G) \mid D \text{ is a diagram of } L\},$$

  where G is the all-A state graph of D and girth(G) is the length of the shortest cycle of G.

- L is adequate if both $g_A(L)$ and $g_A(\overline{L})$ are greater than one, and L is semi-adequate if one of $g_A(L)$ or $g_A(\overline{L})$ is greater than one.

- Large girth simplifies formulas for the head and tail of the colored Jones polynomial.

Khovanov upper bound on girth

Suppose that the Khovanov homology of L lies on more than 2 diagonals. Let m be the difference between the first i-grading on which a third diagonal supports $Kh(L)$ and the first i-grading on which that $Kh(L)$ is supported. Then $g_A(L) \leq m$.

Example 9_{42}

9_{42}	-4	-3	-2	-1	0	1	2
7							1
5							1_2
3					1	1	
1				1	1_2 1		
-1				1_2 1	1		
-3		1	1				
-5		1_2					
-7	1						

$$g_0(L) \leq 4$$

Questions

- Can we show that $H(G)$ contains only 2-torsion?
- How can we generate odd torsion in $Kh(L)$?
- What does torsion in $Kh(L)$ tell us about the link?
- Can we explain the difference in torsion in $Kh(L)$ and $Kh'(L)$?
- Exploiting torsion in functorality.

Thank you!