Chromatic homology, Khovanov homology, and torsion

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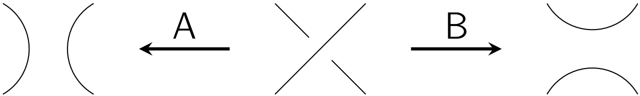
Overview

- Chromatic homology is a categorification of the chromatic polynomial. Khovanov homology is a categorification of the Jones polynomial.
- There is a partial isomorphism between the Khovanov homology of a link and the chromatic homology of an all-A state graph of the link.
- We show that the chromatic homology of a graph contains only torsion of order 2.
- In gradings where the partial isomorphism is defined, Khovanov homology has only torsion of order 2.

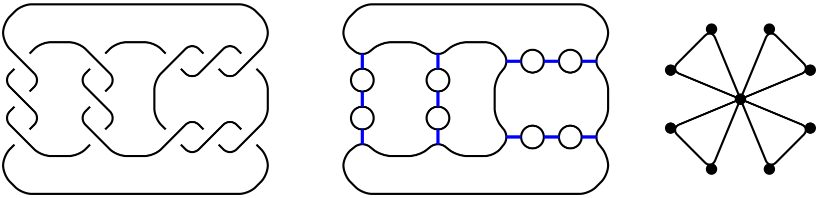
Kauffman states

- Each crossing has an A and a B resolution.



- The collection of simple closed curves in the plane obtained by taking an A or B resolution at each crossing is a *Kauffman state*.
- The all-A state graph $G_A(D)$ of D has vertices corresponding to the components of the all-A state of D and edges corresponding to the crossings of D.

Constructing $G_A(D)$



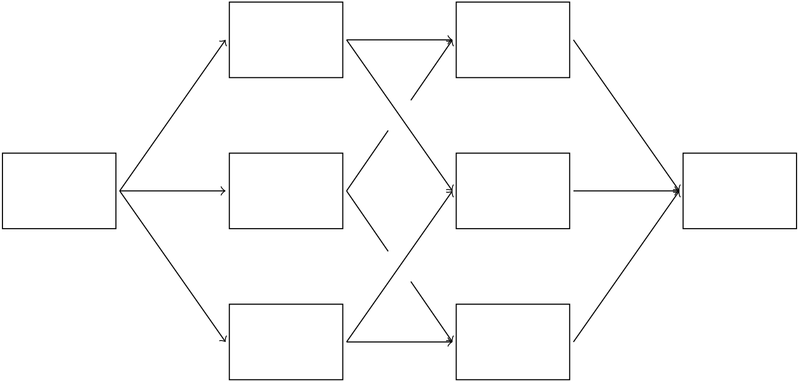
Khovanov and chromatic homology

- Let D be a link diagram, and let G be its all-A state graph.
- The Khovanov homology of D is $Kh(D)$.
- The chromatic homology of G is $H(G)$.
- Both are bigraded:

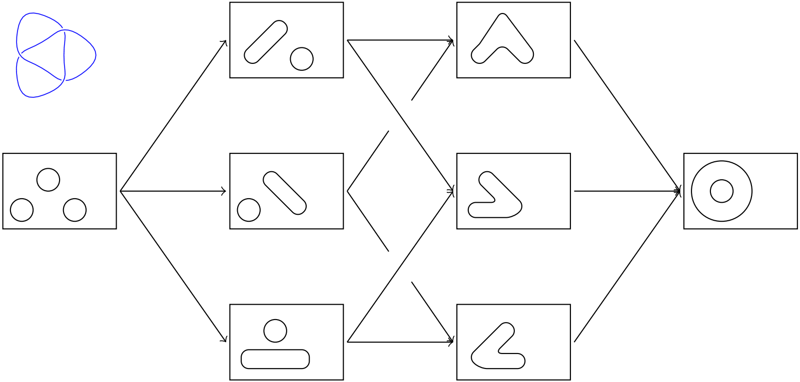
$$Kh(D) = \bigoplus_{i,j} Kh^{i,j}(D) \text{ and } H(G) = \bigoplus_{i,j} H^{i,j}(G).$$

- Let $A = R[x]/(x^2)$ where $R = \mathbb{Z}, \mathbb{Z}_p$, or $\mathbb{Q}$.

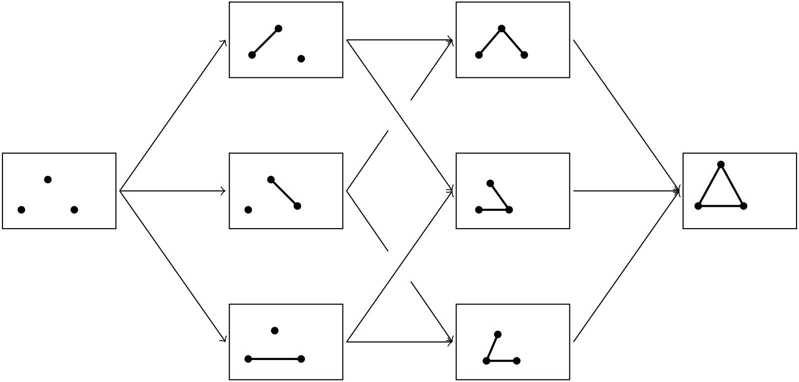
Hypercube



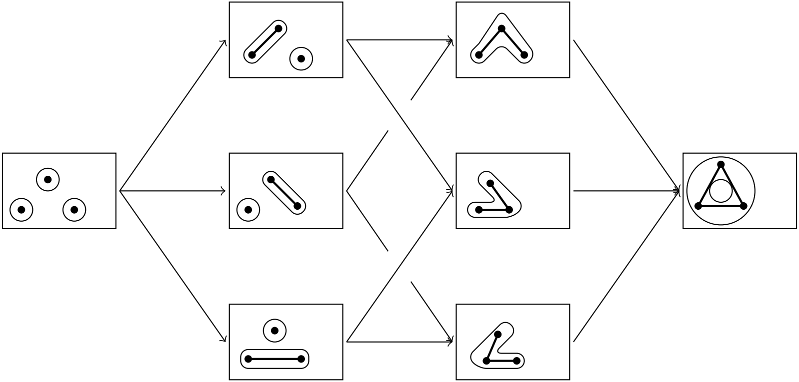
Kauffman states



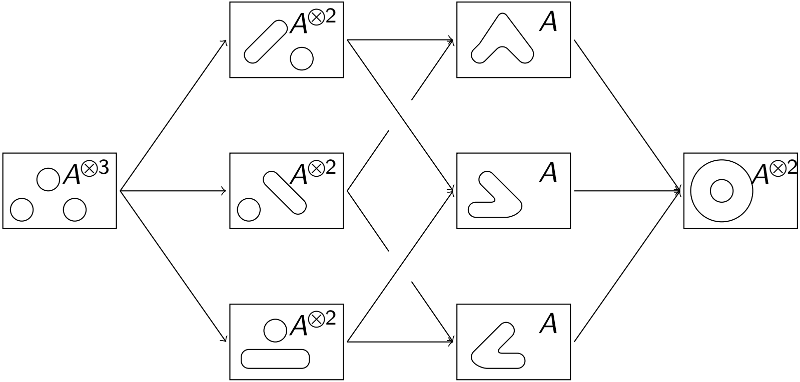
Spanning subgraphs



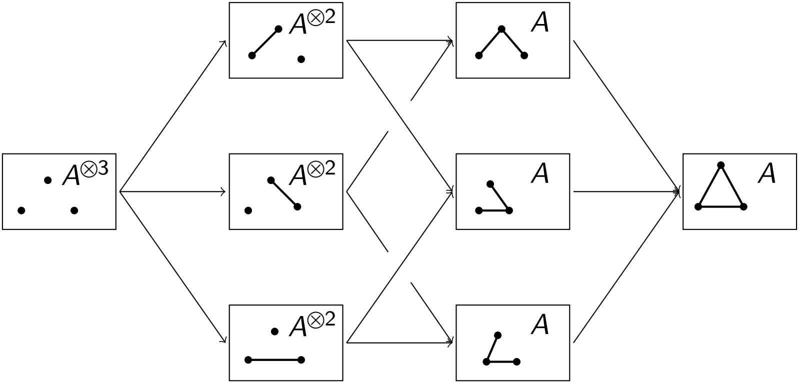
Kauffman states and spanning subgraphs



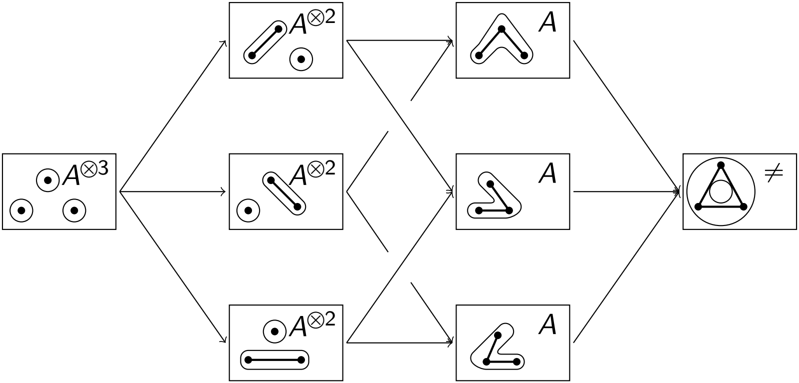
Kauffman states and spaces



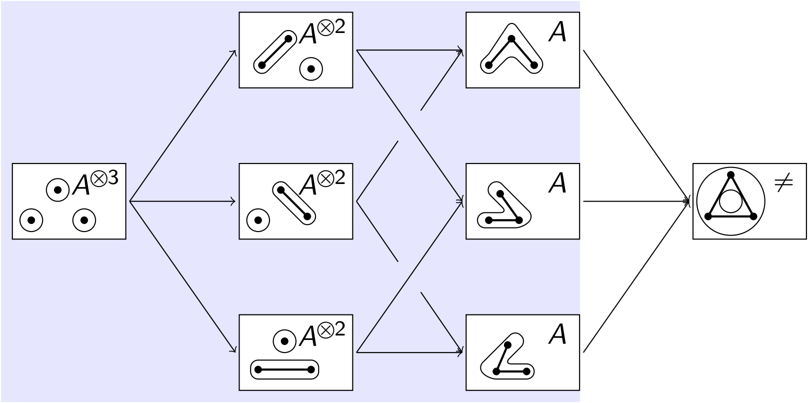
Spanning subgraphs and spaces



Spaces for both



Multiplication until a cycle closes

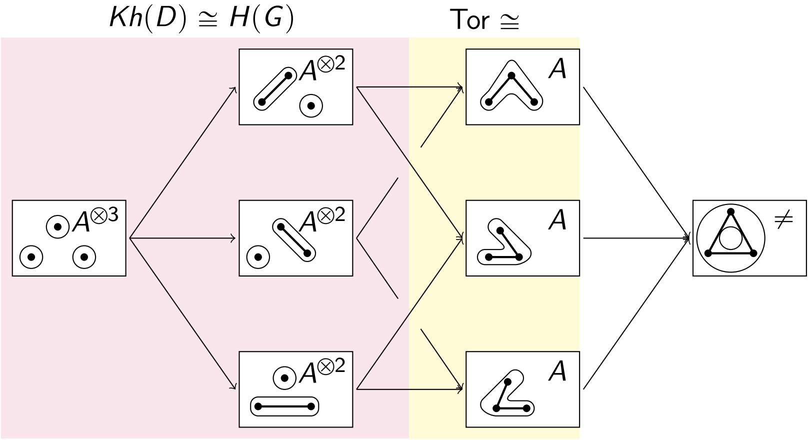


The multiplication map

Define the R-linear multiplication map by

$$m : A \otimes A \to A \qquad m : \begin{cases} 1 \otimes 1 \mapsto 1 & 1 \otimes x \mapsto x \\ x \otimes 1 \mapsto x & x \otimes x \mapsto 0 \end{cases}$$

Partial Isomorphism Picture



Comparing $Kh(D)$ and $H(G)$

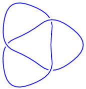
Theorem (Helme-Guizon, Przytycki, Rong - 2006)

If the length g of the shortest cycle in G is greater than one, then there is an isomorphism between $Kh(D)$ and $H(G)$ in the first $g-1$ supported i-gradings and an isomorphism of $\operatorname{Tor} Kh(D)$ and $\operatorname{Tor} H(G)$ in the gth i-grading.

Example: 3_1

j \ i	0	1	2
3	$\mathbb{Z}$		
2		$\mathbb{Z}_2$	
1		$\mathbb{Z}$	

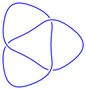
j \ i	-3	-2	-1	0
-1				$\mathbb{Z}$
-3				$\mathbb{Z}$
-5		$\mathbb{Z}$		
-7		$\mathbb{Z}_2$		
-9	$\mathbb{Z}$			



Example: 3_1

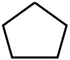
j \ i	0	1	2
3	$\mathbb{Z}$		
2		$\mathbb{Z}_2$	
1		$\mathbb{Z}$	

j \ i	-3	-2	-1	0
-1				$\mathbb{Z}$
-3				$\mathbb{Z}$
-5		$\mathbb{Z}$		
-7		$\mathbb{Z}_2$		
-9	$\mathbb{Z}$			

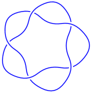


Example: 5_1

j\i	0	1	2	3	4
5	$\mathbb{Z}$				
4		$\mathbb{Z}_2$			
3		$\mathbb{Z}$	$\mathbb{Z}$		
2				$\mathbb{Z}_2$	
1				$\mathbb{Z}$	



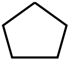
j\i	-5	-4	-3	-2	-1	0
-3						$\mathbb{Z}$
-5						$\mathbb{Z}$
-7				$\mathbb{Z}$		
-9				$\mathbb{Z}_2$		
-11		$\mathbb{Z}$	$\mathbb{Z}$			
-13		$\mathbb{Z}_2$				
-15	$\mathbb{Z}$					

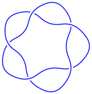


Example: 5_1

j\i	0	1	2	3	4
5	$\mathbb{Z}$				
4		$\mathbb{Z}_2$			
3		$\mathbb{Z}$	$\mathbb{Z}$		
2				$\mathbb{Z}_2$	
1				$\mathbb{Z}$	

j\i	-5	-4	-3	-2	-1	0
-3						$\mathbb{Z}$
-5						$\mathbb{Z}$
-7				$\mathbb{Z}$		
-9				$\mathbb{Z}_2$		
-11		$\mathbb{Z}$	$\mathbb{Z}$			
-13		$\mathbb{Z}_2$				
-15	$\mathbb{Z}$					





Torsion in chromatic homology

Theorem (L - Sazdanović)

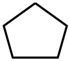
The chromatic homology of a graph contains only torsion of order two.

The shape of $H(G)$

Theorem (Helme-Guizon, Przytycki, Rong - 2006)

Let G be a connected graph with n vertices. Then $H^{i,j}(G) = 0$ unless $n-1 \leqslant i+j \leqslant n$, and $\mathrm{Tor}\, H^{i,j}(G) = 0$ unless $i+j=n$.

j\i	0	1	2	3
5	$\mathbb{Z}$			
4		$\mathbb{Z}_2$		
3		$\mathbb{Z}$	$\mathbb{Z}$	
2				$\mathbb{Z}_2$
1				$\mathbb{Z}$

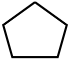


The shape of $H(G)$

Theorem (Chmutov, Chmutov, Rong - 2008)

Let G be a connected non-bipartite graph. Then the summands of $H(G;\mathbb{Q})$ can be arranged in "knight move" pairs.

j\i	0	1	2	3
5	$\mathbb{Q}$			
4				
3		$\mathbb{Q}$	$\mathbb{Q}$	
2				
1				$\mathbb{Q}$



$H(G)$ contains no torsion of odd order

j\i	0	1	2	3
5	$\mathbb{Q}$			
4				
3		$\mathbb{Q}$	$\mathbb{Q}$	
2				
1				$\mathbb{Q}$

Suppose that $H(G;\mathbb{Q})$ is above.

$H(G)$ contains no torsion of odd order

j\i	0	1	2	3
5	$\mathbb{Q}$			
4				
3		$\mathbb{Q}$	$\mathbb{Q}$	
2				
1				$\mathbb{Q}$

j\i	0	1	2	3
5	$\mathbb{Z}_p$			
4				
3		$\mathbb{Z}_p$	$\mathbb{Z}_p$	
2				
1				$\mathbb{Z}_p$

Then $H(G;\mathbb{Z}_p)$ has at least these summands.

$H(G)$ contains no torsion of odd order

j\i	0	1	2	3
5	$\mathbb{Q}$			
4				
3		$\mathbb{Q}$	$\mathbb{Q}$	
2				
1				$\mathbb{Q}$

j\i	0	1	2	3
5	$\mathbb{Z}_p$			
4				
3		$\mathbb{Z}_p$	$\mathbb{Z}_p$	
2				
1				$\mathbb{Z}_p$

j\i	0	1	2	3
5	$\mathbb{Z}$			
4				
3		$\mathbb{Z}$	$\mathbb{Z}$	
2				
1				$\mathbb{Z}$

The free part of $H(G)$ looks as above.

$H(G)$ contains no torsion of odd order

j\i	0	1	2	3
5	$\mathbb{Q}$			
4				
3		$\mathbb{Q}$	$\mathbb{Q}$	
2				
1				$\mathbb{Q}$

j\i	0	1	2	3
5	$\mathbb{Z}_p$			
4		$\mathbb{Z}_p$		
3		$\mathbb{Z}_p$	$\mathbb{Z}_p$	
2				
1				$\mathbb{Z}_p$

j\i	0	1	2	3
5	$\mathbb{Z}$			
4		$\mathbb{Z}_{p^k}$		
3		$\mathbb{Z}$	$\mathbb{Z}$	
2				
1				$\mathbb{Z}$

Suppose that $H(G)$ has torsion of order p^k for some odd p. Let the pictured $\mathbb{Z}_{p^k}$ summand be in the maximum i-grading of any p^m torsion in $H(G)$.

$H(G)$ contains no torsion of odd order

j\i	0	1	2	3
5	$\mathbb{Q}$			
4				
3		$\mathbb{Q}$	$\mathbb{Q}$	
2				
1				$\mathbb{Q}$

j\i	0	1	2	3
5	$\mathbb{Z}_p$			
4		$\mathbb{Z}_p$		
3		$\mathbb{Z}_p$	$\mathbb{Z}_p$	
2			$\mathbb{Z}_p$	
1				$\mathbb{Z}_p$

j\i	0	1	2	3
5	$\mathbb{Z}$			
4		$\mathbb{Z}_{p^k}$		
3		$\mathbb{Z}$	$\mathbb{Z}$	
2				
1				$\mathbb{Z}$

Theorem. $H(G;\mathbb{Z}_p)$ can be arranged in knight move pairs.

$H(G)$ contains no torsion of odd order

j\i	0	1	2	3
5	$\mathbb{Q}$			
4				
3		$\mathbb{Q}$	$\mathbb{Q}$	
2				
1				$\mathbb{Q}$

j\i	0	1	2	3
5	$\mathbb{Z}_p$			
4		$\mathbb{Z}_p$		
3		$\mathbb{Z}_p$	$\mathbb{Z}_p$	
2			$\mathbb{Z}_p$	$\mathbb{Z}_p$
1				$\mathbb{Z}_p$

j\i	0	1	2	3
5	$\mathbb{Z}$			
4		$\mathbb{Z}_{p^k}$		
3		$\mathbb{Z}$	$\mathbb{Z}$	
2				$\mathbb{Z}_{p^\ell}$
1				$\mathbb{Z}$

The universal coefficient theorem implies that $H(G;\mathbb{Z}_p)$ and $H(G)$ looks like above.

Progress so far

- $H(G)$ contains no torsion of odd order.
- All torsion in $H(G)$ must be of order 2^k for some k.
- It remains to show that $k = 1$.

The vertical map

- For a connected graph G, there is an isomorphism

$$\nu_{\downarrow}^{*} : H^{i,n-i}(G;\mathbb{Z}_2) \to H^{i,n-i-1}(G;\mathbb{Z}_2).$$

$\mathbb{Q}$	
	$\mathbb{Q}$

knight move pair

$\mathbb{Z}_2$	
$\mathbb{Z}_2$	$\mathbb{Z}_2$
	$\mathbb{Z}_2$

tetromino

$\mathbb{Z}$	
	$\mathbb{Z}_{2^k}$
	$\mathbb{Z}$

with $\mathbb{Z}$ coefficients

Bockstein spectral sequence

The $\mathbb{Z}_2$-Bockstein spectral sequence satisfies the following.

- The E_1 page of the Bockstein spectral sequence is $H(G;\mathbb{Z}_2)$.
- The E_∞ page of the Bockstein spectral sequence is $[H(G)/\operatorname{Tor} H(G)] \otimes \mathbb{Z}_2$.
- If the Bockstein spectral sequence converges at the 2nd page, then $H(G)$ has no torsion of order 2^k for $k \geqslant 2$.

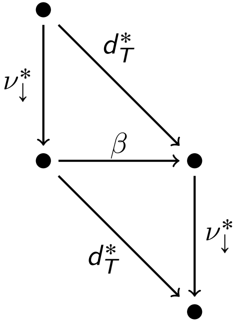
Bockstein example

- $H(G) = \mathbb{Z}^{a_0} \oplus \mathbb{Z}_2^{a_1} \oplus \mathbb{Z}_4^{a_2} \oplus \cdots \oplus \mathbb{Z}_{2^k}^{a_k}$.
- $E_1 = \mathbb{Z}_2^{a_0} \oplus \mathbb{Z}_2^{a_1} \oplus \mathbb{Z}_2^{a_1} \oplus \mathbb{Z}_2^{a_2} \oplus \mathbb{Z}_2^{a_2} \oplus \cdots \oplus \mathbb{Z}_2^{a_k} \oplus \mathbb{Z}_2^{a_k}$.
- $E_2 = \mathbb{Z}_2^{a_0} \oplus \mathbb{Z}_2^{a_2} \oplus \mathbb{Z}_2^{a_2} \oplus \cdots \oplus \mathbb{Z}_2^{a_k} \oplus \mathbb{Z}_2^{a_k}$.
- $E_\infty = \mathbb{Z}_2^{a_0}$.

Goal: If β is the Bockstein map on the E_1 page, then we want to show that the rank of β is the number of tetrominoes N in $H(G; \mathbb{Z}_2)$.

The Turner differential

- There is a differential $d_T : C^{i,j}(G;\mathbb{Z}_2) \to C^{i+1,j-1}(G;\mathbb{Z}_2)$.
- It induces a map $d_T^* : H^{i,j}(G;\mathbb{Z}_2) \to H^{i+1,j-1}(G;\mathbb{Z}_2)$.

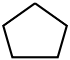


The Bockstein sequence converges at the E_2 page

1. $\nu_\downarrow^* \circ d_T^* = d_T^* \circ \nu_\downarrow^*$.
2. On each diagonal, $\operatorname{rank} d_T^* = N$.
3. $d_T^* = \beta \circ \nu_\downarrow^* + \nu_\downarrow^* \circ \beta$.
4. $\operatorname{rank} \beta = N$.

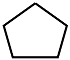
Our example

$j \backslash i$	0	1	2	3
5	$\mathbb{Z}_2$			
4	$\mathbb{Z}_2$	$\mathbb{Z}_2$		
3		$\mathbb{Z}_2$	$\mathbb{Z}_2$	
2			$\mathbb{Z}_2$	$\mathbb{Z}_2$
1				$\mathbb{Z}_2$



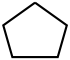
Our example: d_T^*

j\i	0	1	2	3
5	$\mathbb{Z}_2$			
4	$\mathbb{Z}_2$	$\mathbb{Z}_2$		
3		$\mathbb{Z}_2$	$\mathbb{Z}_2$	
2			$\mathbb{Z}_2$	$\mathbb{Z}_2$
1				$\mathbb{Z}_2$



Our example: $\beta \circ \nu_{\downarrow}^{*} + \nu_{\downarrow}^{*} \circ \beta$

j\i	0	1	2	3
5	$\mathbb{Z}_2$			
4	$\mathbb{Z}_2$	$\mathbb{Z}_2$		
3		$\mathbb{Z}_2$	$\mathbb{Z}_2$	
2			$\mathbb{Z}_2$	$\mathbb{Z}_2$
1				$\mathbb{Z}_2$



Consequences

1. Chromatic homology $H(G)$ contains only torsion of order two.
2. Let D be a link diagram with all-A state graph $G_A(D)$ where the shortest cycle in $G_A(D)$ is of length g. The first g homological gradings of $Kh(D)$ have only torsion of order two.
3. Chromatic homology $H(G)$ is determined by the chromatic polynomial.

Thank you!